STABILITY ANALYSIS OF A THIN LIQUID FILM ON AN AXIALLY OSCILLATING CYLINDRICAL SURFACE IN THE HIGH-FREQUENCY LIMIT

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Summary We have considered an axisymmetric thin liquid film on a horizontal circular cylinder undergoing fast axial harmonic oscillation. Using the methods of long-wave theory along with the multi-scale expansions, separation of the relevant fields into fast and slow components and averaging over the fast time, we have derived a nonlinear evolution equation describing the nonlinear spatiotemporal dynamics of the film. We have carried out linear stability and bifurcation analyses based on the derived evolution equation.

INTRODUCTION

External forcing is very well known to introduce a dramatic change in the behavior of mechanical systems. The best example of such striking influence is the Kapitza's inverted pendulum [4, 7]. Various experimental investigations related to drops and films on solid surfaces subjected to vibration were recently reported[1, 5, 6]. Haimovich and Oron [3] considered the nonlinear dynamics of an axisymmetric film on a horizontal cylindrical surface under "low"-frequency harmonic axial vibration and found that the film rupture taking place on a static cylindrical surface [2] is prevented if the product of the dimensionless frequency and amplitude of forcing exceeds a certain constant value depending on the liquid properties and the geometrical parameter. It is the purpose of this work to develop a mathematical model describing the nonlinear dynamics of a thin axisymmetric liquid film coating a horizontal cylindrical surface undergoing harmonic axial vibration in the limit of a "high" frequency using the methods of long-wave theory and multiple scaling, and to carry out the linear and weakly nonlinear stability analyses for a uniform thickness film in its framework.

PROBLEM STATEMENT AND GOVERNING EQUATIONS

We consider an incompressible isothermal thin liquid film of the average thickness h_0 coating a horizontal solid circular cylinder of radius R_c. We assume the solid substrate to perform axial harmonic vibration with the displacement amplitude b and a single frequency ω. The liquid film is assumed to be axisymmetric, so gravity effects can be neglected in comparison with the capillary effects. This would be feasible if the cylinder radius and the mean film thickness are each much smaller than the capillary length of the liquid. The governing equations are therefore the axisymmetric version of the continuity and the Navier-Stokes equations written in cylindrical coordinates.

Averaged and pulsating parts

To investigate the film evolution in the high-frequency limit, we separate the relevant fields into fast and slow components and expand the equations and the boundary conditions using asymptotic series. The problem is solved at zero and first orders with respect to a small film parameter. A resulting set of equations at zero order is related to the pulsating parts of the fields and its solution consists of zero and first order Bessel and modified Bessel functions of a complex argument. The next order problem serves to evaluate the averaged fields and accounts for an interaction of pulsating components giving rise to nonlinearities other than those known in the literature [2]. The evolution equation describing the nonlinear film dynamics is derived based on the continuity equation and averaging the contribution of the interacting pulsating components.

Evolution Equation

The dimensionless evolution equation describing the nonlinear spatiotemporal dynamics of the film is obtained as

$$\frac{1}{2}\frac{\partial}{\partial t}\left(n^2\right)+\frac{\partial}{\partial x}\left[L_1(n)\frac{\partial n}{\partial x}\right]+\frac{\partial}{\partial x}\left[L_2(n)\frac{\partial^3 n}{\partial x^3}\right]=0, \tag{1}$$

where the function $n=n(x,t)$ is related to the film thickness, x and t are, respectively, dimensionless axial coordinate and time, and $L_1(n)$ and $L_2(n)$ are nonlinear functions of n derived here. The function $L_1(n)$ contains the terms accounting for the system forcing and capillarity, whereas $L_2(n)$ is due to capillarity only.

Linear Stability Analysis

Eq.(1) is linearized around the base state $n=n_0$ representing a uniform-thickness film and its stability properties are investigated. The function $L_2(n)$ is always positive, while $L_1(n)$ may be either positive or negative depending on the parameter values. In the case of $L_1(n)\le 0$, the system is linearly stable, otherwise it is linearly unstable in a certain range of wavenumbers. Figure 1 displays in its left panel the variation of the cutoff wavenumber k_c with $a=h_0/R_c$ for the case of a water film of $h_0=0.2$ mm for various values of the forcing frequency ω. In the present case, for all

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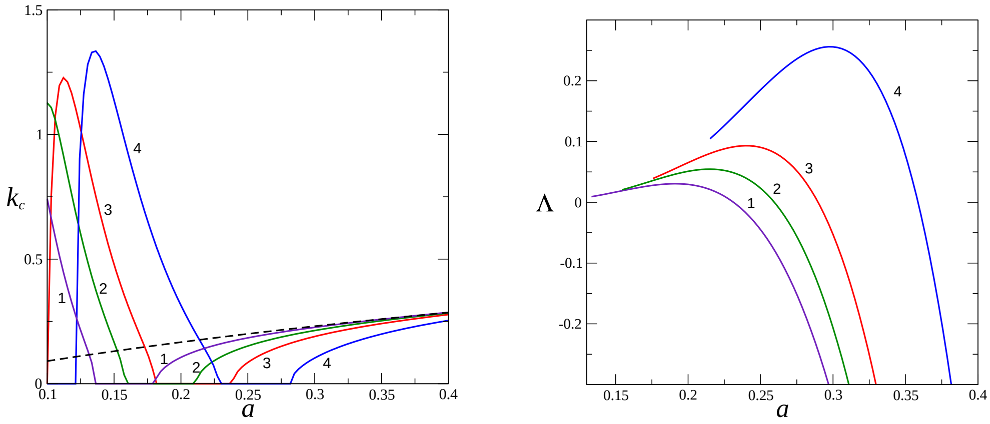

Figure 1. The case of a water film of $h_0 = 0.2$ mm for a fixed forcing amplitude of $b = 1$ mm. Curves 1-4 correspond, respectively, to $\omega = 20$ Hz, $\omega = 25$ Hz, $\omega = 30$ Hz and $\omega = 40$ Hz. Left panel: Variation of the cutoff wavenumber k_c with the parameter a for various values of the forcing frequency as obtained from the linear stability analysis. The dashed curve represents the variation of the cutoff wavenumber k_c with a in the case of the static cylinder $b = 0$. Right panel: Variation of the Landau constant with a.

frequencies the value $L_1(n)$ changes its sign twice in the given range of a and for each of the frequencies presented here, there exists a range of a where $k_c = 0$, and the film is stabilized by the applied forcing. The dashed curve represents the variation of the cutoff wavenumber $k_c = a/(1+a)$ for the unforced system which is always unstable.

Bifurcation Analysis

We carry out the weakly nonlinear analysis based on Eq. (1) and obtain the following Landau equation in terms of the amplitude function $\mathcal{A} = \mathcal{A}(\tau)$:

$$(1+a)\mathcal{A}_\tau = \delta\mathcal{A} - \Lambda|\mathcal{A}|^2\mathcal{A}, \tag{2}$$

where τ is a rescaled time, δ is a constant whose sign reflects the direction of the deviation from criticality, and Λ is the Landau constant. If the value of Λ in Eq. (2) is positive (negative), the corresponding bifurcation from the equilibrium is supercritical (subcritical). In the case of a supercritical bifurcation, saturation of the pattern and the emergence of a nontrivial steady state is expected beyond criticality. On the other hand, if bifurcation is subcritical, we would expect rupture of the film for a weak supercriticality. Variation of the Landau constant Λ with a is shown in the right panel of Fig. 1. It follows from inspection of this figure that for smaller values of a the bifurcation is supercritical, whereas for larger values of a, it is subcritical. In the case of supercritical bifurcation for a specified a, the amplitude of the saturated state is proportional to $\Lambda^{-1/2}$ and decreases with an increase in the frequency.

CONCLUSIONS

We conclude that it is possible to stabilize the liquid film coating on the cylindrical surface in a certain range of a for all wavenumbers. On the other hand, the same parameter set could be also destabilizing for a cylinder of a different radius. Based on the weakly-nonlinear analysis of the system, we find that for fixed values of the mean film thickness and the forcing parameters, the bifurcation is subcritical for smaller radii of the cylinder and film rupture is expected for the corresponding parameter sets, and supercritical for larger radii of the cylinder. In the case of supercritical bifurcation, the forcing frequency has a direct effect on the amplitude of the saturated state which is found to be proportional to $\Lambda^{-1/2}$, which results in smaller amplitudes for higher frequencies. We also find that for fixed values of the film thickness and the forcing frequency, variation of the forcing amplitude does not affect the type of bifurcation.

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