ON ADVANCING CONTACT LINES WITH A 180° CONTACT ANGLE

Eugene Benilov*[a)]

*Department of Mathematics, University of Limerick, Ireland

Summary This work builds on the foundation laid by Benney and Timson (1980 *Stud. Appl. Math.* **63**, 93), who examined the flow near a contact line and showed that, if the contact angle is 180°, the usual contact-line singularity does not arise. Their local analysis, however, does not allow one to determine the velocity of the contact line and their expression for the shape of the free boundary involves undetermined constants.

The present paper considers a two-dimensional steady Couette flow with a free boundary, for which the local analysis of Benney and Timson can be complemented by an analysis of the global flow (provided the slope of the free boundary is small, so the lubrication approximation can be used). We show that the undetermined constants in the solution of Benney and Timson can all be fixed by matching the local and global solutions. The latter also determines the contact line's velocity, which we compute among other characteristics of the global flow.

THE BACKGROUND

It is well-known that the Navier–Stokes equations with the standard set of boundary conditions cannot be used if some of the rigid and free boundaries of the flow intersect, resulting in contact lines. This was first shown by Huh and Scriven [1]: treating the shape of the free boundary as given and assuming that it intersects the rigid boundary at a certain contact angle, they demonstrated that the pressure field has a singularity, making the solution physically meaningless. To eliminate the singularity, numerous researchers replaced the no-slip condition at the rigid boundary with other conditions reflecting various physical models of contact lines.

Note, however, that, if the contact angle is 180°, the pressure-field singularity does not arise even if the no-slip condition *is* imposed on a flow with a contact line.

An attempt to construct an example of such a flow has been made by Benney and Timson [2], who considered a two-dimensional flow on a substrate, with the liquid's free boundary and contact line steadily advancing with a velocity V. Given the absence of gravity, the flow is fully characterized by the capillary number

$$C_a = \frac{\mu V}{\sigma},$$

where μ and σ are the liquid's viscosity and surface tension. Using the co-moving reference frame where the contact line is stationary, while the substrate is moving with the velocity $-V$, [2] examined the problem (including the no-slip condition) *locally*, i.e. expanded the solution in powers of the distance from the contact line. It was shown that the free boundary is described by

$$y = ax^q + \mathrm{O}(x^{q+1}) \qquad \text{as} \qquad x \to 0, \tag{1}$$

where the origin of the coordinate system (x, y) is associated with the contact line, the x-axis is parallel to the substrate, a is a positive constant, and q satisfies

$$\tan q\pi = -2\, C_a. \tag{2}$$

Note that the original version of equation (2) derived in [2] contained a sign error corrected by Ngan and Dussan [3]. This error, however did not seem to have far-reaching consequences, as one can show that (2) admits infinitely many roots such that $q > 1$, so that the contact angle is indeed 180° and the corresponding solution for the pressure is finite.

Still, even though the (amended) analysis of [2] yields an apparently sound solution, Ngan and Dussan insisted that

> *"it is our belief that there is something inherently wrong with this local boundary-value problem* (considered in [2]). *We strongly expect that the solution for the interface shape... is completely determined upon specifying the... contact angle. However, the approach of Benney and Timson does not have this basic characteristic because of the fact that, regardless of the choice of* q [in equation (1)], *the value of the constant* a*... remains undetermined."*

In this work, the argument between [2] and [3] is resolved. We consider an example tractable both locally and globally and demonstrate that a is uniquely determined by matching the former solution to the latter. We also show that matching determines the choice of the root of equation (2) for q. Finally, the global solution yields the velocity of the contact line. Overall, the local analysis of [2] and our global analysis provide a fully consistent description of a flow with a 180° contact angle and no singularity.

There are two reasons why such flows are important. Firstly, the result of [2] appeals to the scientist's general philosophy, according to which a new concept should not be introduced in a problems where the old one (in this case, the no-slip boundary condition) is not exhausted *completely*. Secondly, flows with a 180° contact angle have important applications (e.g. drops with strong surface tension rolling down a sloping substrate – see [4]). Generally, if a problem involves a strong macroscopic effect (e.g. surface tension), it can 'overwhelm' the microscopic physics associated with the contact line and enforce the 180° contact angle.

[a)] Corresponding author. E-mail: Eugene.Benilov@ul.ie.

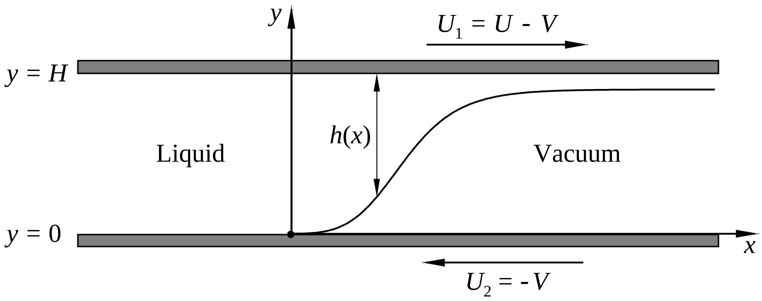

Figure 1. A Couette flow with a free boundary, in the reference frame co-moving with the contact line.

THE SETTING AND THE MAIN RESULT

We consider two horizontal rigid plates separated by a distance H, with the upper plate moving to the right with a constant velocity $U_1 > 0$ and the lower plate moving to the left with a velocity $U_2 < 0$ (see figure 1). Let the region between the plates be partially filled with a liquid of density ρ, dynamic viscosity μ, and surface tension σ. We assume that the free boundary of the liquid-filled region has a contact line on the lower plate, whereas the upper contact line has been dragged by the upper plate to infinity.

We shall use a reference frame co-moving with the contact line, which implies

$$U_1 = U - V, \qquad U_2 = -V,$$

where U and V are the velocities of the upper plate and contact line, both relative to the lower plate. Note that U should be treated as a given parameter, whereas V is to be solved for.

We examine steady two-dimensional flows and assume that the contact line is located at the origin of the coordinate system, with the x-axis directed along the lower plate and the y-axis, vertically. The shape of the free boundary is then described by $y = H - h(x)$, where h is the thickness of liquid-filled region located to the left of, and above, the free boundary (see figure 1). The density of the gas on the other side of the free boundary will be neglected, i.e., effectively, the gas will be replaced with vacuum.

The problem at hand is governed by two non-dimensional parameters – say,

$$\varepsilon = \frac{\mu U}{\rho g H^2}, \qquad \alpha = \frac{\mu U}{\rho g H^2}\left(\frac{\sigma}{\mu U}\right)^{1/3},$$

where g is the acceleration due to gravity. Physically, ε is the ratio of the viscous force to gravity and α characterizes surface tension.

It can be demonstrated that the condition $\varepsilon \ll 1$ guarantees that the slope of the free surface is small and, thus, the lubrication approximation can be used to describe the flow far away from the contact line. The outer flow, however, is still affected by a boundary layer near the contact line, through a boundary condition which follows from the analysis of [2].

The resulting boundary value problem has been analyzed asymptotically for the limits of small and large α, and numerically in the general case. It has been shown that, in a certain range of α, a solution exists describing a steady flow with a a $\mathbf{180^\circ}$ contact angle.

References

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