FILM FLOW OVER SMALL-AMPLITUDE STEP TOPOGRAPHY

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Summary The flow of a liquid film over step topography is studied. The aim is to see if the important qualitative features of such flows can be predicted using an asymptotic model based on the assumption of small topographic height. The case of film flow over a downwards and an upwards step is of particular interest, with the focus being on the shape of the free surface profile as the film negotiates the step. Kallaidasis *et al.* [1] found that for film flow over a downwards step, the surface profile exhibits a capillary ridge immediately prior to the step. The small-height topography asymptotic model reproduces this feature. The effect of an electric field is also investigated with a view to manipulating the surface profile; in particular with a view to ironing out the ridge over the step.

INTRODUCTION

Liquid film flows over topography occur in many industrial applications including, for example, coating technologies and cooling processes. In the former (e.g. [2]-[3]), a coating layer of uniform thickness is normally required and attempts are made to mitigate any disruptions engendered by the topographical features. On the other hand, in the latter case surface waves help to enhance heat or mass transfer and so promote cooling of a target surface. There is a significant body of both theoretical and experimental work devoted to film flow over topography.

The aim of the present work is to derive asymptotic solutions to the Navier-Stokes equations for small-amplitude substrate topography. Of particular interest is the case where a liquid film flows along a flat wall which undergoes a small, sharp step downwards at some point along its length. For such a flow, Kalliadasis *et al.* [1] (and a number of subsequent workers) found that the film noticeably thickens in the immediate region of the step to form what is known as a capillary ridge in the free surface profile. Our goal is to predict this and other features using our asymptotic model. We are also interested to see if the possibility of eliminating the capillary ridge using an electric field, which was demonstrated by Tseluiko et al. [4] using a thin-film model for arbitrary step height, can also be captured by the small-height topography asymptotic model. For finite Reynolds number flow, Bontozoglou & Serifi [5] carried out finite-element calculations and found that the height of capillary ridge increases as the Reynolds number is raised from zero, but then starts to decrease and eventually disappears. Veremieiev *et al.* [6] also found that increasing the Reynolds number raises the height of the ridge, and studied the effect of an electric field on the surface profile at non-zero Reynolds number.

PROBLEM FORMULATION

We consider the steady flow of a liquid film down an uneven wall inclined at an angle θ to the horizontal, as shown in Fig. 1 (left panel). The wall is at $z = s(x)$ and the film surface is at $z = f(x)$. An electric field is applied such that, far from the film surface, it acts purely in the z direction with strength E_0. The wall behaves like an electrode held at a constant voltage. The liquid is treated as a perfect dielectric of permittivity ε_1, and the air above the liquid film is assumed to be a perfect dielectric with permittivity ε_2. Tseluiko et al. [9] noted that the problem for a perfect-dielectric film reduces mathematically to the problem for a perfect-conductor film in the limit $\varepsilon_p \to \infty$, where $\varepsilon_p = \varepsilon_1/\varepsilon_2$.

We wish to find $f(x)$ for chosen $s(x)$, and chosen electric field strength, at arbitrary Reynolds number Re. For a flat wall, $s(x) \equiv 0$, the film is of uniform thickness, h_0, and moves with surface speed $U_0 = \rho g h_0^2 \sin\theta/2\mu$, where ρ and μ denote the density and viscosity of the fluid, respectively, and g is gravity. For a general wall shape, we non-dimensionalise variables using h_0 as the length scale, U_0 as the velocity scale, and $\mu U_0/h_0$ as the pressure scale. The electric potentials in the film and in the air are non-dimensionalised by dividing by $E_0 h_0$.

The no-slip and no-penetration conditions require that $\mathbf{u} = \mathbf{0}$ on the wall, $z = s(x)$. The kinematic condition at the film surface, $z = f(x)$, requires that $\mathbf{u}\cdot\mathbf{n} = 0$, where $\mathbf{n}$ is the unit normal to the film surface pointing into the film. The dynamic balance of stress at the film surface, $z = f(x)$, demands that

$$\sigma_1 \cdot \mathbf{n} = -(\kappa/C + P_a)\mathbf{n} + (\mathbf{M}_2 - \mathbf{M}_1)\cdot\mathbf{n}, \tag{1}$$

where σ_1 is the Newtonian stress tensor, κ is the curvature of the film surface taken to be positive when the surface is concave downwards, as is indicated in Fig. 1, P_a is the dimensionless air pressure, and $\mathbf{M}_j = \varepsilon_p W_e\left(\mathbf{E}_j\mathbf{E}_j - \mathbf{I}|\mathbf{E}_j|^2/2\right)$ are the Maxwell stress tensors in the liquid, $j = 1$, and in the air, $j = 2$. C is a capillary number. The electric field (for a perfect conductor fluid) satisfies Laplace's equation above the fluid and the condition of constant electric potential on the surface, and a matching condition to the normal field at infinity.

We assume small height topography and write $s(x) = \epsilon\sigma(x)$, where $\epsilon \ll 1$. At leading order in ϵ the wall is flat, the free surface is flat, and the flow adopts the classical Nusselt profile, $u = U(y) = y(2 - y)$. To find the correction to the free

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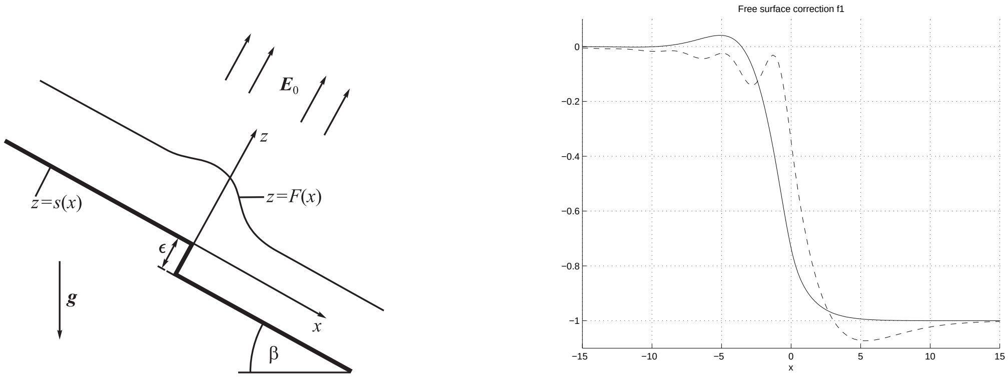

Figure 1. (Left) Illustration of the flow of a thin liquid film over a small downward step of height ϵ in the presence of a normal electric field of strength E_0. The region above the film is occupied by a passive dielectric gas. (Right) Free-surface correction f_1 for $Re = 0$, and $We = 0$ (solid line), $We = 15$ (broken line).

surface provoked by the small-height topography, we expand by writing $f = 1 + \epsilon\, f_1 + \cdots$, and write similar expansions for the flow streamfunction and pressure, and the electric potential, φ, in the air above the film. Eliminating the pressure from the momentum equation in the liquid film, at first order we obtain linearised equation,

$$Re(U\nabla^2\psi_{1x} - U''\psi_{1x}) = \nabla^4\psi_1, \tag{2}$$

with conditions $\psi_{1x} = 0$, $\psi_{1z} = -2\sigma$ at $z = 0$, and

$$\psi_{1x} = -f_{1x}, \quad \psi_{1zz} - \psi_{1xx} = 2f_1, \quad \psi_{1zzz} + 3\psi_{1xxz} = -\frac{1}{C}f_{1xxx} + 2\cot\theta\, f_{1x} + 2\left(1 - \frac{1}{\varepsilon_p}\right)W_e\, \varphi_{1xz}, \tag{3}$$

at $z = 1$. Here We is an electric Weber number (when $We = 0$ there are no electric effects). The function $\varphi_{1xz}|_{z=1}$ is found by solving the first order electric field problem. Both the first order electric field and fluid flow problems can conveniently be solved by first taking a Fourier transform in x; the free surface correction f_1 is then found by inverting the transform.

RESULTS

Here we focus attention on flow over a downwards step so that $\sigma(x) = -H(x)$, where $H(x)$ is the Heaviside function. In Fourier space (2) is an ordinary differential equation in z which we solve using *Matlab* (when $Re = 0$ the equation can be solved exactly); the Fourier transform is inverted numerically using a suitable numerical quadrature rule. Sample solutions at zero Reynolds number, $Re = 0$, are shown in Fig. 1 (right panel). The step is located at $x = 0$. The solid line corresponds to the free surface correction $f1(x)$ in the absence of an electric field, $We = 0$, and exhibits the capillary ridge identified by Kalliadasis *et al.* [1]. The broken line is for the same flow but with $We = 15$, $\varepsilon_p = \infty$ (perfect conductor film). While the electric field does push down the capillary ridge, it also introduces oscillations upstream of the step and a trough just downstream. These features were explained using a thin-film model by Tseluiko *et al.* [8] in terms of the movement of the poles of the inversion kernel for f_1 in the complex Fourier k-plane. The predictions for $Re = 0$ have been successfully compared with the results of numerical simulations for Stokes flow using the boundary element method (Tseluiko *et al.* [7]). Our next goal is to compute small-step predictions for $Re > 0$ and compare with numerical results for film flow over a step of arbitrary height (Veremieiev *et al.* [6]).

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