GENERAL TEMPORAL INSTABILITY CRITERIA FOR STABLY STRATIFIED INVISCID FLOW

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Summary According to this analysis of the Taylor-Goldstein equation theoretically, the stable stratification $N^2 \geq 0$ has a destabilization mechanism, and the flow instability is due to the competition of the kinetic energy with the potential energy, which is dominated by the total Froude number Fr_t. Globally, $Fr_t^2 \leq 1$ implies that the total kinetic energy is smaller than the total potential energy. So the potential energy might transfer to the kinetic energy after being disturbed, and the flow becomes unstable.

INTRODUCTION

The instability of the stably stratified shear flow is one of main problems in fluid dynamics, astrophysical fluid dynamics, oceanography, meteorology, etc. On the one hand, the shear instability is known as the instability of vorticity maximum, after a long way of investigations. It is recognized that the resonant waves with special velocity of the concentrated vortex interact with flow for the shear instability [1]. Other velocity profiles are stable in homogeneous fluid without stratification. On the other hand, Rayleigh (1883) proved out that buoyancy is a stabilizing effect in the statical case. Thus, it is naturally believed that the stable stratification do favor the stability, which finally results in the well known Miles-Howard theorem. According to this theorem, the flow is stable to perturbations when the Richardson number Ri (ratio of stratification to shear) exceeds a critical value $Ri_c = 1/4$ everywhere. In three-dimensional stratified flow, the corresponding criterion is $Ri_c = 1$.
However, the stabilization effect of buoyancy is an illusion [2]. There is a big gap between Rayleigh's criterion and Miles-Howard' criterion, Yih (1980) wondered "Miles' criterion for stability is not the nature generalization of Rayleigh's well-known sufficient condition for the stability of a homogeneous fluid in shear flow" [3]. Recently, Friedlander doubled "the Richardson number criterion is not, in general, a necessary and sufficient condition" [4]. The mystery of the instability is still cover for us.
Following the frame work of [1], this study is an attempt to clear the confusion in theories. We find that the flow instability is due to the competition of the kinetic energy with the potential energy, which is dominated by the total Froude number Fr. And the unexpected assumption in Miles-Howard theorem leads the contradiction to other theories.

STABILITY CRITERION

The Taylor-Goldstein equation for the stratified inviscid flow is employed, which is the vorticity equation of the disturbance. Considering the flow with velocity profile $U(y)$ and the density field $\rho(y)$, and the corresponding stability parameter N (the Brunt-Vaisala frequency),

$$N^2 = -g\rho'/\rho, \tag{1}$$

where g is the acceleration of gravity, the single prime $'$ denotes d/dy, and $N^2 > 0$ denotes a stable stratification. The vorticity is conserved along pathlines. The streamfunction perturbation ϕ satisfies

$$\phi'' + [\frac{N^2}{(U-c)^2} - \frac{U''}{U-c} - k^2]\phi = 0, \tag{2}$$

where k is the real wavenumber and $c = c_r + ic_i$ is the complex phase speed and double prime $''$ denotes d^2/dy^2. For k is real, the problem is called temporal stability problem. The real part of complex phase speed c_r is the wave phase speed, and $\omega_i = kc_i$ is the growth rate of the wave. This equation is subject to homogeneous boundary conditions

$$\phi = 0 \ at \ y = a, b. \tag{3}$$

It is obvious that the criterion for stability is $\omega_i = 0$ ($c_i = 0$), for that the complex conjugate quantities ϕ^* and c^* are also physical solutions of Eq.(**??**) and Eq.(**??**).
Multiplying Eq.(**??**) by the complex conjugate ϕ^* and integrating over the domain $a \leq y \leq b$, we get the following equations

$$\int_a^b [|\phi'|^2 + k^2|\phi|^2 + \frac{U''(U-c_r)}{|U-c|^2}|\phi|^2]\,dy = \int_a^b \frac{(U-c_r)^2 - c_i^2}{|U-c|^4} N^2|\phi|^2\,dy. \tag{4}$$

and

$$c_i \int_a^b [\frac{U''}{|U-c|^2} - \frac{2(U-c_r)N^2}{|U-c|^4}]|\phi|^2\,dy = 0. \tag{5}$$

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As a first step in our investigation, we need to estimate the ratio of $\int_a^b |\phi'|^2 dy$ to $\int_a^b |\phi|^2 dy$. This is known as the Poincaré's problem:

$$\int_a^b |\phi'|^2 dy = \mu \int_a^b |\phi|^2 dy, \qquad (6)$$

where the eigenvalue μ is positively definite for any $\phi \neq 0$. The smallest eigenvalue value, namely μ_1, can be estimated as $\mu_1 > (\frac{\pi}{b-a})^2$ [1].

In departure from previous investigations, we shall investigate the stability of the flow by using Eq.(**??**) and Eq.(**??**). As μ is estimated with boundary so the criterion is global. We will also adapt a different methodology. If the velocity profile is unstable ($c_i \neq 0$), then the equations with the hypothesis of $c_i = 0$ should result in contradictions in some cases. Following this, a sufficient condition for instability can be obtained.

Firstly, substituting Eq.(**??**) into Eq.(**??**), we have

$$c_i^2 \int_a^b \frac{g(y)}{|U-c|^2} |\phi|^2 \, dy = -\int_a^b \frac{h(y)}{|U-c|^2} |\phi|^2 \, dy, \qquad (7)$$

where

$$\begin{aligned} g(y) &= \mu + k^2 + \tfrac{2N^2}{|U-c|^2}, \ and \\ h(y) &= (\mu + k^2)(U-c_r)^2 + U''(U-c_r) - N^2. \end{aligned} \qquad (8)$$

It is noted that $g(y) > 0$ for $N^2 \geq 0$. Then $c_i^2 > 0$ if $h(y) \leq 0$ throughout the domain $a \leq y \leq b$ for a proper c_r and k. Obviously, $h(y)$ is a monotone function of k: the smaller k is, the smaller $h(y)$ is. When $k = 0$, $h(y)$ has the smallest value.

$$h(y) = N^2 [\frac{(U-c_r)^2}{N^2/\mu} + \frac{U''(U-c_r)}{N^2} - 1]. \qquad (9)$$

If we define the total, shear and Rossby Froude numbers Fr_t, Fr_s and Fr_r as

$$Fr_t^2 = Fr_s^2 + Fr_r^2, Fr_s^2 = \frac{(U-c_r)^2}{N^2/\mu}, Fr_r^2 = \frac{U''(U-c_r)}{N^2}, \qquad (10)$$

where the shear Froude number Fr_s is a dimensionless ratio of kinetic energy to potential energy. As U'' plays the same role of β effect in the Rossby wave [1,5], the Rossby Froude number Fr_r is a dimensionless ratio of Rossby wave kinetic energy to potential energy. Then $h(y) \leq 0$ equals to $Fr_t^2 \leq 1$. Thus a general theorem for instability can be obtained from the above notations.

Theorem 1: If velocity U and stable stratification N^2 satisfy $h(y) \leq 0$ or $Fr_t^2 \leq 1$ throughout the domain for a certain c_r, the flow is temporally unstable with a $c_i > 0$.

Physically, $Fr_t^2 \leq 1$ implies that the total kinetic energy is smaller than the total potential energy. So the potential energy might transfer to the kinetic energy after being disturbed, and the flow becomes unstable. On the other hand, when the potential energy is smaller than the kinetic energy ($Fr_t^2 > 1$), the flow is stable because no potential energy could transfer to the kinetic energy.

CONCLUSIONS

In summary, the stably stratification is a destabilization mechanism, and the flow instability is due to the competition of the kinetic energy with the potential energy. Globally, the flow is always unstable when the total Froude number $Fr^2 \leq 1$, where the larger potential energy might transfer to the kinetic energy after being disturbed. Locally, the flow is unstable as the gradient Richardson number $Ri > 1/4$. The approach is very straightforward and can be used for similar analysis. In the inviscid stratified flow, the unstable perturbation must be long-wave scale. This result extends the Rayleigh's, Fjørtoft's, Sun's and Arnol'd's criteria for the inviscid homogenous fluid, but contradicts the well-known Miles and Howard theorems. It is argued here that the transform $F = \phi/(U-c)^n$ is not suitable for temporal stability problem, and that it will leads to contradictions with the results derived from Taylor-Goldstein equation [5].

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