Linear stability analysis of lid-driven cavity flow using a spectral element method

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Summary A program of linear stability analysis based on a Legendre spectral element method is developed, and it is applied to the convection stability of lid-driven cavity flow. The normal mode analysis is used for the disturbances equations. A generalized eigenvalue problem is constituted and the critical Reynolds number is determined by the largest real part of complex growth rate. It is found that the transition of the flow is Hopf bifurcation with the frequency of 0.0703 when the range of wave number is $1 \le k \le 9$.

INTRODUCTION

The linear stability analysis of lid-driven cavity has been the subject of wide-spread research for many years [1-3]. It is a good example for testing the accuracy and efficiency of the numerical method for its simplicity. In the infinitely extended system, the flow is two-dimensional and steady as the control parameter, i.e., the Reynolds number (Re), is sufficiently small. However, the flow will lose stability as increasing Reynolds number beyond the critical Reynolds number ($\mathrm{Re_c}$). In this paper, we focus on the first convective instability and solve numerically the $\mathrm{Re_c}$ by the linear stability analysis based on the Legendre spectral element method (SEM) [4].

MATHEMATICAL FORMULATIONS

Formulations

We consider lid-driven cavity flow in a cavity that is infinitely long in the z direction, and the aspect ratio is 1:1 in x-y plane. The dimensionless form of the control equations are

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

$$\partial \mathbf{u}/\partial t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nabla^2 \mathbf{u}/\mathrm{Re} \tag{2}$$

where $\nabla = \mathbf{i}\partial_{,x} + \mathbf{j}\partial_{,y} + \mathbf{k}\partial_{,z}$, $\mathbf{u} = \mathbf{u}(u, v, w)$. The scales for lengths, time, velocity, and pressure are H, $\mathrm{H/u_0}$, $\mathrm{u_0}$, $\rho \mathrm{u_0^2}$ ($\mathrm{u_0}$ is the lid velocity). Reynolds number is defined as $\mathrm{Re} = \mathrm{Hu_0}/\nu$, where ν is the kinematics viscosity coefficient.

Base flow

The base flow is two-dimensional and steady. In this paper, the base flow is solved by SEM with a two-dimensional unsteady model. The time splitting method [5] is adopted to split the Navier-Stokes equations by time, then one Poisson equation and two Helmholtz equations which can be solved by the SEM [4], are derived. Finally, the base flow is expressed as $\mathbf{x} = \{\mathbf{U}(U, V), P\}$.

Perturbation equations

We perturb the base flow by a disturbance three-dimensional velocity $\hat{\mathbf{u}}\left[= \hat{\mathbf{u}}(\hat{u}, \hat{v}, \hat{w})\right]$ and the pressure $\hat{p}$. The disturbances are expressed as the normal mode, i.e., $\hat{\boldsymbol{\varphi}} = i\hat{\boldsymbol{\varphi}}(x, y)e^{\sigma t + ikz}$ ($\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}(\hat{u}, \hat{v}, \hat{p})$), and $\hat{w} = \tilde{w}(x, y)e^{\sigma t + ikz}$, where k is the spanwise wave number and σ the complex growth rate. If σ is real, the critical Re is that for $\sigma = 0$. If σ is complex, then the neutral condition is that for $\sigma_r = 0$, and the onset of instability is oscillatory with wave speed σ_i. Such a transition is called a Hopf bifurcation. Substituting the total velocities and pressure into Eqs. (1) and (2), and linearizing, we obtain the following equations for the as

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + k\tilde{w} = 0 \tag{3}$$

$$\sigma \tilde{u} = -U\frac{\partial \tilde{u}}{\partial x} - V\frac{\partial \tilde{u}}{\partial y} - \tilde{u}\frac{\partial U}{\partial x} - \tilde{v}\frac{\partial U}{\partial y} - \frac{\partial \tilde{p}}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} - k^2\tilde{u}\right) \tag{4}$$

$$\sigma \tilde{v} = -U\frac{\partial \tilde{v}}{\partial x} - V\frac{\partial \tilde{v}}{\partial y} - \tilde{u}\frac{\partial V}{\partial x} - \tilde{v}\frac{\partial V}{\partial y} - \frac{\partial \tilde{p}}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \tilde{v}}{\partial x^2} + \frac{\partial^2 \tilde{v}}{\partial y^2} - k^2\tilde{v}\right) \tag{5}$$

$$\sigma \tilde{w} = -U\frac{\partial \tilde{w}}{\partial x} - V\frac{\partial \tilde{w}}{\partial y} - k\tilde{p} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \tilde{w}}{\partial x^2} + \frac{\partial^2 \tilde{w}}{\partial y^2} - k^2\tilde{w}\right) \tag{6}$$

The boundary conditions are at x=0, 1 and y=0, 1: $\tilde{u} = \tilde{v} = \tilde{w} = 0$. The weak form and Galerkin method are used for Eqs. (3-6) and then a generalized eigenvalue problem ($\mathrm{A}\boldsymbol{X} = \sigma \mathrm{B}\boldsymbol{X}$) is solved by using the ARPACK library.

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RESULTS

In order to validate our developed program, a comparison of the maximum real and its corresponding imaginary parts at Re=200 is listed in Table 1. It demonstrates that our results are consistent with those in literatures.

Table1. Comparison of the leading eigenvalue versus wave number (k) with the references

k	Present	N. Ramanan [1]	V. Theofilis [2]	Y. Ding [3] (real part)
1	-0.3755±0.00*i*	-0.34±0.00*i*	-0.3297±0.00*i*	-0.3183
2	-0.2424±0.00*i*	-0.23±0.00*i*	-0.2267±0.00*i*	-0.2248
3	-0.2996±0.1055*i*	-0.29±0.11*i*	-0.2954±0.1073*i*	-0.2924
4	-0.2968±0.2812*i*	-0.30±0.28*i*	-0.2956±0.2810*i*	-0.2969
5	-0.3345±0.4261*i*	-0.34±0.43*i*	-0.3404±0.4260*i*	-0.3431
6	-0.3725±0.5689*i*	-0.39±0.58*i*	-0.3844±0.5821*i*	-0.3893
7	-0.3951±0.6558*i*	-0.41±0.67*i*	-0.4013±0.6733*i*	-0.4073
8	-0.4539±0.7030*i*	-0.45±0.72*i*	-0.4587±0.7232*i*	-0.4637
9	-0.5440±0.7395*i*	-0.54±0.76*i*	-0.5473±0.7622*i*	-0.5504

Table2. Four least-stable eigenvalues under three different Re

Re	k=6	k=7	k=8	k=9
1150	-0.02117±0.5040*i*	-0.002±0.4675*i*	-0.00124±0.4426*i*	-0.0206±0.4311*i*
1160	-0.01958±0.5021*i*	-0.0004±0.4661*i*	0.00028±0.4414*i*	-0.01933±0.4301*i*
1170	-0.01796±0.5000*i*	0.00119±0.4647*i*	0.00179±0.4402*i*	-0.0181±0.4290*i*

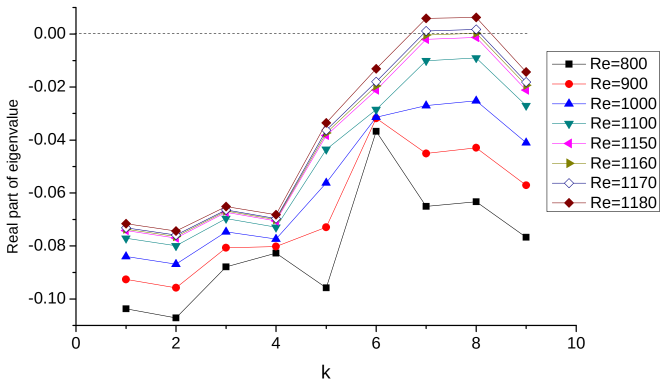

Fig.1 The maximum real part of eigenvalues versus wave number for various Re

Only limited wave numbers $k \in [1,9]$ have been computed and compared with those in literature. Table 2 gives the least-stable eigenvalues of three Re close to the critical value. Figure 1 shows the results $\sigma_r(k, \mathrm{Re})$ for the maximum real part versus wave number. We estimate the critical Re_c (for only $1 \le k \le 9$) as 1158 and the critical wave number as 8. The critical values are very close to those of Ding [3], whose corresponding values are (1025, 7.6). The imaginary part σ_i corresponding to the maximum real part is estimated at 0.4417. Then we have a Hopf bifurcation, the steady solution bifurcates to a periodic solution with a frequency of 0.0703 ($=\sigma_i/2\pi$) while that of Ding is 0.08 [3].

CONCLUSIONS

A program of linear stability analysis based on the Legendre SEM is developed and applied to the problem of lid-driven cavity flow. Under the wave number limitation $1 \le k \le 9$, the critical Re_c is 1158 and critical wave number is 8. The transition of the flow is Hopf bifurcation with the frequency 0.0703. Our results are consistent with those in reference.

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