DEVELOPMENT OF THE FULL LAGRANGIAN APPROACH TO STUDY DIFFERENTIAL CHARACTERISTICS OF PASSIVE ADMIXTURE

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Summary The full Lagrangian approach [1] is developed to study the evolution and singularities of various differential characteristics of a passive admixture in different gas and fluid flows. As a passive admixture, one can regard any physical quantity which is transported by the flow and does not affect the flow parameters. Diffusion is not taken into account. Using the approach proposed, it is possible to calculate the differential characteristics of the passive admixture with respect to Eulerian coordinate from the solution of ordinary differential equations along chosen Lagrangian trajectories.

Let there be a passive admixture, characterized by its density or other parameter ρ_s, transported with the velocity $\mathbf{v}_s$, which does not affect the carrier-flow velocity field $\mathbf{v}$. In a general case, e.g. in disperse flows with passive inertial particles moving with a velocity slip, $\mathbf{v}_s \neq \mathbf{v}$. There is a wide class of flows in which a passive admixture is transported with the flow velocity $\mathbf{v}_s = \mathbf{v}$ (e.g. transport of density heterogeneity in rapid stratified flows). At the initial instant $t = 0$, the distribution $\rho_s(t = 0, \mathbf{r}_0) = \rho_{s0}(\mathbf{r}_0)$ is known at the points $\mathbf{r}_0 = (x_{10}, x_{20}, x_{30})$ of a region G. To calculate Lagrangian trajectories $\mathbf{r}_s(t, \mathbf{r}_0)$ along which ρ_s is transported, the kinematic equation is used:

$$\partial \mathbf{r}_s / \partial t = \mathbf{v}_s \,. \qquad (1)$$

When $\mathbf{v}_s = \mathbf{v}$ or $\mathbf{v}_s$ is a known function, Eq. (1) is sufficient to determine $\mathbf{r}_s(t, \mathbf{r}_0)$. For the case $\mathbf{v}_s \neq \mathbf{v}$, we present main ideas of the method proposed with the reference to an example of a dilute disperse flow with passive inertial particles moving due to the interphase force:

$$\partial \mathbf{v}_s / \partial t = \mathbf{f}_s \,. \qquad (2)$$

Here, $\mathbf{f}_s$ is a known function of the parameters of both phases, e.g. the Stokes drag $\mathbf{f}_s = \beta(\mathbf{v} - \mathbf{v}_s)$, in which β is a constant dependent of physical parameters of the phases. To calculate differential characteristics (gradient, divergence, curl, etc) of ρ_s with respect to Eulerian coordinates $\mathbf{r} = (x_1, x_2, x_3)$ as functions of the Lagrangian coordinates $\mathbf{r}_0$ along the paths of the passive admixture, we find the relation between the Hamilton operator “nabla” with respect to Eulerian coordinates $\nabla = (\partial/\partial x_i)$ and the similar operator with respect to Lagrangian coordinates $\nabla_0 = (\partial/\partial x_{i0})$:

$$\nabla \rho_s = J^{-1} \nabla_0 \rho_s \,, \quad \nabla \cdot \boldsymbol{\rho}_s = J^{-1} \nabla_0 \cdot \boldsymbol{\rho}_s \,, \quad \nabla \times \boldsymbol{\rho}_s = J^{-1} \nabla_0 \times \boldsymbol{\rho}_s \,. \qquad (3)$$

These relations are obtained using the chain rule. Here, in the first formula ρ_s is a scalar function and, in the others, $\boldsymbol{\rho}_s$ is a vector function. In Eqs. (3), $J = (J_{ij}) = (\partial x_i/\partial x_{j0})$ is the Jacobi matrix of the transform from the Eulerian coordinates to the Lagrangian coordinates. Equations for J_{ij} along the admixture trajectories are obtained by differentiation of Eqs. (1), (2) with respect to Lagrangian coordinates [1]:

$$\partial J_{ij} / \partial t = \Omega_{ij} \,, \;\; \partial \Omega_{ij} / \partial t = \partial f_{si} / \partial x_{0j} \,, \;\; i, j = 1,2,3 \,. \qquad (4)$$

The initial conditions for system (1), (2), (4) take the following form:

$$t = 0: \;\; x_{si} = x_{0i} \,, \;\; v_{si} = v_{0i} \,, \;\; J_{ii} = 1 \,, \;\; J_{ij}(i \neq j) = 0 \,, \;\; \Omega_{ij} = \partial v_i(0, \mathbf{r}_0) / \partial x_{0j} \,, \;\; i, j = 1,2,3 \,. \qquad (5)$$

In order to close system (1)–(5), additional equations for Lagrangian differential characteristics $\nabla_0\rho_s$, $\nabla_0 \cdot \boldsymbol{\rho}_s$, and $\nabla_0 \times \boldsymbol{\rho}_s$ from Eq. (3) are required. There is a wide class of flows in which the value of a certain quantity, transported by the flow, retains along the Lagrangian trajectory: $\rho_s(t, \mathbf{r}_0) \equiv \rho_{s0}$. As ρ_s one can take, for example, the density in fast flows of stratified incompressible fluids (Fig. 1). In this case, the additional equation for system (1)–(5) can be obtained by applying the operator ∇_0 to the continuity equation for this quantity. In the case of the density gradient (Fig. 1), it reads $\nabla_0\rho = \nabla_0\rho_0$. In general case, the value of ρ_s is not retained along the Lagrangian trajectory, but it satisfies a certain evolution equation, which provides the equations for differential characteristics with respect to Lagrangian coordinates. For example, in two-phase flows with inertial dispersed particles (Fig. 2) the following equations are used: for calculating the gradients of particle concentration ∇n_s and temperature ∇T_s, gradients $\nabla_0 n_s$ and $\nabla_0 T_s$ are found from the expressions $\nabla_0 n_s = \nabla_0(n_{s0}/\det J)$ and $\partial(\nabla_0 T_s)/\partial t = \theta(J\nabla T - \nabla_0 T_s)$, which are obtained from the equations of continuity $n_{s0} = n_s \det(J)$ and particle heat exchange $\partial T_s/\partial t = \theta(T - T_s)$. Here, T is a known temperature field of the fluid phase and θ is a constant, dependent of physical parameters of the phases. Similarly, to find the vorticity of the particle velocity $\boldsymbol{\omega}_s = \nabla \times \mathbf{v}_s$ one can use the following modification of Eq. (2): $\partial(\nabla_0 \times \mathbf{v}_s)/\partial t = \beta(J\nabla\times \mathbf{v} - \nabla_0 \times \mathbf{v}_s)$.

Thus, the proposed method for calculating differential characteristics of a passive admixture in Eulerian coordinates along Lagrangian trajectories consists in solving the system of ODE and finite relations (1)–(5), closed by deriving additional equations for $\nabla_0\rho_s$, $\nabla_0 \cdot \boldsymbol{\rho}_s$, or $\nabla_0 \times \boldsymbol{\rho}_s$, the latter being obtained from different evolution equations for ρ_s.

The method requires no modification of the algorithm when passing from steady to unsteady flows. It can also be successfully used for investigating flows with the formation of singularities in the differential characteristics. For

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example, in Fig. 1, a $\partial\rho/\partial y \to \infty$ past the body, and in Fig 2, a $\partial n_s/\partial y$, $\partial T_s/\partial y \to \infty$ when $t \to \infty$. Examples of the successful application of the method are illustrated by Figs. 1-2.

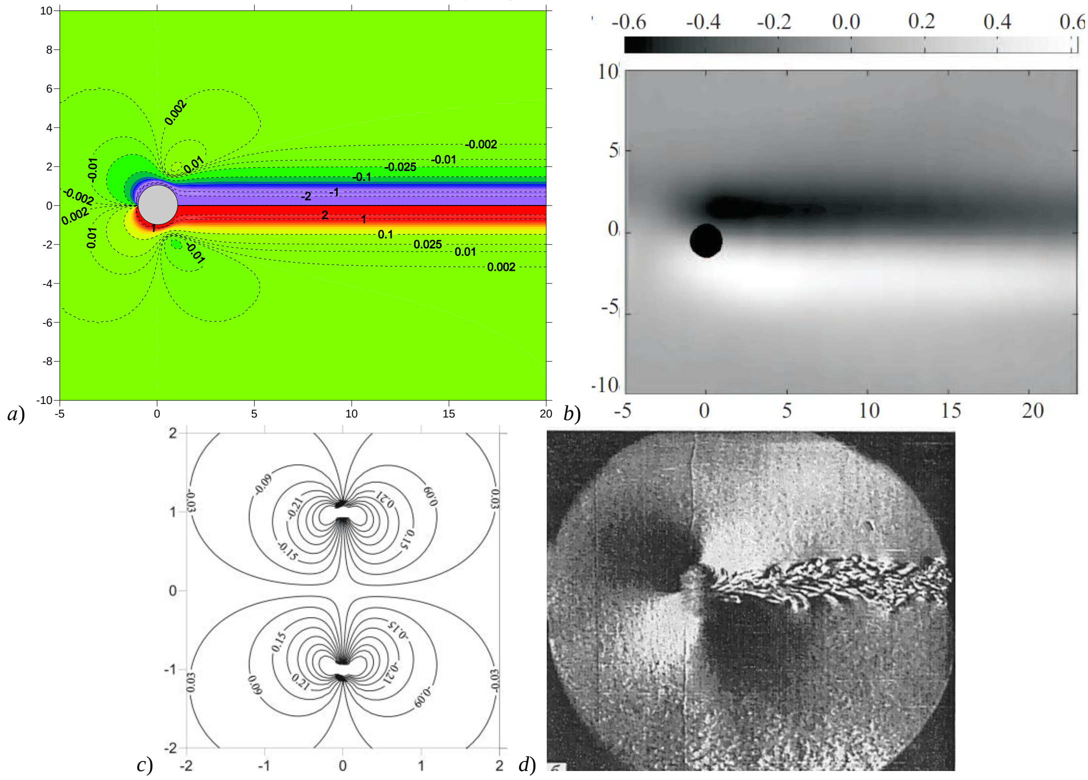

Fig. 1. Rapid stratified flows. Numerical (*a*, *c*) and experimental [2,3] (*b*, *d*) images of the density gradient components: (*a*, *b*) $\partial\rho/\partial y$ around a sphere in a stratified flow for $\nabla_0\rho_0 = (1; 0)$; (*c*, *d*) $\partial\rho/\partial x$ around a vortex ring for $\nabla_0\rho_0 = (0; 1)$.

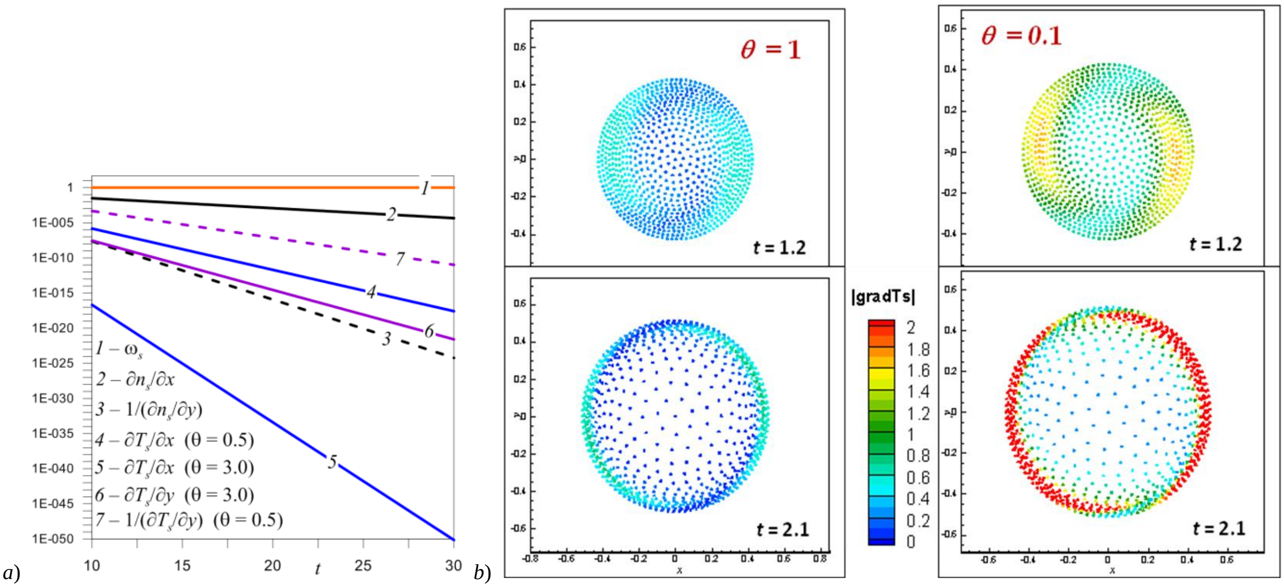

Fig. 2. Numerical calculation of differential characteristics of the particulate phase in two-phase dilute disperse flow with passive inertial particles ($\beta \sim 1$). (*a*) Particulate-phase parameters near a stagnation point in inviscid incompressible plane flow with the carrier phase velocity field $\mathbf{v} = (x + y, -y)$ and the boundary conditions far from the stagnation region: $\nabla_0 n_s = (1; 1)$, $\nabla_0 T_{s0} = (1; 1)$, $\omega_{s0} = 2$. (*b*) Two possible ways of evolution of the temperature gradient $|\nabla T_s|$ of the particle cloud placed in the core of the Lamb-Oseen vortex in viscous fluid at the initial instant $t = 1$ with $\nabla_0 T_{s0} = (1; 0)$.

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References

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