THE CONCEPT OF SELF-CONSISTENT FIELD APPLIED TO THE INVISCID SUSPENSIONS

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The concept of self-consistent field, well known in physics, is applied to the problem of hydrodynamic interaction of spherical particles in inviscid potential fluid-flows. The classic boundary-value problem is reduced to the determination of the tensor coefficients in the resulting exact solution for N particles in a liquid, based on an infinite linear algebraic system of equations. The procedure of the approximate solving of this system is constructed for the case of dilute suspensions. The potential of the developed method is illustrated by solving some typical problems.

The boundary-value problem for N particles in a liquid is essentially a many-body problem, which has not yet been solved exactly even in classical mechanics. This leads to the construction of various approximate methods. One of the first proposed models was the so-called cell model, which essentially reduces the problem of many bodies to the motion of a single particle inside the liquid cell. This model was used by Zuber [1] to determine the virtual mass of gas bubbles that form an inviscid infinite homogeneous suspension. Subsequently, Zuber's result was specified in the works of Wijngaarden [2] and Felderhof [3], based on more rigorous models that take into account the hydrodynamic interaction between bubbles in suspension.

In [4] there was developed a new method for solving the many-body problem of spherical shape in an ideal fluid based on the concept of self-consistent field. This method has allowed to obtain a number of generalizations of known results [5-8] for the particle dynamics in the inviscid suspensions. In addition, it has been effective for the solution of a class of problems that not previously considered in the frame of other methods because of the mathematical problems of fundamental nature.

As shown in [4], the classical boundary problem for N spherical particles in an ideal incompressible fluid can be reduced to solving an infinite algebraic system of equations for the tensor coefficients in the exact solution for the flow potential obtained for the initial boundary-value problem. In dimensionless variables, it has the form:

$$\varphi = \varphi_0 + \sum_{i=1}^{N}\sum_{n=0}^{\infty}\frac{n\,\alpha^{2n+1}}{n+1}C^{(i)}_{\gamma_1\cdots\gamma_n}\frac{X^{(i)}_{\gamma_1}X^{(i)}_{\gamma_2}\cdots X^{(i)}_{\gamma_n}}{R_i^{2n+1}},\quad X^{(i)}_{\gamma} = x_\gamma - x^{(i)}_\gamma,\quad R_i = \sqrt{\left(X^{(i)}_\gamma\right)^2} \tag{1.1}$$

$$C^{(i)}_{\gamma_1\cdots\gamma_n} = \frac{1}{n!}\frac{\partial^n}{\partial x_{\gamma_1}\partial x_{\gamma_2}\dots\partial x_{\gamma_n}}\left\{\varphi_0 - X^{(i)}_q U^{(i)}_q\right\}\Big|_{R_i=0} + \sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{k=0}^{\infty}\frac{k\,\alpha^{2k+1}}{k+1}C^{(j)}_{\beta_1\dots\beta_k}Q^{(j,i)}_{\beta_1\dots\beta_k,\gamma_1\cdots\gamma_n} \tag{1.2}$$

$$Q^{(j,i)}_{\beta_1\dots\beta_k,\gamma_1\cdots\gamma_n} = \frac{1}{n!}\frac{\partial^n}{\partial x_{\gamma_1}\partial x_{\gamma_2}\dots\partial x_{\gamma_n}}\left\{\frac{X^{(j)}_{\beta_1}X^{(j)}_{\beta_2}\cdots X^{(j)}_{\beta_k}}{R_j^{2n+1}}\right\}\Bigg|_{R_i=0},\quad \alpha = \frac{a}{L},$$

where φ_0 is the potential of the external flow, a the radius of spherical particles, $x^{(i)}_\gamma$ the radius-vector of the centre of the particle with index $i = 1,\dots,N$. As the scale values we take some characteristic velocity U_0 and the characteristic distance between centers of adjacent particles L.

Formula (1.1) is an exact solution for the required flow potential, provided that the tensor coefficients $C^{(i)}_{\gamma_1\cdots\gamma_n}$ satisfy an infinite linear algebraic system of equations (1.2). In general case, when the particle velocities are not specified, the system (1.2) is not closed and must be supplemented with equations of motion of particles. In our case it's convenient to use the appropriate equations in the form of van Beek [9]:

$$(\lambda-1)\frac{dU^{(i)}_\beta}{dt} = \frac{3}{2}\frac{\partial C^{(i)}_\beta}{\partial t} + 3\left(C^{(i)}_\gamma + U^{(i)}_\gamma\right)C^{(i)}_{\gamma\beta} + 3\sum_{n=2}^{\infty}\frac{n\,n!\,\alpha^{2n-2}}{(2n-1)!!}C^{(i)}_{\gamma_1\cdots\gamma_n}C^{(i)}_{\gamma_1\cdots\gamma_n\beta},\qquad \lambda = \frac{\rho_p}{\rho}, \tag{1.3}$$

It's possible to find the approximate solution of the system (1.2) and (1.3) for dilute suspensions, when parameter $\alpha \ll 1$. In this case, we can find a solution for the unknown functions in the form of expansions in powers of this parameter. Substitution of these representations into (1.2) and (1.3) leads to a set of recurrent relations, where each subsequent approximation on the small parameter α for all functions is determined through the previous one. Thus, knowing the zeroth approximation, it is possible to determine all functions in an arbitrary approximation in the small parameter. Application of the averaging over the ensemble [10,11] to the resulting solution makes it possible to obtain the dynamic characteristics of the average for large systems of dispersed particles.

The developed method was applied to a set of characteristic problems in dynamics of particles during their joint motion in ideal-fluid flow. In [5] there was considered the problem on the relative motion of phases and on the virtual mass of identical spherical particles of an arbitrary mass immersed in a fluid occupying the semi-infinite space bounded by a

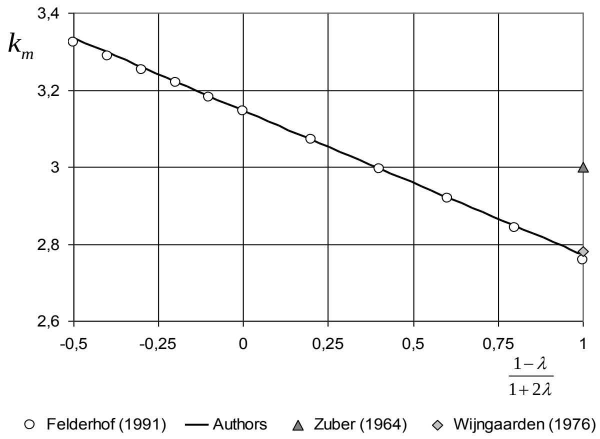

Fig. 1. Analytical dependence k_m on parameter $\frac{1-\lambda}{1+2\lambda}$ for identical spherical particles in inviscid suspension.

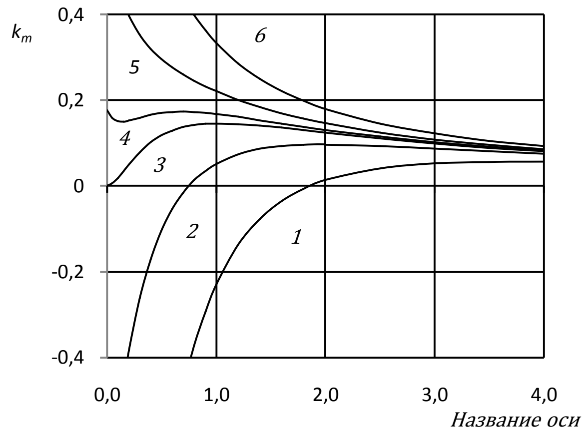

Fig. 2. Plots of the dependence k_m on σ in the formula for the virtual mass of the spherical body for various values of the parameter λ: (*1*) 0, (*2*) 0.5, (*3*) 1, (*4*) 1.2, (*5*) 2.0, and (*6*) ∞.

plane infinite wall, when it impulsively set from the state of rest into the state of motion with specified velocity U. The use of the developed method for velocity u and virtual mass m of particles infinitely removed from the wall to within the first degree on the volume concentration c of the dispersed phase leads to the result of the form

$$u = \frac{3U}{1+2\lambda}(1 + k(\lambda)\cdot c), \quad \frac{m}{\rho\tau} = \frac{1}{2}(1 + k_m(\lambda)\cdot c) \tag{1.4}$$

where τ is the volume of particle. Functions $k(\lambda)$ and $k_m(\lambda)$ have the simple analytical form [5]. The dependence $k_m(\lambda)$ is presented in figure 1 in comparison with the known results of different authors.

In [6], on the basis of the method of self-consistent fields [4], there was considered the problem on the virtual mass of a spherical body moving with a specified acceleration in a mono-dispersed suspension of spherical particles for an arbitrary ratio σ between the sizes of particles and the body. The result for virtual mass of spherical body to within the first degree on the volume concentration of the dispersed phase has the form (1.4) but function k_m depends on two variables – on the ratio σ between the sizes of particles and the body and the ratio λ between the densities of dispersed particles and the body. Function $k_m(\sigma, \lambda)$ has also the simple analytical form [6]. The plots of this function are presented in figure 2.

In [8] the developed method is applied to the problem of virtual mass of a cluster consisting of a large number of spherical particles hard fixed to each other. For cluster of spherical shape, which we can consider as a model of porous sphere, the formula for its virtual mass has the form (1.4) where function k_m depends only on the ratio between the radii of porous sphere and particles forming the porous sphere and tends to constant value as the ratio tends to infinity [8].

As shown in [5-8] the method of self-consistent fields [4] can be used to study a wide class of problems in the dynamics of inviscid dilute suspensions and can be developed on the case of more concentrated one.

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