RING-PATTERN FORMATION IN THE ROTATING STELLAR STRUCTURES

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Summary Aihara's work on account of formation of Saturn's rings is revisited. Based on Chandrasekhar's formalism mathematically rigorous treatments, *i.e.*, generation of a general solution to the first-order PDE and use of similarity solution, lead to an analytic closed-form solution consisting of a couple of special functions. The solution shows the existence of many potential wells in the equatorial plane around the stellar core. This theory gives a clue to explanations for the formation of Saturn's rings.

INTRODUCTION

Chandrasekhar[1] paved the way to construct a basis for physics of stellar structures by use of fluid dynamics, gas dynamics and the gravitation potential. Aihara[2] made use of Chandrasekhar's formalism to annotate Saturn's rings. We revisit Aihara's work and show we can get the analytic closed-form solution.

THEORY

The framework of this theory is within the classical celestial physics[1]: we assume that stellar materials are the perfect gas that changes in a polytropic manner; we also treat stellar materials as to obey compressible fluid dynamics; the entire system is governed by law of the gravitation.
Figure 1 shows the coordinate systems to be used in our study. In Cartesian system Zenith is located at $(0, 0, \infty)$, whilst Nadir is at $(0, 0, -\infty)$. We fix the origin to the centre of our stellar structures, and hence in the equatorial plane z is equal to naught. Another independent variable is the time t.
The basic equations consist of the Euler's equations, the equation of state in polytropic gas and the equation of the gravitational potential, respectively given by

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{u} = 0, \qquad (1)$$

$$\frac{\mathrm{D}}{\mathrm{D}t} \rho \mathbf{u} = -\operatorname{grad} p - \rho \operatorname{grad} \Omega, \qquad (2)$$

$$p = K\rho^{\gamma'}, \qquad (3)$$

and

$$\nabla^2 \Omega = 4\pi G \rho, \qquad (4)$$

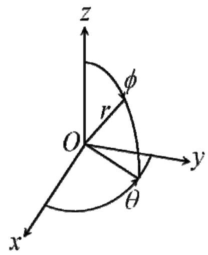

Fig. 1 Cartesian and spherical coordinates; the polar angle θ and the azimuth angle ϕ.

where the variables ρ, $\mathbf{u}$, p and Ω respectively designate the mass density, the velocity vector, the pressure and the gravitational potential, whilst the parameters K, γ' and G respectively designate a product of the gas constant and the polytropic temperature, the polytropic specific-heat ratio and the gravitation constant.

THEORETICAL RESULTS AND DISCUSSION

The final form of the basic equation to be solved

We shall consider the steady-state stellar structure with rotation about z-axis: we neglect $\partial/\partial t$ terms in equations (1) and (2); we make use of the spherical coordinate and assume the velocity vector has only one non-zero component in θ-direction, *i.e.*, u_θ. Introducing one-component assumption upon $\mathbf{u}$ in equation (1), we find $\rho\, u_\theta$ is independent of θ. Using this information as well as one-component assumption upon $\mathbf{u}$, θ-direction equation in (2) leads us to the fact that p and Ω are also independent of θ. Eliminating p, by use of (3), from r- and ϕ-direction equations in (2), we obtain a set of simultaneous PDEs upon ρ and Ω. This set has a general solution:

$K \dfrac{\gamma'}{\gamma' - 1} \rho^{\gamma' - 1} + \Omega = \Psi(\xi)$, where $\xi = r \sin\phi$; there is a boundary condition for an arbitrary function Ψ: $\dfrac{\mathrm{d}\Psi}{\mathrm{d}\xi} = \dfrac{{u_\theta}^2}{r}$.

This is the centrifugal acceleration that must be in equilibrium with the centripetal gravitation. The argument above shows that ρ and Ω have similarity solutions with respect to ξ.

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Using the general solution above, we can eliminate ρ from (4); introducing the compound variable ξ into the Laplacian in (4), we obtain an ODE for the gravitation potential in a legible form with rescaled ξ.

$$\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\Omega=\left(\Psi-\Omega\right)^{1/\gamma'-1}. \qquad (5)$$

The equation (5) is an extension of the Lane-Emden equation. This is nonlinear but we can get a mathematically rigorous and analytic solution of (5) in case of $\gamma' = 2$, which is a plausible value for our solar system. So we shall solve (5) for this case.

We shall use the solution to the classical Lane-Emden equation for formation of the central stellar core. This is given by the spherical Bessel function of the first kind of degree zero, *i.e.*, $-j_0(\xi)$. It is shown in Fig.2. The solution is valid within the radius π. Although there exist both positive and negative regions for $\xi > \pi$, it is not feasible to generate spherical shell layers containing vacuum gaps among themselves. Therefore there is one core sphere only.

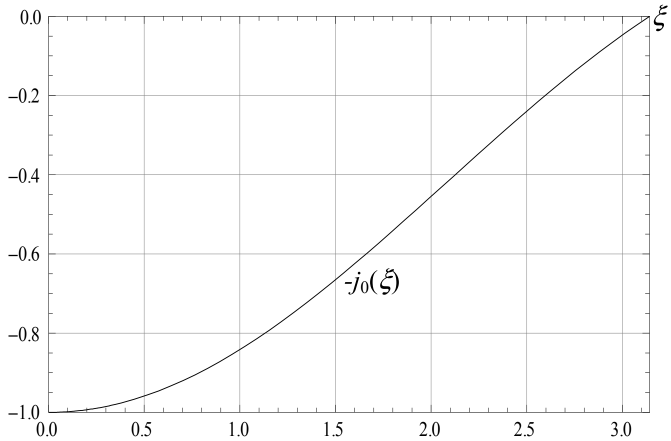

Fig. 2 The normalised spherical potential $-j_0(\xi)$.

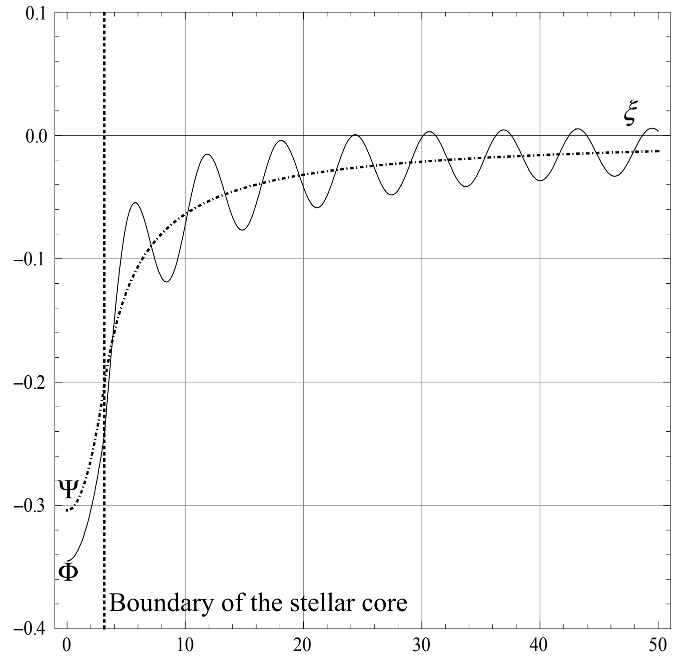

Fig. 3 The normalised cylindrical potentialΦ (solid line) with centrifugal acceleration Ψ (dash-dot line).

Description of the rotation

To close our problem we need to give a physical explanation to the function Ψ: rigid rotation in the stellar core and inverse-square law for outer space. After some computation we get the following form:

$$\Psi(\xi)=\begin{cases}-\dfrac{\kappa}{\xi} & \text{for } \xi>\pi,\\[2mm] \dfrac{\kappa}{2\pi^3}\xi^2-\dfrac{3\kappa}{2\pi} & \text{for } \xi\in[0,\pi],\end{cases} \qquad (6)$$

where κ is a gravitation-related constant; the rotation is continual at the surface of the stellar core. Substitution of (6) for Ψ in (5) leads us to the equation for the gravitational potential with rotation.

Gravitational potential

Assuming a quadratic function of ξ within the stellar core, we get the particular solution as follows.

$\Phi(\xi)=\dfrac{1}{\pi^4}\xi^2-\dfrac{4}{\pi^4}-\dfrac{3}{\pi^2}$, where $\Phi = 2\Omega/\pi\kappa$. The stellar core is spherical but the above potential is cylindrical.

In the outer space the centrifugal force due to the stellar core *in the equatorial plane only*. Then the basic equation becomes a Struve's differential equation of degree zero, and hence the solution is given by

$\Phi(\xi)=-\mathbf{H}_0(\xi)+\left\{\mathbf{H}_0(\pi)-\dfrac{4}{\pi^4}-\dfrac{2}{\pi^2}\right\}\dfrac{\mathrm{Y}_0(\xi)}{\mathrm{Y}_0(\pi)}$, where $\mathbf{H}_0$ and Y_0 denote the Struve function of degree zero and Bessel function of the second kind of degree zero, respectively. The homogeneous solution is added to assure the continuity of the potential at the surface of the stellar core. Its graphical representation is given in Fig.3. There are many potential wells that accommodate dense mass distributions. In case of the cylindrical potential Φ in the equatorial plane it is possible to allow the existence of vacuum gaps between stellar mass distributions, which may become ring structures.

CONCLUSION

We revisit Aihara's work on formation of Saturn's rings and reformulate it in a mathematically rigorous way. This leads us to the analytic closed-form solution that gives us a clue to explanations to formation of Saturn's rings.

References

[1] S. Chandrasekhar, An Introduction to the Study of Stellar Structure, (1958), Dover Pub., pp.507.

[2] Y. Aihara, Acta Astronaut., **11**(1984), pp.313-317.