MIXED CROSS SPIRALS AS INTERIM SOLUTIONS IN THE TAYLOR COUETTE SYSTEM

Sebastian Altmeyer*[a)], Christian Hoffmann**
*Department of Mathematics, Kyungpook National University, 1370 Sangyeok-Dong, Buk-Gu, Deagu, 702-701, Korea
**Institut für theoretische Physik, Universität des Saarlandes, 66123 Saarbrücken, Germany

Summary We present numerical results of secondarily forward bifurcating, stationary flow states that mediate transitions between travelling helical waves (spirals, SPIs). These so-called Mixed Cross Spirals (MCSs) can be seen as nonlinear superpositions of SPI solutions with different helicity and pitch. Thereby, the contribution of the respective SPI to the entire MCS varies continuously with the control parameters and MCSs can connect the bifurcation branches of the involved SPIs.

SYSTEM AND THEORETICAL DESCRIPTION

We report numerically obtained results in the Taylor-Couette system with concentric, co- and counter-rotating cylinders, fixed radius ratio $\eta = r_1/r_2 = 0.883$, no-slip boundary conditions at the cylinder surfaces, and axial periodic boundary conditions determining the axial wave number that is fixed to $k = 3.927$ ($\lambda = 1.6$). The fluid in the annulus between the cylinders (gap width $d = r_2 - r_1$) is considered to be isothermal and incompressible with kinematic viscosity ν. Cylindrical coordinates r, φ, z are used to decompose the velocity field into a radial component u, an azimuthal one v, and an axial one w: $\mathbf{u} = u\,\mathbf{e}_r + v\,\mathbf{e}_\varphi + w\,\mathbf{e}_z$. The system is governed by the Navier-Stokes equations

$$\partial_t \mathbf{u} = \boldsymbol{\nabla}^2\mathbf{u} - (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} - \boldsymbol{\nabla}p. \tag{1}$$

Lengths are scaled by the gap width d and times by the radial diffusion time d^2/ν. The Reynolds numbers $R_i = r_i\Omega_i d/\nu, i \in \{1,2\}$ enter into the boundary conditions for v: R_i is just the reduced azimuthal velocity of the fluid at the cylinder surfaces.

Numerical methods

Numerical simulations have been done with our 'G2D2' code as presented in detail in [1]. G2D2 implements a Galerkin expansion in two dimensions, φ and z, and finite differences of second order in r and of first order in t.

$$f(r,\varphi,z,t) = \sum_{m,n} f_{m,n}(r,t)\, e^{i(m\varphi+nkz)}, \quad f \in \{u,v,w,p\}. \tag{2}$$

Here, $f_{m,n}(r,t)$ are the amplitudes of the m-th azimuthal and the n-th axial Fourier mode.

Numerical analysis and visualization methods

For visualisation purposes, we found the azimuthal vorticity component $\Omega_\varphi = (\boldsymbol{\nabla}\times\mathbf{u})\cdot\mathbf{e}_\varphi = \partial_z u - \partial_r w$ to be an adequate and convenient means to identify and recognise the geometry of complex vortex structures via iso-vorticity surfaces. These so-obtained tubes represent the topology of vortex centres and thus determine the most important properties of the structures discussed here, e.g. symmetry, pitch, complexity etc. They further describe the strength of the vortices and give a rough impression of the vortex deformation.

Structural classification

In order to classify the structures, we often refer to their significant Fourier mode indices abbreviated with (c.f. Eq. 2) $(m,n) := f_{m,n}\, e^{i(m\varphi+nkz)}, f \in \{u,v,w,p\}$. The pure spiral fields do not depend on φ, z, t separately but only on the phase combination $\Phi := M\varphi + Kz - \omega t$ with wave numbers M and K. Thus the reduced axial pitch p of a SPI can be defined as

$$p := \frac{z_0(2\pi) - z_0(0)}{\lambda} = -M\frac{k}{K} = -\mathrm{sgn}(K)\, M, \tag{3}$$

and gives the number of axial periodicity lengths being covered while travelling in positive azimuthal direction along the helicoidal vortex tube once around the cylinder. Hence, $p\lambda/2\pi = -M/K$ is the slope of the lines of constant phase in the $\varphi - z$ plane of an azimuthally unrolled cylindrical surface. We identify the different spirals investigated here by the following combination of letters and numbers: e.g., L1-SPI is a left winding spiral with azimuthal wave number $M = 1$ (thus $p = -1$) while R2-SPI stands for an R-SPI with azimuthal wave number $M = 2$ (thus $p = 2$). The symbols 'L1' and 'R2' correspond to the pitch of the SPI.

BIFURCATION SCENARIOS

The case of MCS appearing as bypass solution bifurcating out and ending in the *same* solution branch was already discussed in [2]. Therein the contributions (major and minor) of the entired SPI are different, in general. For MCS as interim solution it is nessessary that the both significant SPI mode amplitudes are identical in at least one point (mixed riboon solution). A suitable bifurcation scenario for MCS as interim solution is presented in Fig. 1. Here, the bifurcation branches of a LA-SPI (orange) and RB-SPI (red) (M=A and B, addition to their corresponding RIB states A-RIB, B-RIB

a) Corresponding author. E-mail: sebastianaltmeyer@t-online.de.

(green)) are connected by the interim solution of MCS states (LARB-MCS and RBLA-MCS (maroon), depending on the major and minor part). Thus, the MCS connect two states with different helicity and propagation direction. In general $A \neq B$, the special case $A = B$ presents a cross spiral solution [3].

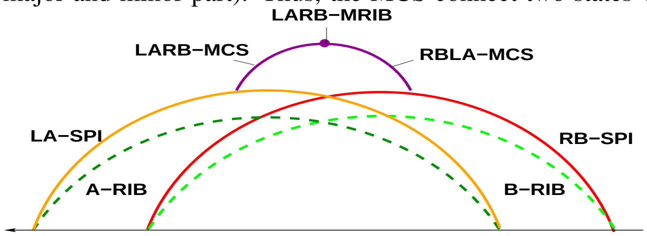

Figure 1. Schematic illustration for a suitable control parameter, e.g. R_2, of the bifurcation scenario for an MCS as interim solution connecting two different SPI branches with azimuthal wave numbers $A \neq B$. Here, the lines only present existing solutions and do not say anything about the stability.

SPATIOTEMPORAL BEHAVIOR

In order to get an impression of the spatial shape of MCS, Fig. 2 presents iso-vorticity surface plots of a L3R5-MCS (middle) and its both SPI contribution parts of L3-SPI (left) and R5-SPI (right).

Obviously, the L3R5-MCS in Fig. 2(middle) consists of a stronger L3 and a weaker R5 component. However, the vorticity maxima in Figs. 2(left and right), and therefore also their contributions to the entire structure, differ significantly, so that the L3 contribution dominates the whole L3R5-MCS structure and the R5 portion only generates a weak modulation. Here, the entire structure propagates upwards, i.e. in positive axial direction, due to the major L3 component, whereas the modulation travels downwards, due to the minor R5 component.

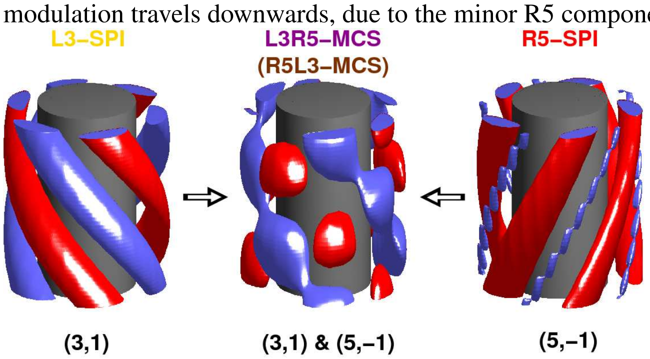

Figure 2. Iso-surfaces of azimuthal vorticity of a MCS (L3R5- and R5L3-MCS, middle) which mediates the transition between two SPI of different azimuthal wave numbers M, i.e. the M=3 left-winding L3-SPI (left) and the M=5 right-winding R5-SPI (right). The iso-vorticity values for all plots are chosen to lie near the half of the respective global maximum of the azimuthal vorticity.

It is remarkable that the blue vortex tubes of counter-clockwise (while looking into positive φ direction) rotating vortices in Fig. 2(right) are significantly smaller than the red, clockwise rotating ones. Since in general, the vorticity is strongest along a distinct line in the interior of a vorticity tube, the absolute maximum of the vorticity inside the larger red tubes is greater than that in the smaller blue ones. Furthermore, the vorticity is positive [negative] in the red [blue] vortex, thus $|\min \Omega_\varphi| < |\max \Omega_\varphi|$. Probably, this is a consequence of the non-vanishing intrinsic axial net flow generated by both spiral components (c.f. [4]) that influences clockwise and counter-clockwise rotating vortices in different ways.

CONCLUSIONS

The transition between different SPI with the same axial wave number $k = 3.927$ is shown to be mediated by secondarily bifurcating MCS. These, here discussed MCS exhibit a *direct* way for a transition without involving any other primary bifurcating solutions. Generally, the occurrence of MCS, i.e. of transitions between SPI of different azimuthal wave

$$\textbf{L−SPI} \xleftrightarrow{\text{LR−MCS}} \textbf{MRIB} \xleftrightarrow{\text{RL−MCS}} \textbf{R−SPI}$$

numbers M seems to be a consequence of the periodic boundary conditions which, in contrast to rigid axial lids, allow for multi-stable SPI with different M and also for multi-stability of different MCS. Due to wave number selection effects in longer axial periodic systems, multi-stability regions seem to disappear mostly and it is an open question whether such regions then exist at all.

References

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