AN EXACT AXISYMMETRIC SPIRAL SOLUTION OF THE INCOMPRESSIBLE 3D EULER EQUATIONS

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Summary A general steady spiral solution of incompressible inviscid axisymmetric flow was obtained analytically. It is a continued tornado-like vortex solution, where there are two intrinsic length scales. One is the radius of maximum circular velocity r_m, the other is the radius of the vortex kernel $r_k = \sqrt{2}r_m$. The explicit vortex solution might be used to describe the 3D structure of the tropical cyclones, tornados and mesoscale eddies in the geophysical flows.

INTRODUCTION

Typhoon (tropic cyclone) is a typical vortex in the nature flow. Its 3D (three-dimensional) structure remains mysterious as the lack of observation data, especially for vertical structure. In the numerical simulations, some 2D (two-dimensional) classical vortices, e.g., Rankine vortex, are usually employed as bogus typhoons. And they are also applied to the theoretical investigations [1].

If the nonlinear advection term vanishes everywhere at certain conditions, the flow is called the Beltrami flow ($u\times\omega = 0$), where u is velocity and $\omega = \bigtriangledown \times u$ is vorticity. Thus the Euler equations could be simplified to linear equations, and some exact solutions might be obtained. The known limited exact solutions of the Euler and Navier-Stokes equations were reviewed in [1]. Such solutions from simplicity to complexity include concentric flows, parallel flows, linear flows, the generalized Beltrami flows ($\bigtriangledown \times (u \times \omega) = 0$) in axisymmetric case.

In this case, the Euler equations can be simplified to the Bragg-Hawthorne equation, or the Long-Squire equation [2]. In general, such equation is nonlinear, only the simple linear Bragg-Hawthorne equation can be solved explicitly. In this short paper, we solved the 3D Euler equation, following the way of [3]. And some explicit solutions were obtained for the typhoon investigations [4].

SPIRAL SOLUTIONS

General solution

We considered the steady solution of the incompressible Euler equations for axisymmetric flow at present study. It is convenient to use a cylindrical coordinate system (r, θ, z) with the velocity components (V_r, V_θ, V_z), and all the velocity components are the functions of r and z but θ, due to the axisymmetric. As $V_r = V_r(r, z)$, $V_\theta = V_\theta(r, z)$ and $V_z = V_z(r, z)$, the governing equations, including mass-conservation and momentum equations, are

$$\frac{\partial(rV_r)}{\partial r} + \frac{\partial(rV_z)}{\partial z} = 0 \qquad (1)$$

$$V_r\frac{\partial V_\theta}{\partial r} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_rV_\theta}{r} = 0 \qquad (2)$$

$$V_r\frac{\partial V_r}{\partial r} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad (3)$$

$$V_r\frac{\partial V_z}{\partial r} + V_z\frac{\partial V_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (4)$$

It should be noted that there is no length scale in Eqs.(1-4), thus the solution can be uniformly stretched by simply multiplying a complex constant. It was noted that Eq.(1) and Eq.(2) can be solved through the introduction of the axisymmetric stream function $\Psi(r, z)$. We tried to find the solution of the above equations by separation of variables $\Psi(r, z) = R(r)H(z)$. One such solution can be written as,

$$V_r = \frac{R(r)}{r}H'(z),\ V_\theta = \lambda\frac{R(r)}{r}H(z),\ V_z = -\frac{R'(r)}{r}H(z) \qquad (5)$$

where $'$ presents first deviation and λ is a complex constant. In this way, equation (1) and equation (2) are satisfied automatically. A solution of Eq.(5) is,

$$R(r) = ar^2e^{-\frac{1}{8}\gamma r^2}, H(z) = e^{kz} or H(z) = \cos(kz) \qquad (6)$$

where a, k and γ are real constants. Thus, Eq.(6) gives the exact solutions of the flow velocity. And other new solutions can also be obtained by combining the different solution at different regions, like that of the Rankine vortex.

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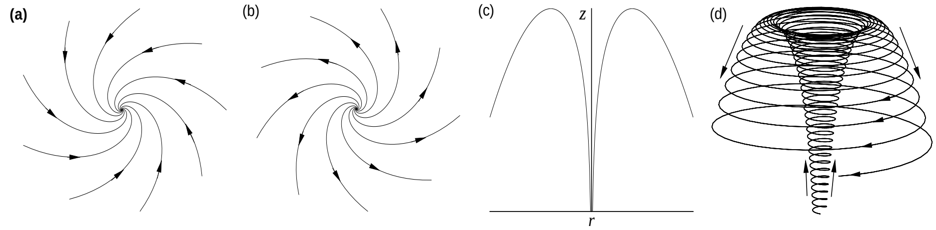

Figure 1. Paths of the fluid material elements for the tornado-like vortex.

Tornado-like vortex

If we take $H(z) = e^{kz}$, the velocity solution is,

$$V_r = kre^{kz-\frac{1}{8}\gamma r^2}, V_\theta = \lambda re^{k_r z-\frac{1}{8}\gamma r^2}, V_z = -2(1-\frac{1}{8}\gamma r^2)e^{k_r z-\frac{1}{8}\gamma r^2}. \tag{7}$$

The solution is finite within the whole domain, if we choose appropriate $k_r > 0$ for $z < 0$ and $k_r < 0$ for $z > 0$. The circular velocity V_θ is the same with that of the Taylor vortex for any fixed z. The circular velocity maximum is $V_{\theta m} = \lambda r_m e^{kz-1/2}$ at $r_m^2 = 4/\gamma$. And the vertical velocity V_z changes it direction at $r_k^2 = 8/\gamma$, which we call as the radius of vortex kernel r_k. Such vertical velocity distribution is something like that of the Sullivan vortex. The paths of the fluid material elements are illuminated in Fig 1. In $r-\theta$ plane, the paths are cyclonic logarithmic spirals (Fig.1a) or anticyclonic logarithmic spirals (Fig.1b) with $\ln r = k\theta/\lambda$. In $z-r$ plan (Fig.1c), the fluid material element moves spirally along the surface decided by $z = (\frac{1}{8}\gamma r^2 - 2\ln r)/k + const$. Figure 1d also shows the path in the 3D space. In the inner region $r < r_k$, the flow cyclonically ascends from below to above as shown by upward arrow. In the outer region $r > r_k$, the descends apart from the kernel. So the mass conserves in this case.

$$\frac{2\pi r_k^2}{e}e^{kz} = -\int_0^{r_k} 2\pi r V_z dr = \int_{r_k}^{\infty} 2\pi r V_z dr \tag{8}$$

There are two intrinsic length scale in the solution. One is the radius of maximum circular velocity r_m, the other is the radius of the vortex kernel $r_k = \sqrt{2}r_m$. According to the shape of the paths, we called it the umbrella vortex. This new vortex solution is very like that of the tornado structure, about which we know quite limited.

As mentioned above, the solutions are the generalized Beltrami flow $\bigtriangledown \times (u \times \omega) = 0$. They can also be the solutions of Crocco equation

$$u \times \omega = -T\nabla S, \tag{9}$$

where T is temperature and S is entropy of the flow. If $T\nabla S$ is potential, then the above velocity solutions yield the Crocco equation. As $\nabla \times (T\nabla S) = \nabla T \times \nabla S = 0$, then the isothermal surfaces must be parallel to the isentropic surfaces.

CONCLUSIONS

A general exact axisymmetric spiral solution was obtained for 3D incompressible Euler equations. It is a continued tornado-like vortex solution, where there are two intrinsic length scales r_k and r_m. The explicit vortex solution might be used to describe the 3D structure of the tropical cyclones, tornados and mesoscale eddies in the geophysical flows.

The method used in present work is very straightforward, and it could also be applied to other complex flows, e.g., the geophysical flows in a rotating frame (f-plane), and even for the non-steady viscous flows.

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