A FIRST INTEGRAL OF NAVIER–STOKES EQUATIONS: THIN FILM FLOW APPLICATIONS

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Summary As is well known, Bernoulli's equation is obtained as the first integral of Euler's equations in the absence of vorticity; in the case of non–vanishing vorticity, a first integral from Euler's equations is found using the so called Clebsch transformation [1] for inviscid flows. By contrast and to date, a generalisation of this procedure to viscous flows has failed to materialise. In the present work a first integral of Navier–Stokes equations is constructed for the case of two–dimensional incompressible viscous flow by making use of an alternative representation of the fields in terms of complex coordinates and introducing a potential representation for the pressure. The associated boundary conditions are also considered.

The capability of the first integral is demonstrated by applying Ritz's direct method to the problem of gravity driven film flows over corrugated substrates and to Reynolds number effects on a shear flow confined between two plates.

GENERAL THEORY

Complex formulation of the Navier–Stokes equations

The theory presented below is at first restricted to the two-dimensional Navier–Stokes equations for steady and incompressible flow. Extension of the theory to three-dimensional flows is under construction and the existing state of the theory will be presented at the congress. Accordingly:

$$\varrho\left[u_1\frac{\partial u_k}{\partial x_1}+u_2\frac{\partial u_k}{\partial x_2}\right]=-\frac{\partial p}{\partial x_k}+\eta\left[\frac{\partial^2 u_k}{\partial x_1^2}+\frac{\partial^2 u_k}{\partial x_2^2}\right]-\frac{\partial U}{\partial x_k}\,,\qquad k=1,2 \tag{1}$$

where u_1, u_2 denote the velocity field, p the pressure and $U(x_1, x_2)$ the potential energy density of a conservative external force. Rewriting these equations in terms of a complex coordinate ξ and a complex velocity field u, namely

$$2\xi := x_2 + \mathrm{i}x_1\,,\qquad u := u_1 + \mathrm{i}u_2\,, \tag{2}$$

leads to the complex equation

$$\frac{\partial}{\partial\bar{\xi}}\left[\mathrm{i}\varrho\frac{u^2}{2}+\eta\frac{\partial u}{\partial\xi}\right]=\mathrm{i}\frac{\partial}{\partial\xi}\left[p+U+\varrho\frac{\bar{u}u}{2}\right]. \tag{3}$$

The continuity equation is identically fulfilled by introducing a streamfunction according to $u = \dfrac{\partial\psi}{\partial\xi}$.

First integral

The complex Navier–Stokes equation in its present form (3) is not integrable due to its formal structure: A derivative with respect to $\bar{\xi}$ equates to a derivative with respect to ξ. In order to resolve this issue, the expression in square brackets on the right hand of (3) is represented [2] by means of a potential φ as

$$p+U+\varrho\frac{\bar{u}u}{2}=\frac{\partial^2\varphi}{\partial\bar{\xi}\partial\xi}\,, \tag{4}$$

transforming (3) into an integrable form:

$$\frac{\partial}{\partial\bar{\xi}}\left[\mathrm{i}\varrho\frac{u^2}{2}+\eta\frac{\partial u}{\partial\xi}-\mathrm{i}\frac{\partial^2\varphi}{\partial\xi^2}\right]=0\,. \tag{5}$$

Integration with respect to $\bar{\xi}$ delivers the first integral of Navier–Stokes equations

$$\mathrm{i}\varrho\frac{u^2}{2}+\eta\frac{\partial u}{\xi}-\mathrm{i}\frac{\partial^2\varphi}{\partial\xi^2}=f(\xi)=0 \tag{6}$$

where the corresponding integration function $f(\xi)$ has been set to zero due to a proper regauging of the potential φ. Using the streamfunction ψ, the above formulation leads to an effective reduction of the Navier–Stokes equations (1) by one order. Depending on the individual type of problem, the field equations (3) have to be supplemented by boundary

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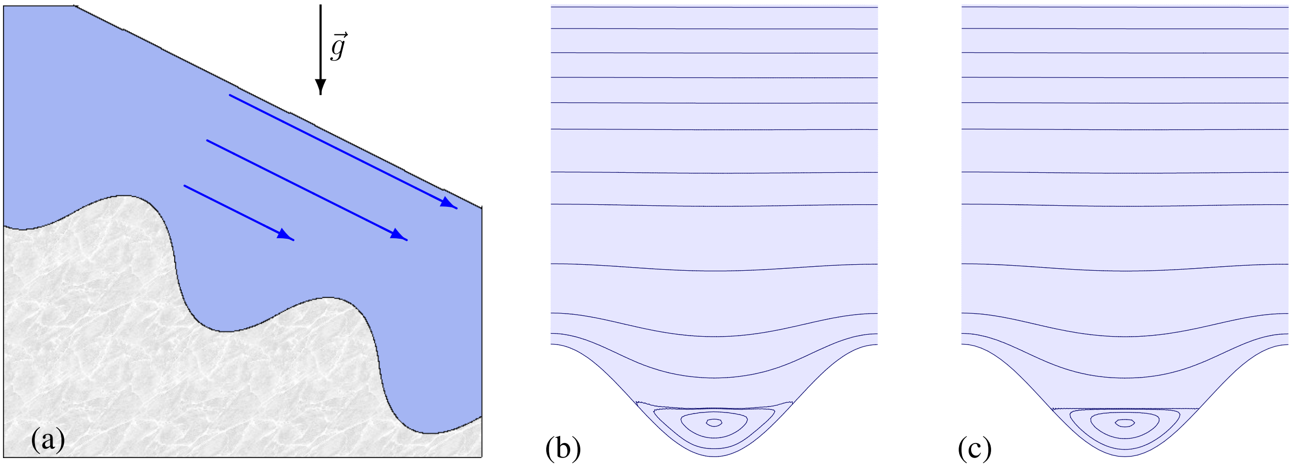

Figure 1. Sketch of the flow (a) and resulting streamlines (b) compared to exact solution (c).

conditions. In case of solid walls the no–slip boundary conditions for the velocity have to be considered in their usual form, whereas for the potential φ no additional conditions have to be taken into account.
In case of a free surface the situation is different: the straight forward analysis of the dynamic boundary conditions leads to a set of conditions for the first order derivatives of the potential φ which finally take the form

$$\int U\left(f_k(s)\right) n_i(s)\,\mathrm{d}s + 2\varepsilon_{ij}\frac{\partial\varphi}{\partial x_j} - \sigma t_i(s) = 0\,, \qquad i = 1,2 \tag{7}$$

where s denotes the arc length, n_i the two components of the normal vector and t_i the corresponding components of the tangent vector along the free surface. The epsilon–tensor is denoted by ε_{ij}.

APPLICATION TO THIN FILM FLOWS

Gravity–driven film over a strongly corrugated wall, see Fig. 1(a), under Stokes flow conditions is considered as an example in order to demonstrate the capabilities of the approach. This type of flow has been investigated in detail elsewhere, see e.g. [3], and therfore provides a good 'benchmark–test' for the first integral. It is shown in [4] that under Stokes flow conditions a variational formulation of the first integral is available allowing for the implementation of Ritz's direct method for above problem. Using sine and cosine functions as test functions in mean flow direction and polynomial test functions in the perpendicular direction, approximate solutions of high accuracy can be obtained: the example shown in 1(b) has been obtained with 6 Fourier modes and polynomial functions up to 13th order. For comparison purposes, 1(c) shows the same flow calculated using the complex variable (CV) method which, according to [3] leads to solutions that correspond identically to their equivalent finite element solutions. As can be observed here, remarkably good agreement is achieved: by refining the procedure using more coefficients, the accuracy and hence agreement can be improved further.

Ritz's direct method can be extended to the case of non–Stokes flow as demonstrated in [4], where Reynolds number effects on a shear flow confined between two horizontally aligned rigid plates, the upper one flat and moving and the lower one stationary with a sinusoidal profile, is investigated.

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