ONSET OF THREE-DIMENSIONAL THERMAL CONVECTION IN AN INCLINED RECTANGULAR OR CUBIC CAVITY

Yoshinari Fukazawa* & Mitsuaki Funakoshi* [a)]

* *Department of Applied Analysis and Complex Dynamical Systems, Graduate school of Informatics, Kyoto University, Kyoto 606-8501, Japan*

Summary Onset of thermal convection of a fluid in an inclined rectangular or cubic cavity is examined numerically using a Galerkin spectral method, under the assumptions of rigid walls of perfect thermal conductance and of a vertically linear external temperature profile. For a cubic cavity, its inclination can cause the decrease in critical Rayleigh number. The exchange of the most unstable mode associated with the variation in the inclination angle of rectangular cavities is observed.

FORMULATION

There have been many studies on the onset of three-dimensional thermal convection from a motionless state in a rectangular or cubic cavity whose upper and lower walls are horizontal. However, even if the cavity is inclined, a motionless state is possible if we impose the condition of perfect thermal conductance with a vertically linear temperature profile on all the rigid walls. Therefore, in the present paper, we examine the onset of three-dimensional thermal convection in an inclined cubic or rectangular cavity under the above boundary condition. Our main purpose is the examination of the variation in critical Rayleigh number and exchange of the most unstable mode associated with the variation in inclination angles.

We consider the three-dimensional motion of an incompressible viscous fluid in an inclined rectangular cavity. Cartesian coordinates (ξ^*, η^*, ζ^*) with their origin at the center of the cavity, shown in Fig. 1(a), are defined so that mutually-perpendicular three sides of the cavity are parallel to the ξ^*, η^*, and ζ^*-axes. Coordinates (ξ^*, η^*, ζ^*) are related to Cartesian coordinates (x^*, y^*, z^*) in which z^*-axis is vertically upward in the following way. Coordinates (ξ^*, y^*, z'^*) are first introduced by the rotation of coordinates (x^*, y^*, z^*) about y^*-axis by angle ϕ, as shown in Fig. 1(b).

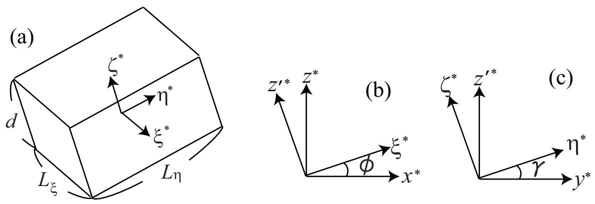

Figure 1. Geometrical configuration and coordinate systems.

Coordinates (ξ^*, η^*, ζ^*) are then defined by the rotation of coordinates (ξ^*, y^*, z'^*) about ξ^*-axis by angle γ, as shown in Fig. 1(c). The shape of cavity is determined by two aspect ratios $A_\xi \equiv L_\xi/d$ and $A_\eta \equiv L_\eta/d$, where L_ξ, L_η, and d respectively denote the length of cavity in the ξ, η, and ζ directions. The attitude of cavity is determined by two angles ϕ and γ. We assume that the external temperature field T^*_{ex} decreases linearly in vertically upward direction as written by $T^*_{ex} = -\beta z^* + T^*_0$, where $\beta(>0)$ is an external temperature gradient, and T^*_0 is the external temperature at $z^* = 0$. All the walls of cavity are assumed to be rigid and of perfect thermal conductance. We also adopt the Boussinesq approximation. Non-dimensionalized governing equations under this approximation are written as

$$\nabla \cdot \boldsymbol{v} = 0, \qquad \Big(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\Big)T = \Delta T, \tag{1}$$

$$\frac{1}{P}\Big(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\Big)\boldsymbol{v} = -\nabla p + \Delta \boldsymbol{v} - \Big(\frac{gd^3}{\kappa\nu} - R(T - T_0)\Big)(\boldsymbol{e}_\xi \sin\phi + (\boldsymbol{e}_\eta \sin\gamma + \boldsymbol{e}_\zeta \cos\gamma)\cos\phi), \tag{2}$$

where $\nabla \equiv \boldsymbol{e}_\xi \frac{\partial}{\partial \xi} + \boldsymbol{e}_\eta \frac{\partial}{\partial \eta} + \boldsymbol{e}_\zeta \frac{\partial}{\partial \zeta}, \Delta \equiv \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \zeta^2}$, and $\boldsymbol{e}_\xi$, $\boldsymbol{e}_\eta$, and $\boldsymbol{e}_\zeta$ respectively represent the unit vectors in the ξ^*, η^*, and ζ^* directions. $\xi \equiv \frac{\xi^*}{d}, \eta \equiv \frac{\eta^*}{d}, \zeta \equiv \frac{\zeta^*}{d}, t \equiv \frac{\kappa}{d^2}t^*, \boldsymbol{v} \equiv \frac{d}{\kappa}\boldsymbol{v}^*, T \equiv \frac{T^*}{\beta d}, p \equiv \frac{d^2}{\rho_0 \kappa \nu}p^*$, and $\boldsymbol{v}^*$ is fluid velocity, T^* is temperature, p^* is pressure, t^* is the time, ρ_0 is the density of fluid for $T^* = T^*_0$. Also ν and κ respectively denote the kinematic viscosity and the thermal diffusivity of fluid. g is the acceleration of gravity and $T_0 = \frac{T^*_0}{\beta d}$. Two non-dimensional parameters R and P respectively denote the Rayleigh number and the Prandtl number defined by $R \equiv \frac{\alpha g \beta d^4}{\kappa\nu}$ and $P \equiv \frac{\nu}{\kappa}$, where α is the coefficient of thermal expansion. Non-dimensionalized boundary conditions are written as

$$\boldsymbol{v} = 0,\ T = T_{ex} \quad \text{on} \quad \xi = \pm\frac{A_\xi}{2},\ \eta = \pm\frac{A_\eta}{2},\ \zeta = \pm\frac{1}{2}, \tag{3}$$

where $T_{ex} = -(\xi\sin\phi + (\eta\sin\gamma + \zeta\cos\gamma)\cos\phi) + T_0$.

Governing equations (1) and (2) with boundary condition (3) have the solution of a motionless state with temperature expressed as T_s and pressure p_s. For R equal to the critical Rayleigh number R_c, infinitesimal deviations of velocity, temperature, and pressure from those in the motionless state, defined by $\hat{\boldsymbol{u}} \equiv \boldsymbol{v}, \hat{\theta} \equiv T - T_s, \hat{q} \equiv p - p_s$, of neutrally stable disturbances satisfy

$$\nabla \cdot \hat{\boldsymbol{u}} = 0, \qquad \Delta\hat{\theta} + \hat{\boldsymbol{u}} \cdot (\boldsymbol{e}_\xi \sin\phi + (\boldsymbol{e}_\eta \sin\gamma + \boldsymbol{e}_\zeta \cos\gamma)\cos\phi) = 0, \tag{4}$$

[a)] Corresponding author. E-mail: mitsu@acs.i.kyoto-u.ac.jp.

$$-\nabla\hat{q}+\Delta\hat{\boldsymbol{u}}+R_c\hat{\theta}(\boldsymbol{e}_\xi\sin\phi+(\boldsymbol{e}_\eta\sin\gamma+\boldsymbol{e}_\zeta\cos\gamma)\cos\phi)=0, \tag{5}$$

$$\hat{\boldsymbol{u}}=0,\ \hat{\theta}=0 \quad \text{on} \quad \xi=\pm\frac{A_\xi}{2},\ \eta=\pm\frac{A_\eta}{2},\ \zeta=\pm\frac{1}{2}. \tag{6}$$

We use Galerkin spectral method to solve eqs. (4)–(6) numerically. Here a divergence-free set of basis functions satisfying all boundary conditions, produced from Chebyshev polynomials, is used. By solving the eigenvalue problem derived from this method, we obtain the value of R_c as well as neutrally stable disturbances.

From the symmetry of eqs. (4)–(6), it is sufficient to solve these equations only for the $\hat{\boldsymbol{u}}$ and $\hat{\theta}$ of some symmetries. For $\phi=\gamma=0$, it is sufficient to solve these equations only for each of eight modes denoted as $\hat{\theta}(e,e,e)$, $\hat{\theta}(e,e,o)$, $\hat{\theta}(e,o,e)$, $\hat{\theta}(e,o,o)$, $\hat{\theta}(o,e,e)$, $\hat{\theta}(o,e,o)$, $\hat{\theta}(o,o,e)$, and $\hat{\theta}(o,o,o)$ modes. Here in $\hat{\theta}(e,e,o)$ mode, $\hat{\theta}$ is an even function of ξ, η and an odd function of ζ, and the symmetry of $\hat{\boldsymbol{u}}$ is determined from this property of $\hat{\theta}$. The symmetry of other modes is defined similarly. For $\phi=0$ and $\gamma\neq 0$, it is sufficient to solve eqs. (4)–(6) only for each of four modes denoted as $\hat{\theta}(e,e), \hat{\theta}(e,o), \hat{\theta}(o,e)$, and $\hat{\theta}(o,o)$ modes. Here in $\hat{\theta}(e,o)$ mode, $\hat{\theta}$ is an even function of ξ and an odd function with respect to the transformation $(\eta,\zeta)\to(-\eta,-\zeta)$, for example. For $\phi\neq 0$ and $\gamma\neq 0$, it is sufficient to solve eqs. (4)–(6) only for each of two modes denoted as $\hat{\theta}(e)$ and $\hat{\theta}(o)$ modes, where in $\hat{\theta}(e)$ mode, $\hat{\theta}$ is an even function with respect to the transformation $(\xi,\eta,\zeta)\to(-\xi,-\eta,-\zeta)$, for example. Among the above eight, four, or two modes, the mode of smallest R_c is called the most unstable mode.

RESULTS

In the description of results of stability analysis, we use $\tilde{\phi}\equiv(180/\pi)\phi$ and $\tilde{\gamma}\equiv(180/\pi)\gamma$ instead of ϕ and γ. For $A_\xi=A_\eta=1$ and $\tilde{\phi}=0$, with the increase in $\tilde{\gamma}$ from 0 to 45, R_c decreases and takes the minimum value of $R_c=6714.9$ at $\tilde{\gamma}=45$, as shown in Fig. 2. Therefore, the inclination of a cubic cavity causes the decrease in R_c. For $\tilde{\gamma}$ satisfying $0<\tilde{\gamma}<90$, $\hat{\theta}(e,o)$ mode always gives the minimum value of R_c among four modes. For $\tilde{\gamma}=0$ or $\tilde{\gamma}=90$, because one of $\hat{\theta}(e,o,e)$ and $\hat{\theta}(o,e,e)$ modes is obtained by rotating the other mode about the vertical axis by 90 degrees, both $\hat{\theta}(e,o)$ and $\hat{\theta}(o,e)$ modes in Fig. 2 give the same value of $R_c=6797.7$, the minimum value of R_c among eight modes in the non-inclined case. This value agrees well with the value obtained by Mizushima and Matsuda[1] for a non-inclined cubic cavity. The flow pattern of most unstable mode [$\hat{\theta}(e,o)$ mode] for $\tilde{\gamma}=45$ is similar to one-vortex structure whose axis is horizontal and agrees with the axis of the rotation of cavity, ξ-axis (x-axis).

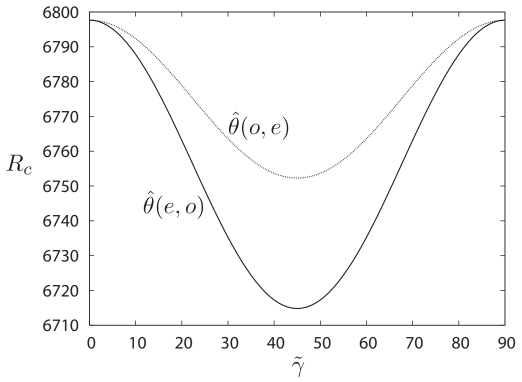

Figure 2. Dependences of R_c for $\hat{\theta}(e,o)$ and $\hat{\theta}(o,e)$ modes on $\tilde{\gamma}$. $A_\xi=A_\eta=1$ and $\tilde{\phi}=0$.

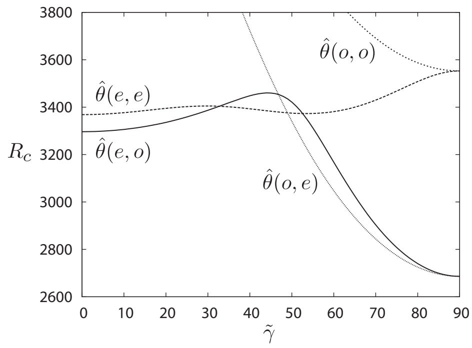

Figure 3. Dependence of R_c on $\tilde{\gamma}$. $A_\xi=1, A_\eta=3$, and $\tilde{\phi}=0$.

For $A_\xi=A_\eta=1$ and $\tilde{\phi}=45$, with the increase in $\tilde{\gamma}$ from 0 to 45, R_c decreases to 6712.4 for $\tilde{\gamma}$ satisfying $0\leq\tilde{\gamma}\leq 29$ and increases for $\tilde{\gamma}$ larger than 29. For $\tilde{\gamma}$ satisfying $0<\tilde{\gamma}<90$, $\hat{\theta}(o)$ mode always gives the lower value of R_c than $\hat{\theta}(e)$ mode. From the above results, the value of R_c for a cubic cavity can be decreased at least to 6712.4 from the value 6797.7 in the non-inclined case by its inclination.

For $A_\xi=1$, $A_\eta=3$, and $\tilde{\phi}=0$, the exchange of the most unstable mode occurs twice, as shown in Fig. 3. That is, with the increase in $\tilde{\gamma}$ from 0 to 90, the most unstable mode changes from $\hat{\theta}(e,o)$ mode to $\hat{\theta}(e,e)$ mode at $\tilde{\gamma}=33$, and then changes to $\hat{\theta}(o,e)$ mode at $\tilde{\gamma}=49$. It is noted that although $\hat{\theta}(e,e)$ mode is the most unstable for intermediate values of $\tilde{\gamma}$, this mode is not the most unstable for $\tilde{\gamma}$ close to 0 or 90.

Similar exchange of the most unstable mode with the increase in $\tilde{\gamma}$ from 0 to 90 with $\tilde{\phi}$ fixed to 0 is observed also for $A_\xi=1$, $A_\eta=2$ and for $A_\xi=1$, $A_\eta=4$. However, the exchange occurs only once for $A_\xi=1$, $A_\eta=2$, and three times for $A_\xi=1$, $A_\eta=4$.

References

[1] Mizushima J. and Matsuda O.: Onset of 3D thermal convection in a cubic cavity. *J. Phys. Soc. Jpn.* **66**: 2337–2341, 1997.