NEW EXACT COHERENT STATES IN PLANE POISEUILLE FLOW

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Summary Two new classes of travelling wave solutions are found in plane Poiseuille flow by continuing the stationary and travelling hairpin vortex states in plane Couette flow. One of them arises from a saddle-node bifurcation, characterized by two mirror-symmetries with respect to planes perpendicular to the spanwise direction. The second new class solution bifurcates by breaking the mid-plane symmetry of the first class. Both classes exist below the critical Reynolds number known to date.

BACKGROUND

For many decades plane Poiseuille flow (PPF) has been serving as a testing ground for theories that attempt to explain observed transitions to turbulence in shear flows. It is known that the laminar PPF is linearly stable until the Reynolds number, R, exceeds its critical value 5772.22 [1] and that a spanwise-independent two-dimensional nonlinear travelling wave solution bifurcates subcritically [2]. The subcritical branch experiences a saddle node at $R = R_{sn}^{2D} = 2939.0$ and turns back in the direction of increasing R. Secondary bifurcations occur on the lower branch and solutions in the form of three-dimensional travelling wave emerge [3], [4]. Despite the claim in [4] that the 3D solutions exist below the saddle node point Reynolds number, R_{sn}^{2D}, of the 2D branch, accurate calculations [5] indicate the contrary. This is due to the highly truncated level of calculations in [4]. We, therefore, believe that there exist no solutions below R_{sn}^{2D} that can be traced back to the unperturbed flow at the linear critical point. Note that this does not exclude the possibility that other types of solution may exist below R_{sn}^{2D}. In fact, by considering a homotopy continuation of the 3D steady solutions of Nagata [6] in plane Couette flow (PCF), Waleffe [7] obtained 3D travelling-wave solutions, which are found to emerge as a result of a saddle-node bifurcation at $R = 977.6$. We call this solution W03. Recently, stationary [8] and travelling [9] hairpin vortex states have been found in PCF. It is known that the stationary hairpin vortex state bifurcates via a saddle node bifurcation and the travelling hairpin vortex state bifurcates supercritically from the branch of the stationary hairpin vortex state. The bifurcation of the travelling wave hairpin state is a pitchfork type. In this Short Paper, we attempt to transform these hairpin vortex states in PCF to PPF by homotopy continuation.

ANALYSIS

Homotopy

We consider the motion of an incompressible fluid with the density ρ and the kinematic viscosity ν in a channel bounded by two horizontal walls of infinite extent separated by the distance $2d$. The motion of the fluid is induced by moving the walls in opposite directions, with both boundaries moving at a constant speed, U_c. In addition, we apply a constant pressure gradient, G, in the direction of the motion of the top wall. We use a Cartesian coordinate system (x, y, z) with the origin on the mid-plane of the channel, where the x and y-axes are directed in the stream- and spanwise directions, and the z-axis is normal to the channel walls. The motion of the fluid is governed by the continuity equation, the Navier-Stokes equation and the no-slip boundary condition on $z = \pm 1$. The basic flow is a combination of a PCF component, $-R_c z$, and a PPF component, $R_p(1 - z^2)$, where the two Reynolds numbers, R_c and R_p, are defined by $R_c = U_c d/\nu$ and $R_p = U_p d/\nu$ with $U_p = Gd^2/(2\rho\nu)$. In order to establish a homotopy continuation from PCF to PPF, we consider the basic flow $U_B(z)$ given by

$$U_B(z) = -\sqrt{1-\epsilon}R_c z + \sqrt{\epsilon}R_p(1 - z^2), \tag{1}$$

where $\epsilon \in [0, 1]$ represents a homotopy parameter.

Symmetries

For a velocity deviation, $\boldsymbol{u} = (u, v, w)$, from the basic state, periodicity in the x and y-directions with the wavenumber pair (α, β) is assumed. We also assume a travelling wave form for the deviation, so that $\boldsymbol{u}$ is expressed as $\boldsymbol{u}(x - ct, y, z)$, where c is the phase velocity in the x-direction. The numerical procedures to obtain nonlinear solutions are the same as those used in [9]. It is known that the stationary hairpin vortex state in PCF satisfies the following three symmetries:

$$S_1[u, v, w](x, y, z) = [u, -v, w](x + \pi/\alpha, -y, z), \tag{2}$$

$$S_2[u, v, w](x, y, z) = [-u, v, -w](-x + \pi/\alpha, y + \pi/\beta, -z), \tag{3}$$

$$S_3[u, v, w](x, y, z) = [u, v, w](x + \pi/\alpha, y + \pi/\beta, z). \tag{4}$$

Symmetry S_1 is a shift-reflection symmetry, S_2 is a shift-rotation symmetry and the composition of S_1 and S_2 corresponds to a rotation symmetry about the point $(0, \pi/(2\beta), 0)$: $\boldsymbol{u}(x, y, z) = -\boldsymbol{u}(-x, -y + \pi/\beta, -z)$. It is shown in [9] that the travelling wave hairpin vortex state in PCF satisfies only S_1 and S_3. It is also shown in [9] that the composition of S_1 and

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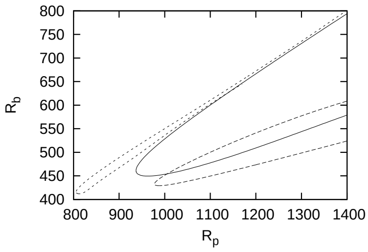

Figure 1. The Reynolds number, R_p, vs. the bulk Reynolds number, R_b, for the wavenumber pair (α, β) which give the minimum value for R_p The solid, dashed and dotted curves correspond to MS with $(R_p, \alpha, \beta) = (937.1, 1.47, 3.06)$, W03 with $(R_p, \alpha, \beta) = (977.6, 1.02, 2.64)$ and AS with $(R_p, \alpha, \beta) = (805.5, 1.33, 1.73)$, respectively.

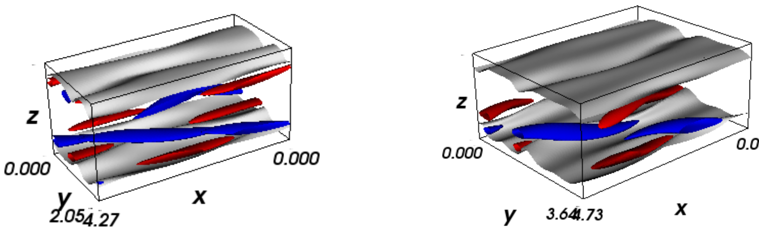

Figure 2. The flow patterns of the new exact coherent states. (left): MS at $(R_p, \alpha, \beta) = (937.1, 1.47, 3.06)$ and (right): AS at $(R_p, \alpha, \beta) = (805.5, 1.33, 1.73)$. Iso-surfaces of $u = 300$: gray , $\omega_x = 400$: red, $\omega_x = -400$: blue .

S_3 represents a mirror-symmetry about the two vertical planes $y = \pm\pi/(2\beta)$. With the hairpin vortex states in PCF as seeds, we start the homotopy continuation by increasing ϵ from 0 to 1, while keeping, say, $(\alpha, \beta) = (1.0, 2.0)$, $R_c = 250$ and $R_p = 1300$. As soon as a PPF component comes into existence, $i.e.$, as ϵ departs from 0, stationary hairpin vortex states cease to be stationary, becoming travelling waves. All the four homotopy routes, that started with the upper and lower branch stationary hairpin vortex states and two travelling wave hairpin vortex states at $\epsilon = 0$, possess symmetries S_1 and S_3 for $0 < \epsilon < 1$. It is found that the two homotopy routes out of four manage to reach the PPF limit where, in addition to S_1 and S_3, a new symmetry S_4,

$$S_4[u, v, w](x, y, z) = [u, v, -w](x, y, -z), \tag{5}$$

due to the absence of the PCF component is acquired. The symmetry S_4 corresponds to a mirror-symmetry about the mid-plane $z = 0$. The other two homotopy routes join at $\epsilon < 1$ failing to reach the PPF limit. By using the two solutions on $\epsilon = 1$, where $(R_c, R_p) = (0, 1300)$, as seeds, new solution branches in PPF are obtained. Fig.1 presents this new solution, referred to as MS, together with W03. Also shown in FIg.1 is a second new solution branch, referred to as AS, which bifurcates from MS by breaking the mid-plane symmetry S_4.

CONCLUSIONS

As Fig.2 shows both MS and AS are characterized by two quasi-streamwise low-speed streaks within one spanwise period. The low-speed streaks are aligned with the vertical planes of mirror symmetry, with their width varying in a varicose fashion in the streamwise direction. These streaks appear close to both top and bottom channel walls for MS and only the bottom channel wall for AS. A pair of quasi-streamwise vortices forms a Λ-shaped vortex: vortices are up-lifted downstream while attached by one foot to each of two neighboring varicose bulges of the streamwise low-speed streaks. This structure is quite different from that of W03: only one quasi-streamwise low-speed streak is included in one spanwise period. Furthermore, the low-speed streak is sinusoidal in the streamwise direction, flanked by staggered quasi-streamwise vortices that lie closer to the mid-plane than those of MS.
We believe that the existence of the new exact coherent states contributes towards the understanding of laminar-turbulent transition in plane Poiseuille flow.

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