BUBBLE DYNAMICS IN A COMPRESSIBLE VISCOELASTIC MEDIUM

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Summary We theoretically and numerically study the dynamics of a spherical bubble in a compressible viscoelastic (tissue-like) medium. The damping and frequency of the bubble oscillations are predicted using perturbation analysis (linear and higher-order multiple scales). This analysis, combined with numerical results, is used to elucidate the underlying physics, namely how the viscoelastic properties and compressibility affect the bubble response.

INTRODUCTION

Cavitation plays an important role in a wide range of biomedical applications, such as diagnostic and therapeutic ultrasound, drug delivery, and traumatic brain injury. In these problems, cavitation usually takes place in biological liquids or soft tissue, both of which exhibit different levels of viscoelasticity. The study of cavitation in soft tissue is challenging due to the complex physics associated with viscoelasticity, multiphase and compressible flow. Insights into cavitation can be gained through a reduced-order model that describes the response of a single spherical bubble in an infinite medium subjected to a far-field pressure change. The few past studies of spherical bubble dynamics in viscoelastic media are phenomenological studies that have primarily focused on polymeric liquids governed by a Maxwell model. In the present work, analytical and numerical approaches are followed to predict the frequency and damping of bubbles oscillating in a viscoelastic medium, and explain the underlying physics. The canonical Rayleigh collapse problem, in which the collapse is driven by an instantaneous increase in the pressure of the surrounding medium [1], is used to study the bubble dynamics.

MODEL EQUATIONS AND APPROACH

The Keller-Miksis equation [2] is used to study the dynamics of an air bubble in a compressible material. The surrounding viscoelastic medium is assumed to behave linearly and have some viscosity (Reynolds number Re), elasticity (Cauchy number Ca) and relaxation time (Deborah number De). The dimensionless equations of motion are:

$$\left(1-\frac{\dot{R}}{C}\right)R\ddot{R}+\frac{3}{2}\left(1-\frac{1}{3}\frac{\dot{R}}{C}\right)\dot{R}^2=\left(1+\frac{\dot{R}}{C}\right)\left(\frac{p_{Go}}{R^{3\gamma}}-\frac{2}{WeR}+3\varsigma-1-p_R\right)-\frac{\dot{R}}{C}\left(\frac{3\gamma p_{Go}}{R^{3\gamma}}-\frac{2}{WeR}\right)+3\frac{R}{C}\dot{\varsigma}, \quad (1)$$

$$De\dot{\tau}_{rr}|_R+\tau_{rr}|_R=-\frac{4}{3Ca}\left(1-\frac{1}{R^3}\right)-\frac{4}{Re}\frac{\dot{R}}{R}, \qquad De\dot{\varsigma}+\varsigma+De\frac{\dot{R}}{R}\tau_{rr}|_R=\frac{1}{3}\left[-\frac{4}{3Ca}\left(1-\frac{1}{R^3}\right)-\frac{4}{Re}\frac{\dot{R}}{R}\right], \quad (2)$$

with radius $R(t)$, dimensionless sound speed C, initial pressure of the non-condensible gas $p_{Go}=1+2/We$, Weber number We, step increase in pressure p_R and specific heats ratio γ. The dot denotes time derivative and $\varsigma=\int_R^\infty \tau_{rr}(r,t)dr/r$. The non-dimensionalization is performed using the bubble initial radius (5 μm), the density of tissue and a characteristic speed defined as the square root of ambient pressure divided by density. The viscoelastic medium is essentially a standard linear solid; one recovers the Voigt model for $De=0$ and the Maxwell model for $Ca=0$. Two approaches are followed to solve these equations. Analytical solutions are obtained using a linear analysis for the Voigt model and a higher-order multiple scales method [3], though the derivations are omitted for conciseness. The numerical solution, obtained with a fifth-order accurate Runge-Kutta method with adaptive time-step control, is considered exact.

RESULTS

First, a linear perturbation analysis for the radius is performed for the Voigt model. In this case, a single differential equation is solved. For zeroth order, the expression for the final radius is recovered, *i.e.*, the expansion is performed about the final equilibrium radius, which is a nonlinear function of Ca. The first-order equation takes the form of a simple harmonic oscillator with a damping constant that consists of a viscous and a compressible components, and a natural frequency that includes an additional elastic term. A comparison of this analytical solution with the numerical solution is shown in Fig. 1(a). The analytical solution agrees well with the exact solution. Although it may be expected that increasing the elasticity makes the surroundings stiffer and thus increases the damping, this exact behavior does not occur. Instead, the damping decreases (middle result in Fig. 1(a)), before increasing again. This behavior may be explained by considering the damping coefficient in Fig. 1(b): the analysis predicts that the damping consists of a sum of compressible and viscous components, which is confirmed by the numerical solution. It is clear that the total damping at small Ca is

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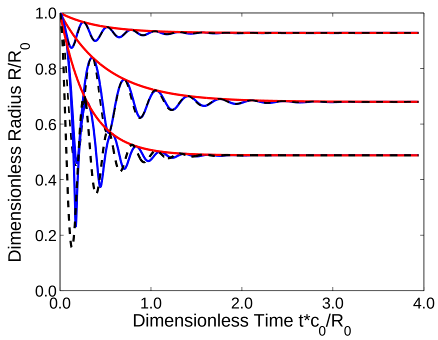
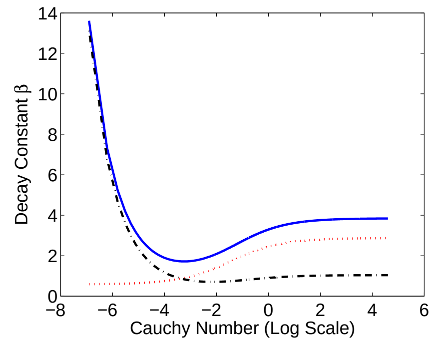
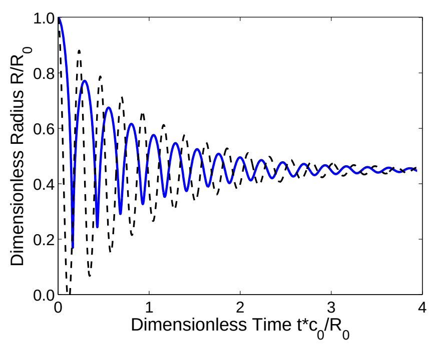


(a) History of the bubble radius for different elasticities using the Kelvin-Voigt model ($G = 10, 1, 0.1$, MPa from top to bottom). Solid lines: numerical results; dashed lines: analytical results.

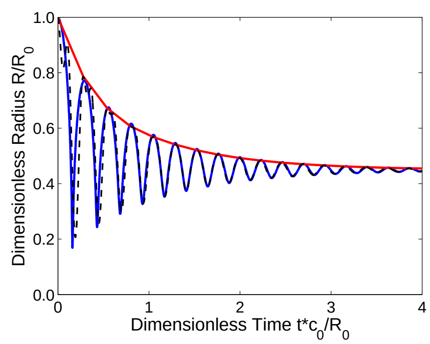


(b) Damping coefficient vs. Ca. Solid line: full damping coefficient; dash-dotted line: compressible damping only; dotted line: viscous damping only.

Figure 1. Results for the Kelvin-Voigt Model.



(a) Linear analysis.



(b) Multiple scales analysis.

Figure 2. History of the bubble radius for the Maxwell model ($De = 1$). Thick solid line: numerical solution; dashed line: multiple scales result; thin solid line: analytical decay curve.

compressibility-dominated, while at larger Cauchy number it is viscosity-dominated. However, because the compressible component decreases earlier (as a function of Ca) than the increase in the viscous component, there is a minimum in the damping. Soft tissue elasticities reported in the literature fall within this range.

For the Maxwell model, a linear analysis fails, as shown in Fig. 2(a): both the damping and the oscillation frequency are incorrect. To accurately represent the nonlinearities of the system of three coupled differential equations, the method of multiple scales is used. The radius is expanded to third order with terms that are functions of different time scales. Results from this analysis are shown in Fig. 2(b). Relaxation effects enter the problem purely through the viscous damping; the final radius does not depend on De. In addition, the natural frequency consists of a constant first-order term, a constant high-order term and a time-dependent high-order term that decays exponentially with time. These higher-order effects, which are not negligible, are well captured by the analytical results.

CONCLUSIONS

A theoretical model is developed to study the dynamics of a spherical gas bubble in a tissue-like medium. The viscoelastic properties and models strongly affect the bubble dynamics, *i.e.*, damping and frequency. A linear perturbation analysis represents the behavior in a Kelvin-Voigt medium well, while a higher-order multiple scales analysis is required for the more nonlinear Maxwell model. The analysis is currently being applied to the standard linear solid model, and departures from sphericity are being investigated; these results will be discussed at the meeting.

References

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