THREE-DIMENSIONAL THIN FILM FLOW OVER TOPOGRAPHY: FULL NAVIER-STOKES SOLUTIONS

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Summary The problem of steady-state gravity-driven film flow over topography is explored via solutions of the the full Navier-Stokes system of equations. Besides providing a unique benchmark with which to assess the limitations of related models based on the long-wave approximation, a defining feature is the lifting of many of the constraints associated with the latter while in addition enabling the internal flow topology to be explored as the Reynolds number and capillary number are varied.

INTRODUCTION

Numerous manufacturing processes require the deposition of thin liquid films, involving a balance between viscous, surface tension and gravity forces, on a variety of substrates containing topographic features. These encompass a wide range of applications and about which much of the underpinning basic science is now reasonably well documented.

While investigation of the problem of gravity-driven thin film flow over topography based on the solution of the full Navier-Stokes (N-S) equations has remained restricted to two-dimensions, due in the main to the computational resource and effort required in three-dimensions, to do so overcomes many of the restrictions associated with simpler models based on the long-wave approximation, such as the lubrication theory, [1], or the integral-boundary-layer approach, [2]. It allows the removal in one go of any constraints, other than physically realisable ones, concerning choice of capillary number, film thickness or topography aspect ratio. In addition, the internal velocity field forms part of the solution enabling the internal flow topology to be explored as the Reynolds number and capillary number are varied. Solving the three-dimensional flow over localised topographical features as encountered in practice represents a significant step forward while allowing the internal flow topology that arises to be contrasted with that appearing in two-dimensional flow over spanwise topography [3].

PROBLEM SPECIFICATION AND METHOD OF SOLUTION

The problem of interest consists of a continuous thin fluid film, of asymptotic thickness H_0, flowing down a rigid substrate, containing a trench topography and inclined at an angle θ to the horizontal, the volumetric flow rate being Q_0 per unit width and the characteristic velocity $U_0 = 3Q_0/2H_0$ – see [3] or [4]. The fluid involved is considered to be incompressible with constant density, ρ, viscosity, μ, and surface tension, σ.

The strategy adopted to obtain a solution of the full N-S equations involves using a Bubnov-Galerkin mixed-interpolation finite element scheme [5], free-surface parametrisation based on the Arbitrary Lagrangian-Eulerian method of spines [6], the use of a direct parallel multifrontal method [7], as contained in the MUMPS library [8], and utilisation of the memory-efficient out-of-core approach for storing matrix cofactors on the hard drive. The three-dimensional solutions obtained in this way represent the first of their kind, enabling both the internal flow topology and free-surface disturbance generated to be explored simultaneously. Accordingly, they provide a unique benchmark with which to assess the limitations of related models based on the long-wave approximation.

RESULTS

Comparison of the free-surface disturbance generated by the N-S solutions with their corresponding ones obtained using long-wave approximation models, lubrication theory [3] and depth-averaged from [4], reveals that for a three-dimensional inertial flow over a localised square trench topography the depth-averaged results are found to be in very good agreement, while the lubrication model underpredicts the free-surface behaviour. As expected, the disagreement between the long-wave approximation solutions and their full N-S counterpart is amplified as the capillary number, $\mathrm{Ca} = \mu U_0/\sigma$, or the ratio of trench depth to length is increased, both of which are exacerbated the higher the value of the associated Reynolds number, $\mathrm{Re} = \mu U_0 H_0/\rho$.

Analysing the internal flow topology which arises, reveals that across the trench topography the three-dimensional thin film flow is topologically different internally from its two-dimensional counterpart as reported in [3]: whereas for the latter eddies are always centres (closed elliptic trajectories), for the former, at the mid-plane, they may instead be foci or nodes (not closed logarithmic spiral or power law trajectories), limit cycles are possible as well, see [9].

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In addition, the three-dimensional flow case leads to different flow topologies, dependent not only on the trench geometry in both the streamwise and spanwise directions but also on the capillary and Reynolds numbers; Figure 1, with Ca $=$ 0.001, is a typical case. For Re $= 0$ the trajectories of the flow inside the trench form closed eddies; for Re $= 10$, the trajectories are no longer closed forming a stable focus, instead they encroach into the trench near the left hand symmetry mid-plane (see the blue and red trajectories) swirl around several times during which they are displaced laterally away from the mid-plane and exit the trench close to its (right hand) side before travelling downstream; similar features have been reported for flow in a three-dimensional lid-driven cavity, see [10]. In contrast, increasing Ca to 1.0 when Re $= 0$, the critical point in the mid-plane of the trench is an unstable focus, the trajectories swirl around but out of the eddy exiting the trench near the symmetry mid-plane. This is typical of the case for film thickness where the size of the free surface disturbance generated impacts strongly on the internal flow topology in the vicinity of the trench, independent of inertia effects.

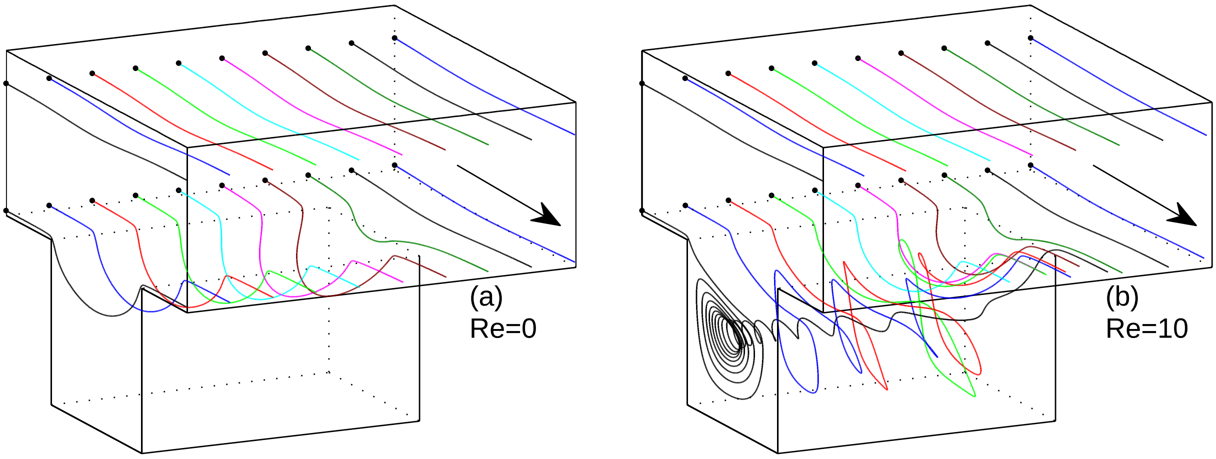

Figure 1. Three-dimensional flow topology, obtained by integrating along path lines, for the case of a localised rectangular trench topography with nondimensional (scaled with respect to H_0) streamwise length $l_t = 1.5$, spanwise width $w_t = 3.0$, depth $s_0 = 1.0$ and capillary number Ca $= 0.001$: Re $= 0$ (left) and Re $= 10$ (right). Starting positions are denoted as filled circles located adjacent to the substrate at $z = 0.03$ and to the free surface at $z = 0.8$. For illustrative purposes different colours are used for streamlines corresponding to different starting positions. The symmetry mid-plane is on the left hand side, the cell's closed side is on the right. The arrow shows the direction of flow.

Other new results that emerge from the N-S solutions together with an extensive investigation apropos flow topology, will be previewed and discussed at the congress. For example, for a sufficiently small ratio of trench depth to length, when the two corner eddies present are small compared to the length of the trench, the internal velocity profile across the film becomes effectively self-similar parabolic across the entire flow domain.

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