PREDICTING SUDDEN CHANGES IN POLLUTANT PLUMES USING LCS-CORE ANALYSIS

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Summary We have recently introduced a methodology, *LCS-core analysis*, to predict major short-term changes in pollutant plumes such as those produced by oil spills in the ocean and volcanic eruptions in the atmosphere. LCS-core analysis is based on objective (frame-independent) identification of key material curves, widely known as Lagrangian Coherent Structures (LCSs), which drive tracer mixing in unsteady two-dimensional flows defined over a finite-time interval. LCS-core analysis has the potential to forecast imminent shape changes in the pollutant plume, even before an instability builds up and brings large masses of water or air into motion. Furthermore, LCS-core analysis provides a model-independent forecasting scheme that relies only on already observed or validated flow velocities at the time the prediction is made. We illustrate all this by making high-precision forecasts of major instabilities that occurred in the shape of the oil slick produced by the *Deepwater Horizon* explosion in the Gulf of Mexico and the ash cloud produced by the Eyjafjallajökull volcano eruption in Iceland.

INTRODUCTION

Concealed in the flow are distinguished material surfaces (curves in two-space dimensions) which form the centerpieces of observable coherent patterns acquired by fluid particle (passive tracer) trajectories. Such distinguished material surfaces, broadly known as *Lagrangian Coherent Structures* (*LCSs*), correspond to material surfaces such that, overall over a finite-time interval, normally attract or repel nearby fluid particle trajectories at the locally strongest rate [2, 1]. Attracting LCSs thus constitute the centerpieces of stretching, while repelling LCSs are responsible for producing folding. It turns out that some of these LCSs admit highly attracting cores that lead to inevitable material instabilities even under future uncertainties or unexpected perturbations to the observed flow. These LCS cores thus have the potential to forecast imminent shape changes in a tracer patch, even before the instability builds up and brings large masses of fluid into motion. Exploiting this potential, Olascoaga and Haller [3] develop a methodology, called *LCS-core analysis*, which provides a model-independent forecasting scheme that relies only on already observed or validated flow velocities at the time the prediction is made.

LAGRANGIAN COHERENT STRUCTURES (LCSs)

Assume that the flow takes place on a region of the plane over a finite-time interval. Let $u(x,t)$ denote the (presumed smooth) velocity of the fluid, where x is position and t is time. The evolution of fluid particles obeys the *nonautonomous dynamical system*

$$\dot{x} = u(x,t). \tag{1}$$

The *flow map*, given by

$$F_{t_0}^t : x_0 \mapsto x_t = x(t; x_0, t_0), \tag{2}$$

takes a fluid particle from position x_0 at time t_0 to position x_t at time t.
Consider now a collection of fluid particles at time t_0 defining a curve γ_0. This curve is advected by the flow map into a time-evolving material curve $\gamma_t = F_{t_0}^t(\gamma_0)$. To measure how attracting or repelling γ_t is, we can consider the evolution of the unit normal to each $x_0 \in \gamma_0$, n_0, under the action of the linearized flow map, $DF_{t_0}^t(x_0)$. More precisely, let $\rho_{t_0}^t(x_0)$, which can be called a *normal repulsion rate*, denote the length of the projection of the vector $DF_{t_0}^t(x_0)n_0$ in the direction normal to $x_t \in \gamma_t$, n_t. Namely,

$$\rho_{t_0}^t(x_0) := \langle n_t, DF_{t_0}^t(x_0)n_0 \rangle, \tag{3}$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product. If $\rho_{t_0}^{t_0+\tau}(x_0) > 1$ for each $x_0 \in \gamma_0$, then γ_t is, overall over the finite-time interval $[t_0, t_0+\tau]$, normally repelling or attracting along all fluid particle trajectories starting on γ_0 depending on whether these are computed in forward ($\tau > 0$) or backward ($\tau < 0$) time, respectively.
Using the above normal repulsion notion, [1] formally defines *repelling* (*attracting*) *LCS* over $[t_0, t_0+\tau]$, where $\tau > 0$ ($\tau < 0$), as a normally repelling (attracting) material curve γ_t whose normal repulsion rate is pointwise maximal along such material curve at time t_0, γ_0, among all nearby normally repelling (attracting) material curves.

LCS CORES

To identify the most influential LCS in spreading a passive tracer patch (e.g., a pollutant plume) we seek to isolate special ensembles of fluid particle trajectories, which we will refer to as *LCS cores*, that have the property of attracting nearby trajectories over any arbitrary short subinterval of a given finite-time interval. This property is different than that of fluid particle trajectories that give rise to attracting LCS, which, attracting nearby trajectories overall over a finite-time interval at locally the highest rate, constitute the centerpieces of stretching. In other words, LCS cores are composed of uniformly

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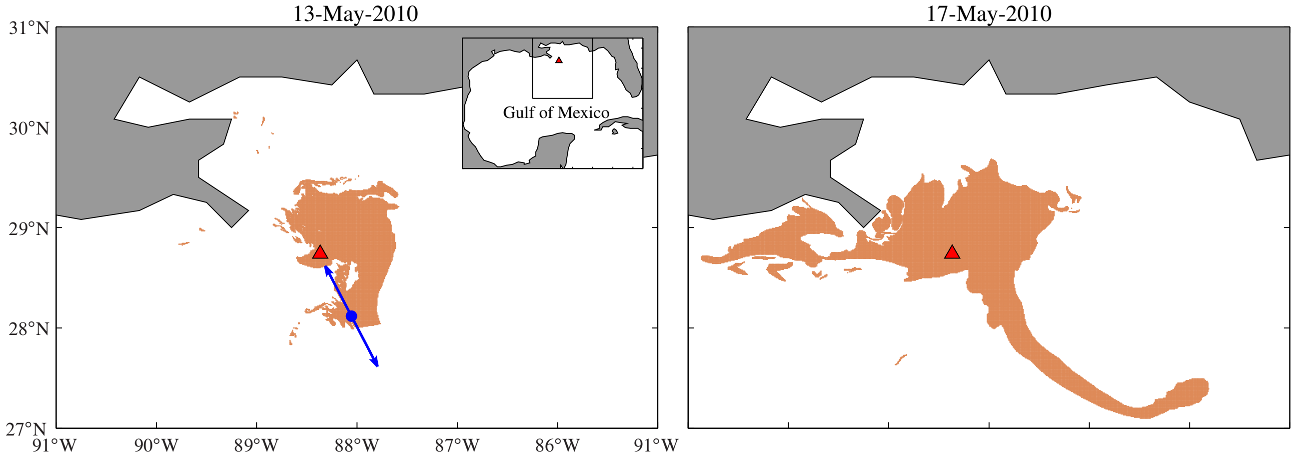

Figure 1. LCS-core analysis applied to the *Deepwater Horizon* oil spill in the Gulf of Mexico. The spill site is indicated by a triangle.

attracting fluid particle trajectories, and thus can be regarded as generalized saddle-type points. As such, marked fluid particles within an LCS core at certain time will be repelled away from the core at subsequent times delineating the LCS of interest. Hence, a passive tracer patch situated in a region traversed by an attracting LCS containing an LCS core will be persistently stretched along the LCS, which will act as the centerpiece of the coherent pattern formed.

To locate LCS cores we make use of the normal repulsion notion discussed above. Concretely, note the convenient relationship

$$\partial_t \rho_{t_0}^t(x_0) = \rho_{t_0}^t(x_0) r(x_t, t) \tag{4}$$

where $r(x_t,t) := \langle n_t, S(x_t,t)n_t\rangle$ and $S(x_t,t) := \frac{1}{2}\left(\nabla u(x,t) + \nabla u(x,t)^{\mathrm{T}}\right)_{x=x_t}$, which can be called *Lagrangian strain rate* and *Lagrangian strain rate tensor*, respectively. Integrating (4) from t_0 through t,

$$\rho_{t_0}^t(x_0) = \mathrm{e}^{\int_{t_0}^t r(x_\zeta,\zeta)\,\mathrm{d}\zeta}. \tag{5}$$

In order for a fluid particle trajectory, which starts at t_0 on an attracting LCS, to be uniformly attracting over $[t_0, t_0+\tau]$, $\tau < 0$, it must have $\rho_{t_0}^t(x_0) > 1$ uniformly increasing over $[t_0, t_0+\tau]$. From (5) it clearly follows that such trajectory must have $r(x_t, t) < 0$ for all $t \in [t_0, t_0+\tau]$.

LCS-CORE ANALYSIS

Our *LCS-core analysis* starts by extracting, at t_0, attracting LCS (i.e., with $\tau < 0$) on the fluid domain where changes in the shape of a tracer patch are sought to be predicted. The procedure follows by locating LCS cores by seeking for fluid particle trajectories such that $r(x_t,t) < 0$ for all $t \in [t_0, t_0+\tau]$, $\tau < 0$. If an LCS core happens to be located within a tracer patch, the direction along which the tracer patch will stretch beyond t_0 will be given by the (averaged) direction of maximum stretching at the LCS core, given by the eigenvector with minimum eigenvalue of the *Cauchy–Green strain tensor*,

$$C_{t_0}^{t_0+\tau}(x_0) = DF_{t_0}^{t_0+\tau}(x_0)^{\mathrm{T}} DF_{t_0}^{t_0+\tau}(x_0) \tag{6}$$

This way a change in the shape of the tracer patch beyond t_0 can be anticipated. Very importantly, this does not rely on a velocity prediction beyond t_0: the velocity must be known only up to the time the prediction is made, t_0.

EXAMPLE

An example of LCS-core analysis is shown in Fig. 1 for the case of the *Deepwater Horizon* oil spill in the Gulf of Mexico. On the prediction date, 13 May 2010, an LCS core is found to be located within oil slick, which beyond that date develops a prominent finger. The direction of the fingering instability is well anticipated by the direction of stretching associated with the LCS core found.

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