LARGE DROPS OF A POWER-LAW FLUID IN A THIN FILM ON A VERTICAL FIBRE

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Summary On a vertical fibre, drops in a thin film either accelerate and grow or fall at a constant speed, depending on the value of a non-dimensional gravity. We make an asymptotic analysis of the large drops falling at constant speed, to determine the relation between the speed and the gravity. We show a different behaviour of the relations for shear-thinning and shear-thickening fluids.

INTRODUCTION

We study a thin liquid film on a vertical fibre. Without gravity, there is a Rayleigh-Plateau instability in which surface tension reduces the surface area of the initially cylindrical film. Spherical drops cannot form because of the fibre, and instead, the film forms bulges of roughly twice the initial thickness. Large bulges then grow very slowly through a ripening mechanism. A small non-dimensional gravity moves the bulges. They leave behind a thinner film than that in front of them, and so grow. As they grow into large drops, they move faster and grow faster. When gravity is stronger, the bulges grow only to finite amplitude solitary waves, with equal film thickness behind and in front. We study these solitary waves, and the effect of shear-thinning and shear-thickening of the fluid. In particular, we will be interested in solitary waves of large amplitudes, which occur near the boundary between large and small gravity.

We use lubrication theory, scaling axial distance x by the radius of the fibre, the film thickness by its initial uniform value, and time by the growth rate of the linear Rayleigh instability. The governing equation is then

$$h_t + \left(h^{2+\frac{1}{n}}(G + (h + h_{xx})_x)^{\frac{1}{n}}\right)_x = 0$$

where h is the film thickness and n is the power-law index relating shear stress to shear rate. The parameter $G = \rho g a^3/\gamma h_0$ measures the non-dimensional strength of gravity, and is the ratio of the hydrostatic pressure difference over a radius to the capillary pressure. Here, ρ is the density, g the gravitational acceleration, a the radius of the fibre, γ the surface tension, and h_0 the initial film thickness.

RESULTS

Figure 1 plots the maximum amplitude $h_{\max}$ of the solitary waves propagating at speed c without change of form as a function of the strength of gravity G, for several values of the power-law index n. At large values of G, the amplitude decreases to a small finite value. For shear-thickening fluids with $n > 1$, the maximum amplitude grows indefinitely as G decreases to a critical value. In contrast, shear thinning fluids with $n < 1$ have a finite value of the maximum amplitude at a minimum value of G, and there is a range of G above this minimum where there are two branches of solitary waves. On one branch, the amplitude increases indefinitely as G increases to the end of the range. We are interested in the large amplitude solitary waves, as G approaches its critical value.

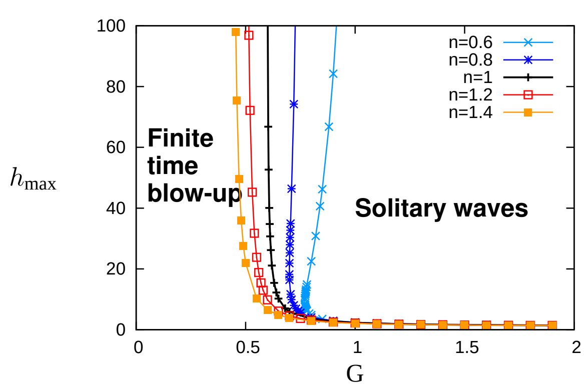

Figure 1

It is convenient to make an asymptotic expansion in terms of the large propagation speed c of the large amplitude solitary waves. The large solitary waves have three regions, a large drop of order one length, plus two short transition regions to the uniform layer in front and behind. To leading order, the large drop is at constant capillary pressure, and so takes the form

$$h_0 = \tfrac{1}{2}h_{\max}\,(1 - \cos x)\,, \quad \text{in} \quad 0 < x < 2\pi.$$

The short transition regions around $x = 0$ and $x = 2\pi$ are governed by Bretherton's equation modified for power-law fluids,

$$h_{\xi\xi\xi} = \frac{(h-1)^n}{h^{2n+1}}, \quad \text{where} \quad \xi = c^{n/3}(x - x_0).$$

The solution has a behaviour similar to that found by Bretherton. As one comes out of the transition region into the large drop, the solution takes the form

$$h \sim \tfrac{1}{2}P\xi^2 + R_\pm, \quad \text{as} \quad \xi \to \pm\infty,$$

where numerical solution gives the functions $P(n)$ and $R_\pm(n)$. Matching the curvature gives a relation between the amplitude and the speed,

$$h_{\text{max}} = 2Pc^{2n/3}.$$

The difference between $R_+(n)$ and $R_-(n)$, the apparent film thicknesses seen by the large drop behind and in front, requires a small correction h_1 to the drop accounting for hydrostatic pressure variations:

$$G_0 + (h_1 + h_{1xx})_x = 0.$$

The pendant drop solution is

$$h_1 = R_+ + G_0(\sin x - x), \quad \text{in} \quad 0 < x < 2\pi,$$

which yields the critical value of the parametmer G:

$$G_0(n) = (R_+(n) - R_-(n))/2\pi.$$

At this stage, we have yet to determine the speed c. It will depend on the difference of the parameter G from its critical value, in $G = G_0 + c^{-\alpha(n)}G_1$, where $\alpha(n) = (2n^2 - n)/3$. We need to find the change h_2 in the shape of the drop due to propagation:

$$G_1 + (h_2 + h_{2xx})_x = \frac{1}{h_0^{1+n}}.$$

The solution is singular as the transition regions are approached. This singular behaviour matches exactly one term in the transition regions. However a constant term $D_\pm(n)$ in the solution, a correction to the singular term, cannot be matched to anything in the transition regions. Instead, the difference in the constant terms must be accommodated by a hydrostatic pressure contribution through G_1. This gives

$$G_1(n) = (D_+(n) - D_-(n))/2\pi,$$

and so we have determined the speed c.

In the above, we have described the leading order behaviour. We have made a formal matched asymptotic expansion, for the drop and for the transition regions, proceeding to six terms in various fractional powers of c^{-1}. A successful matching of some 30 contributions determines all the constants of integration.

We plot in figure 2 the speed of propagation of the solitary waves raised to the power $-\alpha(n)$ as a function of the non-dimensional measure of gravity. Figure 2(a) gives the shear-thinning cases $n < 1$, and figure 2(b) the shear-thickening cases $n > 1$. We plot the numerical solutions and our asymptotic approximations.

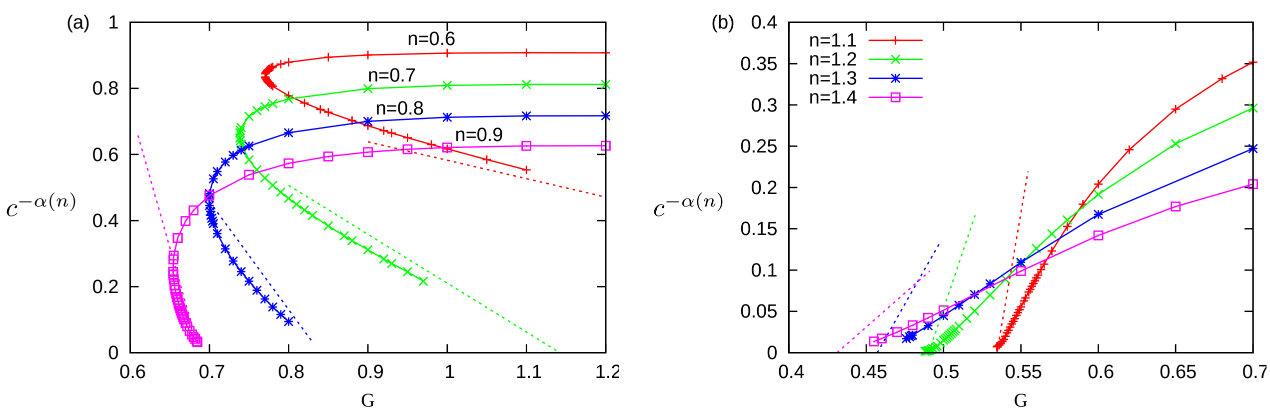

Figure 2 $\quad \alpha(n) = (2n^2 - n)/3$

References

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