OPTIMAL SUPPRESSION OF UNSTEADINESS IN INCOMPRESSIBLE FLOW

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Summary
We describe a methodology for the computation of optimal boundary perturbations that minimize the unsteadiness or oscillation energy in incompressible flow. The treatment is based on a sequential time integration of the linearised Navier–Stokes (LNS) equations and their adjoint. We apply this methodology to flow past a cylinder at Reynolds number 10^2 to calculate the wall-normal boundary perturbation that optimally suppresses vortex shedding. It is observed that vortex shedding can be completely suppressed when the control cost is large enough, and that the resulting stabilized flow is asymptotically stable to both two-dimensional and three-dimensional disturbances at the parameters considered.

INTRODUCTION

Various methodologies exist for the computation of optimal flow disturbances in the form of initial perturbations and external forcing [4]. Quite recently, numerical algorithms were proposed to calculate the optimal boundary perturbation that maximizes the energy of the responding flow field with zero initial perturbations or that minimizes the energy of the responding field with non-zero initial perturbations [2, 1].
The existing algorithms target at the response of the flow field at a fixed time horizon and therefore only a final condition of the flow field is taken into account. But the algorithm we present here targets at the unsteadiness of the flow field modelled as the fluctuation energy over a finite time interval, and therefore involves the full development trajectory of the primary variables.

METHODOLOGY

To simplify notation, we introduce scalar products

$$(\boldsymbol{a},\boldsymbol{b}) = \int_\Omega \boldsymbol{a}\cdot\boldsymbol{b}\,\mathrm{d}V, \qquad \langle \boldsymbol{a},\boldsymbol{b}\rangle = \tau^{-1}\int_0^\tau (\boldsymbol{a},\boldsymbol{b})\mathrm{d}t, \qquad [\boldsymbol{c},\boldsymbol{d}] = \int_{\partial\Omega} \boldsymbol{c}\cdot\boldsymbol{d}\,\mathrm{d}S,$$

where Ω represents the spatial domain, $\partial\Omega$ denotes the boundary segment where the boundary perturbation is introduced and τ is a final time.
We consider Dirichlet-type boundary-normal perturbations, denoted as $\boldsymbol{u}_n(\boldsymbol{x},t)$, where $\boldsymbol{x}$ represents spatial coordinates on the boundary. To reduce the dimension of $\boldsymbol{u}_n(\boldsymbol{x},t)$ after temporal-spatial discretization, we decompose the temporal and spatial dependency as $\boldsymbol{u}_n(\boldsymbol{x},t) = \hat{\boldsymbol{u}}_n(\boldsymbol{x})f(t,\omega)$ where $\hat{\boldsymbol{u}}_n(\boldsymbol{x})$ is the spatial dependency to be optimized and $f(t,\omega)$ is a prescribed temporal-dependency function with ω acting as a temporal frequency. In this work, the temporal function is set to $f(t,\omega) = \exp(-t^2)\cos(\omega t)$, which satisfies $\boldsymbol{u}_n(\boldsymbol{x},0) = 0$ so as to avoid temporal discontinuity in the numerical simulation at $t = 0$.
Decomposing the flow field as the sum of an uncontrolled base flow and a perturbation flow induced by the control i.e. $(\boldsymbol{u},p) = (\boldsymbol{U},P) + (\boldsymbol{u}',p')$ and omitting the interaction of perturbations, we obtain the governing equations of the perturbation, the LNS equations, compactly denoted as $\partial_t\boldsymbol{u}' - L(\boldsymbol{u}') = 0$. Similarly to the methodology established in [3] to calculate the optimal boundary perturbation that results in maximum energy in the domain over a finite time horizon, a Lagrangian functional can be defined as

$$\mathcal{L} = (\langle \boldsymbol{u},\boldsymbol{u}\rangle - (\boldsymbol{u}_a,\boldsymbol{u}_a)) + \langle \boldsymbol{u}^*, \partial_t\boldsymbol{u}' - L(\boldsymbol{u}')\rangle + \lambda(E - [\hat{\boldsymbol{u}}_n,\hat{\boldsymbol{u}}_n]),$$

where $\boldsymbol{u}_a$ is the averaged velocity and is calculated as $\boldsymbol{u}_a = \tau^{-1}\int_0^\tau \boldsymbol{u}\mathrm{d}t$, $\boldsymbol{u}^*$ represents the adjoint velocity vector, λ is a Lagrangian multiplier and E is a prescribed control cost. The first term is the unsteadiness or the fluctuation energy to be minimized, the second term applies the constraint of the LNS equations and the third term is the constraint on the control cost or the magnitude of the control perturbation.
The gradient of $\mathcal{L}$ with respect to $\hat{\boldsymbol{u}}_n$ is calculated as $\nabla_{\hat{\boldsymbol{u}}_n}\mathcal{L}_n = \boldsymbol{n}\cdot\tau^{-1}\int_0^\tau (p^*\boldsymbol{n} - Re^{-1}\boldsymbol{\nabla}_{\boldsymbol{n}}\boldsymbol{u}^*)f^*(t,\omega)\,\mathrm{d}t - 2\lambda\hat{\boldsymbol{u}}_n$, where $\boldsymbol{n}$ is a unit outward normal on the surface of the cylinder, Re is the Reynolds number and $f^*(t,\omega)$ is the adjoint operator of $f(t,\omega)$. The adjoint pressure p^* and velocity vector $\boldsymbol{u}^*$ are calculated by integrating the adjoint equation,

$$\partial_t\boldsymbol{u}^* + L^*(\boldsymbol{u}^*) + 2(\boldsymbol{u} - \boldsymbol{u}_a) = 0,$$

where L^* is the adjoint operator of L. This adjoint equation is initialized by $\boldsymbol{u}^*(\tau) = 0$ and integrated backwards from $t = \tau$ to $t = 0$. We note that the full trajectory of the primary variables $\boldsymbol{u}$ are required to solve this adjoint equation and therefore the development history of $\boldsymbol{u}$ has to be saved while solving the LNS equations.

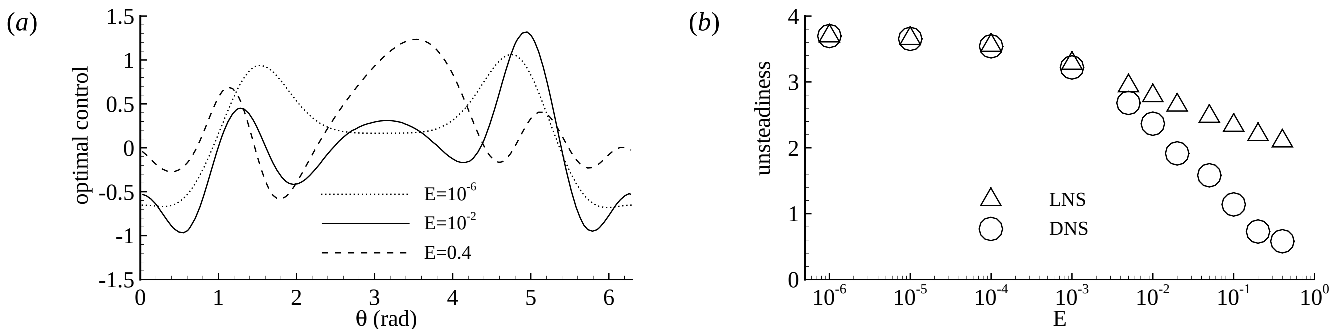

Figure 1. (a) Distribution of the optimal control perturbation $\hat{\boldsymbol{u}}_n$ (normalized so that $[\hat{\boldsymbol{u}}_n, \hat{\boldsymbol{u}}_n] = 1$) on the surface of the cylinder at various prescribed control costs. θ is the radial coordinate of the surface of the cylinder and $0 \le \theta \le 2\pi$. (b) Unsteadiness of the controlled flow field. “LNS" is the minimum value of $\mathcal{L}$ and “DNS" is the oscillation energy calculated through DNS of the base flow perturbed by the optimal control at various values of control cost E.

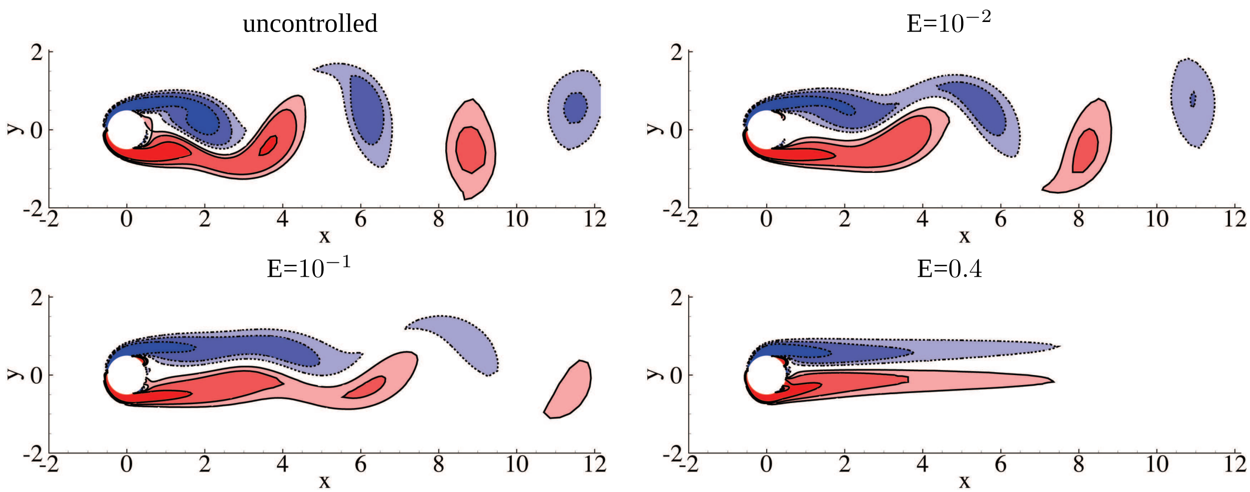

Figure 2. Contours of spanwise vorticity for the controlled and uncontrolled flow around a circular cylinder at $Re = 100$, for $t = T = 80$. Dashed lines and solid lines represent negative and positive vorticity respectively and the contour levels $\in [-2, 2]$.

EXAMPLE RESULTS

The algorithm outlined above is implemented to optimally suppress vortex shedding in the flow past a cylinder. Since $\omega = 0$ is observed to be the most efficient choice of frequency, in this study we adopt time-invariant boundary control. The terminal time and Reynolds number are set to $\tau = 80$ and $Re = 100$ respectively. Optimal control in the form of wall-normal Dirichlet-type boundary perturbations imposed on the surface of the cylinder is illustrated in figure 1(a), where we see that the spatial distribution of the optimal control changes dramatically as the increase of the control cost E. At small values of E, the perturbation is concentrated around the separation points, while at $E = 0.4$, most of the energy of the control is concentrated around the front stagnation point. DNS is conducted to study the flow perturbed by the optimal control perturbation and the unsteadiness is measured and illustrated as “DNS" in figure 1(b), where “LNS" represents the minimum value of $\mathcal{L}$ in the optimization, since the governing equations are LNS equations. We see that the “DNS" results deviate from the optimized “LNS" results at large values of E when the interaction of perturbations cannot be neglected. The flow subjected to the optimal control is illustrated in figure 2, where we see that the vortex shedding is significantly weakened as the increase of control cost and at $E = 0.4$, the flow is completely stabilized. Floquet analysis conducted using this stabilized flow as the base flow reveals that this flow is asymptotically stable to both two-dimensional and three-dimensional perturbations.

References

[1] Barkley D, Blackburn HM & Sherwin SJ. Direct optimal growth analysis for timesteppers. *Int J Num Meth Fluids*, **57**:1437–1458, 2008.
[2] Guégan A, Schmid P & Huerre P. Optimal energy growth and optimal control in swept hiemenz flow. *J Fluid Mech*, **566**:11–45, 2006.
[3] Mao X, Blackburn HM & Sherwin SJ. Optimal inflow boundary condition perturbations in steady stenotic flows. *J Fluid Mech*, Festschrift volume, 2012.
[4] Schmid PJ & Henningson DS. *Stability and Transition in Shear Flows*. Springer, 2001.