STABILIZATION AND UNIVERSALITY IN HELE-SHAW FLOWS WITH MANY INTERFACES

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Summary Saffman-Taylor instability is a well known viscosity driven instability of an interface. Motivated by a need to understand the effect of various injection policies currently in practice for chemical enhanced oil recovery, we study linear stability of displacement processes in a Hele-Shaw cell involving injection of an arbitrary number of fluid phases in succession. This is a problem involving many interfaces. Universal stability results that hold for flows with arbitrary number of interfaces have been obtained.

INTRODUCTION

Two-layer Hele-Shaw flows in which a fluid is displacing another fluid of higher viscosity with both fluid layers extending up to infinity away from the common interface is the most celebrated case in the context of Saffma-Taylor instability [1] of an interface separating these two fluids. If μ_r is the viscosity of the displaced fluid, μ_l ($\mu_l < \mu_r$) is the viscosity of the displacing fluid, U is the constant velocity of the rectilinear flow, and the interfacial tension at the interface is constant T, then the growth rate σ_{st} of the interfacial disturbance having wave-number k is given by

$$\sigma_{\rm st}(k) = \frac{Uk(\mu_r - \mu_l) - k^3 T}{\mu_r + \mu_l}, \tag{1}$$

from which it follows that the maximum growth rate $\sigma_{\rm sm}$ and corresponding dangerous wavenumber k_{sm} are given by

$$\sigma_{\rm sm} = \frac{2T}{(\mu_r + \mu_l)} \left(\frac{U(\mu_r - \mu_l)}{3T} \right)^{3/2}, \qquad k_{\rm sm} = \sqrt{\frac{U}{3T}(\mu_r - \mu_l)} \tag{2}$$

The critical (also known as cut-off) wave number for which the wave is neutral (i.e., growth rate zero) is given by

$$k_{cr} = \sqrt{\frac{U}{T}(\mu_r - \mu_l)}. \tag{3}$$

We obtain extension of the above results and in particular universality of several results including the formula (3) for rectilinear Hele-Shaw flows of many immiscible fluids separated by sharp interfaces. The flow domain is $\Omega := (x, y) = \mathbb{R}^2$ (with a periodic extension of the set-up in the $y-$direction). The fluid upstream (i.e., as $x \to -\infty$) has a velocity $\mathbf{u} = (U, 0)$ and constant viscosity μ_l which occupies the region $x < -L$ in the moving frame (moving with velocity $(U, 0)$). Similarly, the fluid in the region $x > 0$ in the moving frame has viscosity μ_r. There are N interior regions of the equal length L/N in the interval $(-L, 0)$ (see Fig. 1). Each of these regions contains constant viscosity fluids μ_i, $i = 1, 2, ..., N$ such that viscosity increases in the direction of basic flow, i.e., $\mu_r > \mu_1 > \mu_2 > \mu_3 .... > \mu_N > \mu_l$. Thus, this set-up has $(N + 1)$ interfaces located at $x = 0, -L/N, ..., -L$ with corresponding interfacial tension coefficients denoted by T_i, $i = 0, 1, ..., N$ respectively.

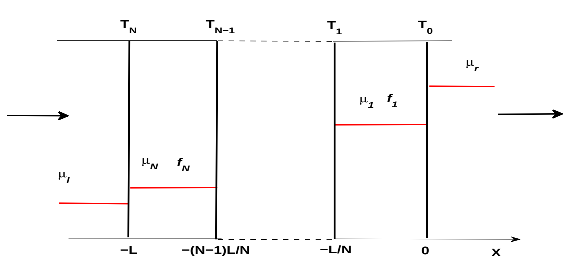

Figure 1. Multi-layer fluid flow

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TECHNIQUES AND SOME RESULTS

Using linear stability analysis of the standard Hele-Shaw model of the fluid flow, several eigenvalue problems, one per layer, are obtained. These are coupled through nonlinear interfacial conditions. These eigenvalue problems are analyzed theoretically and numerically using tools from PDES, optimization and numerical analysis. Among many new results for which we do not have space here, we mention following results.

1. The cut-off wavenumbers $k_i,\ i = 0, 1, \cdots, N$ (one for each dispersion relation), given by

$$k_i = \sqrt{\frac{U(\mu_i - \mu_{i+1})}{T_i}}, \quad i = 0, 1, \ldots . N, \tag{4}$$

   are universal meaning that they hold for flows with any number of interfaces. Notice two important facts about this: (i) cut-off wavenumbers are pure Saffman-Taylor cut-off wavenumbers for individual interfaces; (ii) these cut-off wavenumbers do not depend on the separation distance, L, between every two consecutive interfaces. These properties are universal as well.

2. The unstable bandwidth $k_{\max}$ given by

$$k_{\max} = \max(k_0, k_1, \cdots, k_N), \quad \text{and} \quad k_{\min} = \min(k_0, k_1, \cdots, k_N), \tag{5}$$

   is universal as well and it does not depend on L. The shorter bandwidth $k_{\min}$ given by $(5)_2$ which contains unstable waves having all modes positive is also universal. These bandwidths also do not depend on L.

3. For given interfacial tensions, there are specific sets of viscosities of the internal layers that give the smallest unstable bandwidth which happens to be same as (3) except that T in formula (3) is now sum of all interfacial tensions now. These results are universal as well and do not depend on L.

4. We will also present upper bounds on the modal and maximum growth rates in the multi-layer case which can be viewed as generalizations of formulas (1) and (2).

5. For the three-layer case with monotonic viscous profile for the middle layer, we show that the exponential profile is optimal over most of the parametric range and, when it is not, the growth rate of the optimal exponential profile is very close to that for the true optimum. In fact, there is little variation in the growth rates for all the profiles considered (with the possible exception of constant viscosity) unless L is very small.

6. Numerical experiments reveal that there are some very simple rules for the selection of most optimal configuration for the multi-layer case with monotonic variable viscous profiles in each layer.

These results are useful for stabilization and understanding of flow features in unstable multi-layer Hele-Shaw flows. Application of these new results to stabilization of multi-layer Hele-Shaw flows has been made. We will also show numerical results based on reservoir simulations of enhanced oil recovery processes to show relevance of results based on Hele-Shaw model to actual simulation results based on Buckley-Leverett model. Most of the results to be presented are unpublished and builds on our earlier work (see [2], [3]).

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