MATHEMATICAL MODELING OF THE ADMIXTURE TRANSPORT IN FREE-SURFACED STREAMS

Konstantin Nadolin[*a)], Igor Zhilyaev[*]
[*] *Faculty of Mathematics, Mechanics and Computer Science, Southern Federal University, Rostov-on-Don, 344090, Russia*

Summary Spreading of an admixture in the stream of the viscous fluid in a natural low-curved open channel is considered. The admixture is assumed to be passive and its dissipation and diffusion are taken into account. The channel bed is assumed to be known and quite smooth and the stream is assumed to be lengthy and shallow. A reduced mathematical model is derived by the small parameter technique, starting from the 3D Reynolds equations for the incompressible fluid (coupled with Boussinesq turbulence hypothesis) and the diffusion equation for the moving medium. The particular feature of this reduced model is taking into account the stream cross-structure that allows us to study the contaminant spreading in a channel with varying width and depth more accurate than with depth-averaged models (for example, to catch the phenomenon of rising the near-surface opposite flow which may be caused e.g. by the wind action). The results obtained on the base of reduced models for super shallow and lengthy stream and for shallow and wide stream have been tested through direct numerical simulation in the FEM package COMSOL/Multyphisics.

INTRODUCTION

One peculiarity of the natural waterways is the considerable difference in size scale of they length, width and depth. For example, the ratio between the characteristic depth and width for the typical lowland river varies from 1:10 to 5:1000. It allows for the use various small parameters techniques [1]–[5] to reduce the original system of differential equations by excluding terms non essential for flows of this shape and to derive the mathematical models, which are easier for study. Note, that systematical analysis of the admixture spreading in the lengthy stream of viscous fluid was initiated by G.I.Taylor [6, 7] and R.Aris [8]. The term "Taylor dispersion" now is widely used in literature as the name for this problem [9, 10]. The main aim of this work is to test simplified mathematical models for spreading process in natural streams, like rivers and channels.

FORMULATION OF THE PROBLEM

A passive admixture that spreading in the open turbulent flow of incompressible viscous fluid in the section of channel or river is studying with following assumptions: curvature of considered part of the stream is weak; the section's length is large with respect to the channel's width and the channel's width, in this section, is large with respect to its depth. The admixture is assumed to be passive and its dissipation and diffusion are taken into account. The passivity of admixture means that the mathematical description of the process splits into two subsystems of equations. The first one — the "hydrodynamics subsystem" — allows to find fluid velocity and pressure fields and the second one — the "concentration subsystem" — allows finding the concentration of admixture in the stream with known velocity.

The objective of the investigation is to provide adequate mathematical models describing the admixture spreading process. These models should be reduced in space and simple enough for numerical or analytical solution, they should be adequate and provide acceptable accuracy, they should provide the opportunities for adjustment with experimental data and can be calibrated in accordance with available experimental data, and they should allow improvements and corrections.

REDUCED EQUATIONS FOR LENGTHY SUPER SHALLOW AND SHALLOW WIDE STREAMS

The technique of deriving the reduced 3D mathematical models of the flow and admixture spreading processes is based on small parameter technique, which has been applied to Reynolds equations (coupled with the Boussinesq turbulence hypothesis) and to the convection-diffusion equation written in the special dimensionless form. This technique was presented in details in [11] where the reduced models classification was introduced.

Reduced equations in dimensionless form for super shallow and lengthy stream and for shallow and wide stream are

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = \frac{\partial}{\partial z}\left(d_{33}\frac{\partial c}{\partial z}\right) - \lambda c, \qquad c\big|_{t=0} = c^0, \qquad \left.\frac{\partial c}{\partial x}\right|_{x=0} = c_0, \qquad \left.\frac{\partial c}{\partial z}\right|_{z=h} = \left.\frac{\partial c}{\partial z}\right|_{z=\xi} = 0 \quad (1)$$

$$\frac{\partial}{\partial z}\left(\nu\frac{\partial u}{\partial z}\right) = -\frac{Re\sin\varphi}{Fr}, \quad u\big|_{z=h} = 0, \quad \left.\frac{\partial u}{\partial z}\right|_{z=\xi} = 0; \qquad \frac{\partial p}{\partial z} = \frac{\cos\varphi}{Fr}, \quad p\big|_{z=\xi} = 0 \quad (2)$$

$$\frac{\partial}{\partial z}\left(\nu\frac{\partial v}{\partial z}\right) = 0, \qquad v\big|_{z=h} = 0, \qquad \left.\frac{\partial v}{\partial z}\right|_{z=\xi} = 0, \quad (3)$$

$$\frac{\partial \xi}{\partial t} + u\big|_{z=\xi}\frac{\partial \xi}{\partial x} + v\big|_{z=\xi}\frac{\partial \xi}{\partial y} - w\big|_{z=\xi} = 0 \quad (4)$$

[a)] Corresponding author. E-mail: nadolin@math.sfedu.ru.

Here c is an admixture concentration; u, v and w are velocity components in longitudinal (x), width (y) and depth (z) directions; p – pressure; ξ – unknown free-surface function; h – known bed shape function.

The equations (1)-(3) include several parameters: d_{33} – dimensionless coefficient of the turbulence diffusion in z direction; λ – admixture decay factor; ν – normalized viscosity that allows to take into account the viscosity variations in turbulent flow according with the Boussinesq turbulence hypothesis; Re – Reynolds number; Fr – Fruid number and φ – slope of the bed (for details see [11]).

RESULTS AND CONCLUSIONS

To test proposed mathematical model some calculations and computer experiments were made on the base of both $k-\varepsilon$ CFD COMSOL model and elaborated computer program for reduced model (1)-(4). For the last one the coupled algorithm of Galerkin aproach and finite difference method on the base of the characteristic lines mesh was utilized.

The admixture spot spreading along stream is shown at the pictures bellow, where the concentration curves are presented. Figure 1 shows the results of COMSOL calculations and figure 2 — the numerical results for the reduced model (1)-(4). Each concentration curve depicts admixture distribution along stream near its' surface at the sequenced time moments: $t_0 = 0$, $t_1 = 10$, $t_2 = 20$, $t_3 = 30$.

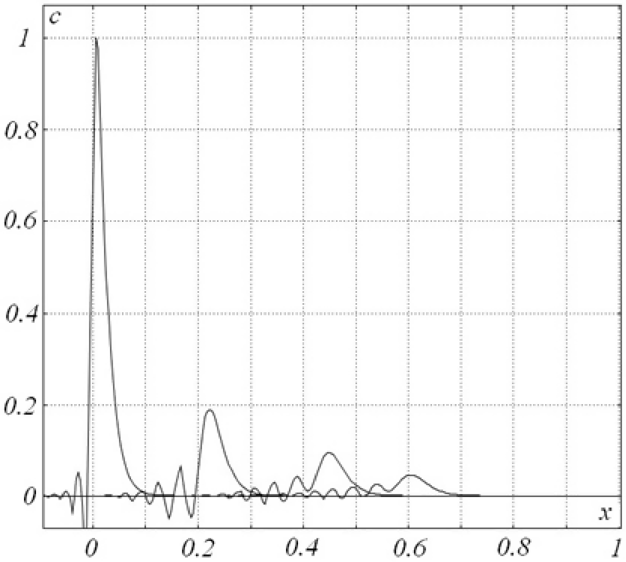

Figure 1. COMSOL calculations

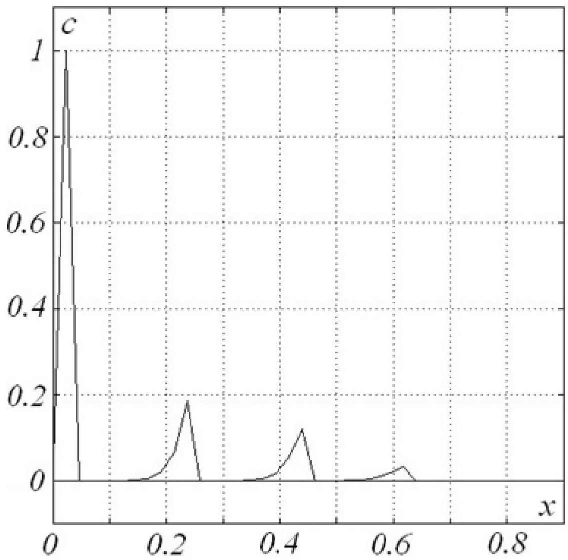

Figure 2. Reduced model calculations

So, the comparison of numerical results shows that reduced models adequately describe the spreading processes in natural streams.

The proposed model are quite simple and allows us (in contrast to conventional one-dimensional model) to analyze the influence of the shape of the channel bed and shoreline and the effect of some external forces (e.g. wind) to the admixture spreading in different points of the flow.

Some computational oscillations were occurring at the backside of concentration curves in COMSOL calculations. But the coupled Galerkin and finite difference method on the base of the characteristic lines mesh demonstrates a stable behavior. It could be successfully applied to numerical study of proposed model.

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