VARIATIONAL FORMULATION OF CONTINUUM MECHANICS BASED ON THE LAGRANGIAN WITH THE FRICTION FORCE

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Summary The direct correlation between classical mechanics of material points (Lagrange principle of classical mechanics) and classical continuum mechanics can be established when the existence of a trajectory and a friction force are added. This new approach to the fluid flow stability leads to the revision of the Rayleigh's or Fjörtoft's criteria and allows us to explain the discrepancy between the theory and experiments. As the example the Couette and Poisseuille flows are investigated and the discrepancies are explained.

CLASSICAL MECHANICS OF CONTINUOUS MECHANICAL SYSTEMS

The most general formulation of the laws governing the motion of all mechanical systems composed from many interacting particles in general material material points is so-called *the principle of the least action* or the *Hamilton principle* [1]. In the application to the continuous system, see Fig. 1, we can evaluate the action S in the fix volume V with the surface ∂V between the instants $t_0,\ t_1$ as follows

$$\begin{aligned} S\left(\mathbf{v},\boldsymbol{\beta},\mathbf{X}\right) &= \int_{t_o}^{t_1}\int_V \rho\left[\frac{\mathbf{v}^2(\mathbf{x},t)}{2}-\Phi(\mathbf{x})-u\left(\rho(\mathbf{x},t),s(\mathbf{x},t)\right)-\mathbf{X}(\mathbf{x},\mathbf{t})\dot{\boldsymbol{\beta}}(\mathbf{x},\mathbf{t})\right]dvdt \\ &= \int_{t_o}^{t_1}\int_V \rho l(\mathbf{v}(\mathbf{x},t),\boldsymbol{\beta}(\mathbf{x},t),\mathbf{X}(\mathbf{x},t))dvdt \end{aligned} \tag{1}$$

Specific lagrangian is $l(\mathbf{v}(\mathbf{x},t),\boldsymbol{\beta}(\mathbf{x},t),\mathbf{X}(\mathbf{x},\mathbf{t}))$. The standard abbreviation $\dot{()}$ is used for material derivative and further, the kinetic energy of material point is $\frac{\mathbf{v}^2}{2}$ and Φ is the potential energy, $\mathbf{X}\dot{\boldsymbol{\beta}}=\sum_{L=1}^{3}X^L\dot{\beta}_L=x^i\dot{\beta}_i(\mathbf{x},t)$ is the energy of interaction with surroundings and $u\left(\rho((\mathbf{x},t)),s((\mathbf{x},t))\right)$ is the internal energy of the material point (M.P.). For solid body is density ρ replaced by the Euler deformation tensor $\mathbf{e}$. The independent quantity is the trajectory $\mathbf{x}(\mathbf{X},t)$ of the M.P. $\mathbf{X}$ and the velocity $\mathbf{v}(\mathbf{x},t)$ and friction force $\boldsymbol{\beta}(\mathbf{x},t)$ satisfy supplementary conditions. We suppose that each material point has the initial position $\mathbf{X}$-at some instant t_0. Its motion is described by the trajectory $\mathbf{x}(\mathbf{X},t)$ (with the inverse mapping $\mathbf{X}\mathbf{x}=\mathbf{x}\mathbf{X}=\mathbf{I}$) and the motion of all material points materializes in the velocity field $\mathbf{v}(\mathbf{x},t)$ [2].

Necessary conditions of the least action

Material point $\mathbf{X}=\tilde{\mathbf{X}}(\mathbf{x},t)$ intersect the geometrical point (G.P.) $\mathbf{x}$ and interacts (exchanges the energy) with the surroundings due to friction force $\dot{\boldsymbol{\beta}}$. The general laws of mechanics and thermodynamics have to be valid for all material points, i.e. for all trajectories which intersect this point $\mathbf{x}=\tilde{\mathbf{x}}(\mathbf{X},t)$. So that the necessary condition of the extremum of the functional (1), with respect to the variations (fluctuations) of the M.P. trajectory are as follows

$$v_i = X^L\frac{\partial\beta_L}{\partial x^i}=\frac{\partial\overbrace{\left(X^L\beta_L\right)}^{\varphi(\mathbf{x},t)}}{\partial x^i}-\beta_L\frac{\partial X^L}{\partial x^i}=v_{i\,\mathrm{pot}}+v_{i\,\mathrm{rot}},\quad v_{i\,\mathrm{pot}}=\frac{\partial\varphi}{\partial x^i},\ v_{i\,\mathrm{rot}}=\beta_i=-\beta_L\frac{\partial X^L}{\partial x^i} \tag{2}$$

The vorticity **rot v** is generated by friction or by the entropy gradient, as it will be shown later and it can induce the instability. The laws of conservation of the energy in the G.P. is

$$l-\frac{p}{\rho}=-X^L\frac{\partial\beta_L}{\partial t}-h_c=0 \quad \text{...energy conservation,}\quad \text{for}\quad h_c=\frac{v^2}{2}+u+\frac{p}{\rho}+\Phi \tag{3}$$

Total enthalpy h_c ***depends on the local friction force, only*** . For the local stationary friction field, i.e., $\partial(\boldsymbol{\beta})/\partial t=0$, the total enthalpy of the fluid is constant. From point of view of classical mechanics of mass points, the total enthalpy is the ***integral of motion*** and h_c can be taken as an ***adiabatic invariant*** [1].

The balance of momentum can be reformulated in the measurable quantities as follows

$$-\dot{\beta}_L\frac{\partial\tilde{X}^L}{\partial x^i}-\frac{\partial\Phi}{\partial x^i}-T\frac{\partial s}{\partial x^i}=0\quad\text{after elimination of }\boldsymbol{\beta},\quad \frac{\partial v_i}{\partial t}-\left(\mathbf{v}\times\mathbf{rot\,v}\right)_i=-\frac{\partial h_c}{\partial x^i}+T\frac{\partial s}{\partial x^i}+\frac{\partial\Phi}{\partial x^i} \tag{4}$$

After standard analysis (adoption **grad**h_c in (3), using (2)) and after reorganization of some terms, we obtain the final form of the momentum equation (4). This equation is an alternative form of the Crocco equation, which is valid for the steady case only [3]. From the comparison, the important role of the **grad**h_c, **grad**s and volume forces for the vorticity generation is evident

$$-\left(\mathbf{v}\times\mathbf{rot\,v}\right)_i=-\frac{\partial h_c}{\partial x^i}+T\frac{\partial s}{\partial x^i}+\frac{\partial\Phi}{\partial x^i}\quad\text{cond. (4)}\quad -\left(\mathbf{v}\times\mathbf{rot\ v}\right)_i=-\frac{\partial h_c}{\partial x^i}+T\frac{\partial s}{\partial x^i}+\frac{\partial t^{li}_{dis}}{\rho\partial x^l}\quad\text{Crocco theorem} \tag{5}$$

The friction (dissipative) forces in actual system are established explicitly by the dissipative part of the stress tensor $\mathbf{t}_{dis}$, nevertheless, even for isentropic flow with h_c=0, the volume force can induce the vortex generation.

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THERMODYNAMIC STABILITY CONDITIONS

As a consequence of the above analysis the thermodynamic stability criterion is formulated by h_c. The entropy is the function of h_c, p and the thermodynamic inequality (II. Law of thermodynamics) is

$$\pi = \rho \underbrace{(T\dot{s} + \frac{1}{\rho}\frac{\partial p}{\partial t} - \dot{h}_c)}_{=0\ldots\text{entropy definition}} - \frac{q^k}{T}\frac{\partial T}{\partial x^k} + \frac{\partial(t^{ki}_{dis} v_i)}{\partial x^k} \geq 0, \quad -\frac{\rho}{2}\dot{\overline{d^2 h_c}} = \pi, \tag{6}$$

for $\pi < 0$ is the thermodynamic inequality violated and the instability can occurs. To show the generality of the thermodynamic stability condition the classical Couette flow between two cylinders [4] was investigated.

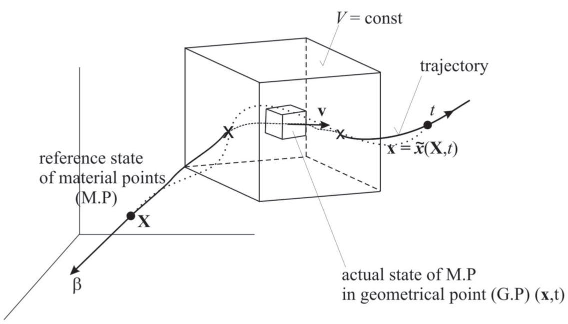

Fig. 1. At the G.P. $(\mathbf{x}, t)$ has the M.P. $\mathbf{X}$ the velocity field $\mathbf{v}\,(\mathbf{x}, t)$ and its interaction force is $\dot{\boldsymbol{\beta}}(\mathbf{x}, t)$.

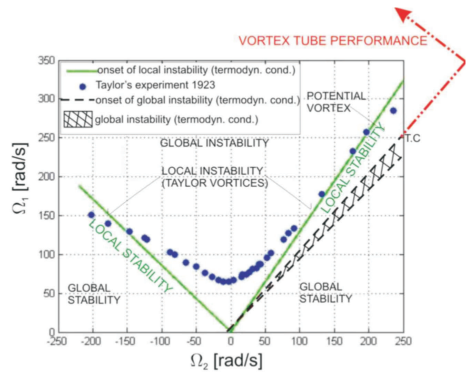

Fig. 2. Onset of coherent structures (instability) for a viscous flow between two rotating cylinders (Couette flow).

Provided that, the flow is isothermal and fluid has the constant density and the viscosity, the exact solution of the Navier-Stokes equations for the flow field between the two rotating cylinders was found like superposition of a solid body vortex and of a potential vortex. The radius of outer cylinder is R_2 and its angular velocity is Ω_2 and for inner cylinder is R_1, Ω_1. For nondimensional quantity $-\infty < \mu = \Omega_2/\Omega_1 < 1, \quad 0 < \eta = R_1/R_2 < 1, \quad 0 < \tilde{r} = r/R_2 < 1$ the Rayleigh criterion gives

$$\Phi(r) = 4A\left(A + \frac{B}{r^2}\right) \overset{r=R_2\tilde{r}}{=} \bar{\Omega}_R(\mu - \eta^2)[\eta^2(1-\mu) + \tilde{r}^2(\mu - \eta^2)] \geq 0, \quad \text{for} \quad \bar{\Omega}_R = \frac{4\Omega_1^2}{\tilde{r}^2(1-\eta^2)^2} > 0. \tag{7}$$

The stability region is for $0 < \mu < 1, \quad \mu = \Omega_2/\Omega_1 > \eta^2$. There are not the conclusion for the region $\mu < 0$, $\mu(\tilde{r}^2 - \eta^2) < \eta^2(\tilde{r}^2 - 1)$. Thermodynamic criterion (6) gives

$$\frac{\pi}{\mu} = \frac{2B}{r^2}\left(A + \frac{3B}{r^2}\right) \overset{r=R_2\tilde{r}}{=} \bar{\Omega}_T(1-\mu)[3\eta^2(1-\mu) + \tilde{r}^2(\mu - \eta^2)] \geq 0 \quad \text{for} \quad \bar{\Omega}_T = \frac{2\Omega_1^2\eta^2}{\tilde{r}^4(1-\eta^2)^2} > 0. \tag{8}$$

The stability region is $-2\eta^2 < \mu \leq 1$, $\mu = \Omega_2/\Omega_1 > \eta^2$ and for $\mu < 0$, $\mu > \eta^2 - 3$. The limits of stability regions and the comparison with experiment are in Fig. 2.

CONCLUSIONS

The important role of total enthalpy follows from the variational analysis (5). The thermodynamic criterion of stability based on the h_c is in comparison with the Rayleigh theory more general. This theory was verified for all modifications of the Couette flow, even for a solid body rotation $\Omega_1 = \Omega_2$, where the Rayleigh condition fails [5]. It was shown that the solid body vortex is at the margin of the stability, which is experimentally observed. Analogously, the potential vortex is by thermodynamic criterion stable, however by Rayleigh criteria is on the onset of stability.

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