EVOLUTION OF ROTATING CHARGED DROPS SUBJECT TO ELECTRIC FIELDS

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Summary In this work we study with the Boundary Element Method the evolution of a viscous and conducting fluid drop that rotates about a fixed axis at constant angular momentum L. This drop, which is surrounded by a dielectric viscous fluid, is also subject to a uniform electric field parallel to the axis of rotation. We are interested in the stability/instability of equilibrium shapes and the formation of singularities. We also develop a theoretical framework to approximate axisymmetric equilibrium solutions.

MATHEMATICAL MODEL

We study the evolution of a viscous conducting fluid drop immersed in a dielectric fluid. This drop holds an amount of charge Q, rotates about an axis with constant angular momentum L and is subject to a uniform electric field with magnitude $\mathcal{E}_\infty$ parallel to its axis of rotation. Our model can be described under low Reynolds and large Ekman number approximations by the Stokes system:

$$\begin{cases} \mu^{(i)}\Delta \mathbf{u}^{(i)} - \nabla p^{(i)} = 0 & , \quad \text{in } \mathcal{D}_i(t) \\ \nabla \cdot \mathbf{u}^{(i)} = 0 & , \quad \text{in } \mathcal{D}_i(t) \end{cases} ,$$

where $i = 1$ represents the fluid drop and $i = 2$ the surrounding fluid. $\mathcal{D}_i(t)$ is the region occupied by the fluid, $\mathbf{u}^{(i)}$ its velocity field, $p^{(i)}$ the pressure and $\mu^{(i)}$ the viscosity. To complete this system one has to consider the balance of stresses at the interface of both fluids:

$$\left(T^{(2)} - T^{(1)}\right)\mathbf{n} = \left(2\gamma\mathcal{H} - \frac{L^2}{2\mathcal{I}^2}\rho^2 - \frac{\sigma^2}{2\epsilon_0}\right)\mathbf{n} \quad , \quad \text{on } \partial\mathcal{D}(t) \ ,$$

where $\mathbf{n}$ is the outward unit normal to $\mathcal{D}_1(t)$, $\mathcal{I}$ the moment of inertia, γ the surface tension, $\mathcal{H}$ the mean curvature at the interface, ϵ_0 the vacuum permittivity and ρ the orthogonal distance from a point at the surface of the drop to the axis of rotation. Since the drop is a conductor, the electric potential $\mathcal{V}$ is constant inside and at the drop surface and all the electric charge is located on the boundary. The electric potential outside the drop verifies:

$$\begin{cases} \Delta\mathcal{V} = 0 & , \quad \text{in } \mathcal{D}_2(t) \\ \mathcal{V} = \mathcal{V}_0 & , \quad \text{in } \partial\mathcal{D}(t) \\ \mathcal{V} \to -\mathcal{E}_\infty z + O(|\mathbf{r}|^{-1}) & , \quad \text{as } \mathbf{r} \to \infty \end{cases} ,$$

and the value $\mathcal{V}_0$ has to be chosen so that the total charge is Q.

EQUILIBRIUM SOLUTIONS

Rotating charged drops

In the absence of an electric field we present an approximation formula for the equilibrium shapes of a conducting drop with charge Q that rotates with constant angular momentum L. We compare this approximate solution with an ellipsoid model, and also, with the numerical results obtained when considering the evolution problem with axial symmetry (see Fig. 1). The equilibrium configurations must verify:

$$\delta p = 2\gamma\mathcal{H} - \frac{L^2}{2\mathcal{I}^2}\rho^2 - \frac{\sigma^2}{2\epsilon_0} \quad , \quad \text{on } \partial\mathcal{D} \ , \tag{1}$$

where $\mathcal{D}$ is the region enclosed by the drop and δp is the pressure difference at the axis of rotation across the drop surface. This equation is the Euler-Lagrange equation of the energy functional:

$$\mathcal{E} = \gamma A + \frac{L^2}{2\mathcal{I}} + \frac{Q^2}{2\mathcal{C}} - \delta p\,(V - 1) \ ,$$

in which $\delta p\,(V-1)$ is the constraint (δp is a Lagrange multiplier) so the volume of the drop is $V = 1$, A is the drop's surface area and $\mathcal{C}$ is the capacitance of the drop, defined as:

$$\mathcal{C} = -\frac{\epsilon_0}{\mathcal{V}_0}\int_{\partial\mathcal{D}} \frac{\partial\mathcal{V}}{\partial\mathbf{n}}\,dS \ .$$

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It is known that for $L = 0$ the sphere is a solution of (1) (see [1]), so one could search for solutions as expansions in powers of L^2 about the sphere when $L \ll 1$. This yields the formula:

$$r(\theta,\varphi) = R + \lambda_1 Y_{2,0}(\theta,\varphi) L^2 + (\lambda_2 Y_{2,0}(\theta,\varphi) + \lambda_3 Y_{4,0}(\theta,\varphi)) L^4 + O\left(L^6\right) \ ,$$

where $R = \sqrt[3]{\frac{3}{4\pi}}$ is the radius of the sphere with volume 1, $Y_{2,0}$ and $Y_{4,0}$ are spherical harmonics and:

$$\lambda_1 = \frac{5\sqrt{5\pi}}{24\gamma(\chi-1)} \quad , \quad \lambda_2 = -\frac{6\lambda_1^2}{\sqrt{5\pi}}\sqrt[3]{\frac{\pi}{12}}\left(\frac{1}{2} + \frac{40\gamma}{7\sqrt{5\pi}}\lambda_1\right) \quad , \quad \lambda_3 = \frac{125\sqrt{\pi}}{112\gamma^2}\sqrt[3]{\frac{\pi}{48}}\frac{1}{(\chi-1)(2\chi-3)} \ ,$$

with $\chi = \frac{Q^2}{64\pi^2\epsilon_0\gamma R^3}$ the Rayleigh fissibility ratio.

We also introduce an ellipsoid model in which the equilibrium configurations are approximated by oblate or prolate spheroids. In this case, we have explicit formulas for the surface area and moment of inertia, and an approximation for the capacitance of the drop [2]. We can thus write the energy of the system in terms of the semi-major axis of the spheroid and find the enegy minimum given a charge Q and an angular momentum L.

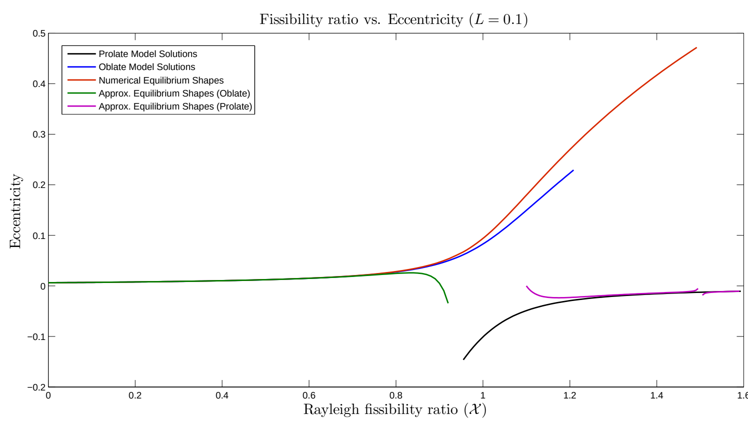

Figure 1. Comparison of the different equilibrium solutions obtained for the case $L = 0.1$.

Rotating drops subject to a uniform electric field

When a conducting and uncharged viscous drop is subject to an electric field it elongates in the direction of the field. If we also consider rotation about an axis parallel to this electric field with constant angular velocity, there exists a linear relationship between the electric field magnitude and the angular velocity which yield spherical equilibrium configurations [3]. We have validated this theoretical result with our numerical model when the angular momentum remains constant.

STABILITY ANALYSIS (3D SIMULATION)

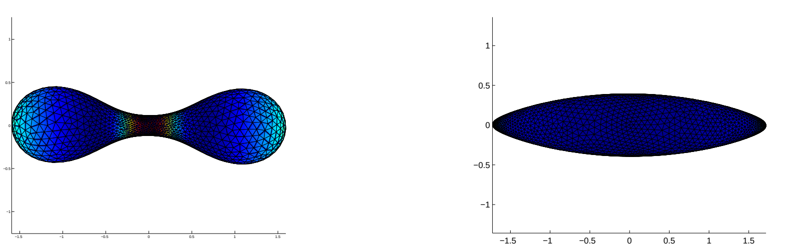

Figure 2. Left: Neck formation (and breakup) for $L = 0.4$ and $Q = 11$. Right: Tip formation for $L = 0.2$ and $Q = 12.5$. Both cases were obtained when evolving an asymmetric perturbation of the corresponding axisymmetric equilibrium configuration.

References

[1] Fontelos M. A., Friedman A.: Symmetry-breaking bifurcations of charged drops. *Archive for Rational Mechanics and Analysis*. **172** (2):267–294, 2004.

[2] Pólya G., Szegő G.: Isoperimetric inequalities in mathematical physics. *Annals of mathematics studies*. Princeton University Press, 1951.

[3] Sozou C.: A rotating spherical liquid drop in an electric field. *J. Fluid Mech.* **56**:305–312, 1972.