HEAT TRANSFER IN DEVELOPED COMPRESSIBLE CONVECTION

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Summary We study the influence of compressibility (density stratification) on fully developed nonlinear convection. The global energy balance and the dynamics of the thermal boundary layers is throughly investigated. The scaling laws of the Nusselt and top and bottom Reynolds numbers with the Rayleigh (and Prandtl) numbers are obtained. It is reported, that the compressibility typically tends to decrease the overadiabatic heat flux in the system and that the dynamics of the top and bottom boundary layers differs significantly. The top boundary layer is typically thicker and the entropy jump across it is larger than in case of the bottom boundary layer and the viscous heating at the top layer is stronger. The top boundary layer is also more prone to instability than the bottom one, since the local Reynolds number based on the boundary layer thickness and the convective velocity is larger in the former case.

INTRODUCTION

The problem of influence of stratification on developed convection is of vital importance from the astrophysical point of view. The giant planets and the stars are typically strongly stratified and the anelastic approximation is commonly used to describe convection in their interiors.

The properties of fully developed compressible convection substantially differ from the Boussinesq case the main two reasons for it being that the thermal and kinetic energies are comparable in the compressible case which means the work of the buoyancy force and the viscous heating are non-negligible compared to the total heat flux in the system and that the convection tends to be most vigorous in the region where the density is smallest, i.e. near the top. We study here the influence of stratification on the Rayleigh (and Prandtl) number dependence of the Nusselt number (the measure of the total convective heat flux through the system) and top and bottom Reynolds numbers in anelastic convection. The study is based on the theory of incompressible convection developed by [1].

FULLY DEVELOPED COMPRESSIBLE CONVECTION UNDER ANELASTIC APPROXIMATION

Consider a layer of compressible fluid between two parallel plates, of thickness d, periodic in horizontal directions, the evolution of which is described by the set of the Navier-Stokes, mass conservation and energy equations under the anelastic approximation,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla\left(\frac{p}{\bar{\varrho}}\right) + \frac{g}{c_p}s\hat{\mathbf{e}}_z + \frac{\mu}{\bar{\rho}}\left[\nabla^2\mathbf{u} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{u})\right], \tag{1}$$

$$\nabla\cdot(\bar{\varrho}\mathbf{u}) = 0, \tag{2}$$

$$\bar{\rho}\bar{T}\left[\frac{\partial s}{\partial t} + \mathbf{u}\cdot\nabla(\bar{s}+s)\right] = k\nabla^2 T + \mu\left[q + \partial_j u_i \partial_i u_j - (\nabla\cdot\mathbf{u})^2\right], \tag{3}$$

$$\frac{p}{\bar{p}} = \frac{\rho}{\bar{\rho}} + \frac{T}{\bar{T}}, \qquad s = c_v\frac{p}{\bar{p}} - c_p\frac{\rho}{\bar{\rho}}, \tag{4}$$

where

$$q = \nabla\mathbf{u}:\nabla\mathbf{u} + \frac{1}{3}(\nabla\cdot\mathbf{u})^2, \tag{5}$$

and $\nabla\mathbf{u}:\nabla\mathbf{u} = \partial_j u_i \partial_j u_i$. The dynamic viscosity $\mu = \bar{\rho}\nu$ and the thermal conductivity k were assumed constant. The basic (static: $\mathrm{d}\bar{p}/\mathrm{d}z = -g\bar{\rho}$, $\nabla^2\bar{T} = 0$) state is

$$\bar{T} = T_B\left(1-\theta\frac{z}{d}\right), \quad \bar{\rho} = \rho_B\left(1-\theta\frac{z}{d}\right)^m, \quad \bar{p} = \frac{gd\rho_B}{\theta(m+1)}\left(1+\theta\frac{z}{d}\right)^{m+1}, \quad \bar{p} = \bar{\rho}R\bar{T}, \quad \bar{s} = c_v\ln\frac{\bar{p}}{\bar{\rho}^\gamma}, \tag{6}$$

$$m = \frac{gd}{R\triangle T} - 1 = \frac{1}{\gamma-1} - \frac{\epsilon}{\theta}\frac{\gamma}{\gamma-1}, \tag{7}$$

$$\bar{s} = c_p\frac{m+1-\gamma m}{\gamma}\ln\left(1-\theta\frac{z}{d}\right) + \mathrm{const}, \qquad \frac{m+1-\gamma m}{\gamma} = \frac{\epsilon}{\theta} = \mathrm{O}(\epsilon), \tag{8}$$

where $\theta = \frac{\triangle T}{T_B}$ is the magnitude of the basic temperature gradient and a measure of compressibility of the fluid, T_B is the temperature at the bottom of the layer, $\triangle T = T_B - T_T > 0$ is the temperature jump across the layer and $0 \leq \theta < 1$. The parameter

$$\epsilon = -\frac{d}{T_B}\left[\left(\frac{\mathrm{d}\bar{T}}{\mathrm{d}z}\right)_B + \frac{g}{c_p}\right] = -\frac{d}{c_p}\left(\frac{\mathrm{d}\bar{s}}{\mathrm{d}z}\right)_B \ll 1 \quad \Longrightarrow \quad \frac{\triangle T}{d} - \frac{g}{c_p} = \frac{T_B}{d}\epsilon, \tag{9}$$

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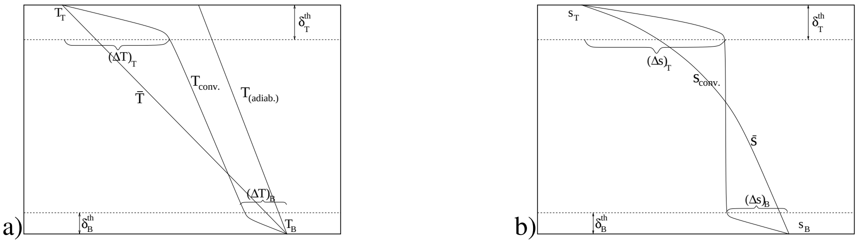

Figure 1. A schematic picture of the temperature profile a) and the entropy profile b) in developed convection.

measures the departure of the system from adiabaticity, which is assumed small and $\gamma = c_p/c_v$ is the specific heat ratio. The subscripts B and T indicate that the value is taken at the bottom or at the top of the layer respectively. We have the following physical picture in mind. The mixing in the bulk is expected to be very efficient in turbulent convection thus the bulk is assumed to be almost adiabatic and the overadiabaticity, which drives the flow, is accounted for by the temperature (entropy) jumps accross the top and bottom thermal boundary layers, i.e. $(\triangle T)_T$ and $(\triangle T)_B$ respectively (cf. Figure 1). Understanding the energy transfer and production in convective flow is the key to understanding the physics of compressible convection. Therefore we have derived the following relations which will allow us to study some general aspects of the dynamics of developed compressible convection.

$$\begin{aligned} F_{total}^{(overadiab.)} &\approx \left\langle \bar{\rho}\bar{T}u_z s\right\rangle_h - k\frac{\mathrm{d}}{\mathrm{d}z}\left(\bar{T} + \langle T\rangle_h - T_{adiab.}\right) + \frac{\triangle T}{d}\int_0^z \langle\bar{\rho}u_z s\rangle_h \,\mathrm{d}z - \mu\int_0^z \langle q\rangle \,\mathrm{d}z \\ &\approx T_B\langle\bar{\rho}u_z s\rangle_h + \frac{T_B}{\bar{T}}\left[-k\frac{\mathrm{d}}{\mathrm{d}z}\left(\bar{T} + \langle T\rangle_h - T_{adiab.}\right)\right] - \mu\int_0^z \frac{T_B}{\bar{T}}\langle q\rangle_h \,\mathrm{d}z. \end{aligned} \tag{10}$$

$$F_{total}^{(overadiab.)}\left(\frac{1}{T_T} - \frac{1}{T_B}\right) \approx \mu\int_0^d \frac{1}{\bar{T}}\langle q\rangle_h \,\mathrm{d}z = Q_2(d). \tag{11}$$

$$\frac{g}{c_p}\langle\bar{\rho}u_z s\rangle = \mu\langle q\rangle, \tag{12}$$

$$F_{total}^{(overadiab.)} = -k\frac{\mathrm{d}}{\mathrm{d}z}\left(\bar{T} + \langle T\rangle_h - T_{adiab.}\right)\Big|_{z=0} = -k\frac{\mathrm{d}}{\mathrm{d}z}\left(\bar{T} + \langle T\rangle_h - T_{adiab.}\right)\Big|_{z=d}. \tag{13}$$

$F_{total}^{(overadiab.)}$ is the total overadiabatic heat flux in the system and equation (12) states that the total work per unit volume of the buoyancy force averaged over the horizontal directions is equal to the total viscous dissipation in the fluid volume. Finally, we define the Nusselt number Nu, the Rayleigh number Ra and the top and bottom Reynolds numbers Re_T and Re_B in the following way,

$$Nu = \frac{F_{total}^{(overadiab.)}}{F_{basic\,state}^{(overadiab.)}} = \frac{(\triangle T)_B\, d}{\epsilon T_B \delta_B^{th}} = \frac{(\triangle T)_T\, d}{\epsilon T_B \delta_T^{th}}, \qquad Ra = \frac{g\triangle s d^3\rho_B^2}{\mu k} \approx \frac{c_p \triangle s\triangle T d^2\rho_B^2}{\mu k}, \qquad Re_i = \frac{U_i d\rho_i}{\mu}, \tag{14}$$

where (9) was used in deriving the second expression for the Rayleigh number in (14) and $i = B, T$ with U_T and U_B being the convective velocities at the top and bottom in the bulk. The total convective heat flux in (14) was estimated with $F_{total}^{(overadiab.)} \approx k(\triangle T)_i/\delta_i^{th}$ and the overadiabatic conductive heat flux in the basic state is $F_{cond} = -k\mathrm{d}\bar{T}/\mathrm{d}z - kg/c_p = \epsilon k T_B/d$. In the above $\triangle s = -c_p\epsilon\ln(1-\theta)/\theta$ is the total entropy jump across the fluid layer.

CONCLUSIONS

We have reported that in a stationary state the heat flux entering at the bottom is equal to the heat flux leaving the system at the top as in the Boussinesq case, but the work done by the buoyancy force and the viscous heating are no longer negligible and are of the same order as the total heat flux in the system, which means that the total heat flux going through every plane $z =$ const. is no longer the same. The heat flux entering at the bottom is typically increased by the bottom boundary layer viscous heating, then the work done by the buoyancy force reduces the flux in the bulk so that it goes below the value at the enter and then it is again boosted by the viscous heating in the top boundary layer to reach the same value at the top boundary as at the bottom boundary. Moreover, the results obtained suggest that increasing the stratification (compressibility) rather tends to decrease the heat flux going through the system.

Furthermore, we also reported that top boundary layer is thicker and has a larger entropy jump across it than the bottom layer and that the convective velocities are always larger close to the top than in the bottom region. These two facts mean that the top layer is much more likely to produce significant viscous heating and is more prone to instability.

References

[1] Grossmann S. & Lohse D.: Scaling in thermal convection: a unifying theory. *J. Fluid Mech* **407**:27–56, 2000.