A LAGRANGIAN TAKE ON CONVECTIVE HEAT TRANSFER

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Summary Heat transfer is customarily investigated from an Eulerian perspective in terms of temperature fields and heat-transfer coefficients. An alternative exists in an approach based on internal heat fluxes and the delineated Lagrangian trajectories. In this Lagrangian picture, heat transfer becomes the "motion" of thermal energy along "thermal paths" in a similar way as fluid flow is the motion of fluid parcels along fluid paths. This fundamental analogy admits application of well-known Lagrangian concepts from mixing studies to thermal analyses. The Lagrangian ansatz is elaborated for convective heat transfer, a case of great practical importance, by definition of relevant heat fluxes and associated thermal paths through decomposition of the temperature field into a conductive and convective contribution. The proposed formalism is illustrated for the visualisation of convective heat transfer in a 2D case study. This affords insight into thermal transport that is complementary to the Eulerian picture and thus demonstrates its great potential.

CONVECTIVE HEAT TRANSFER: LAGRANGIAN REPRESENTATION

Heat transfer inside a domain $\mathcal{D}$ with boundary $\Gamma = \partial\mathcal{D}$ and outward normal $\boldsymbol{n}$ is for solenoidal flow ($\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$) governed by the non-dimensional energy equation

$$\frac{\partial T}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}T = \frac{1}{Pe}\boldsymbol{\nabla}^2 T, \qquad T|_\Gamma = h(\boldsymbol{x}_\Gamma), \qquad T(\boldsymbol{x},0) = T_0(\boldsymbol{x}), \tag{1}$$

with Pe the well-known Péclet number and h a generic Dirichlet boundary condition. The temperature admits decomposition into conductive ($\widetilde{T}$) and convective (T') contributions, $T = \widetilde{T} = T'$, governed by

$$\frac{\partial \widetilde{T}}{\partial t} = \frac{1}{Pe}\boldsymbol{\nabla}^2\widetilde{T}, \qquad \frac{\partial T'}{\partial t} + \boldsymbol{\nabla}\cdot\left(\boldsymbol{u}T' - Pe^{-1}\boldsymbol{\nabla}T'\right) = -\boldsymbol{u}\cdot\boldsymbol{\nabla}\widetilde{T}, \tag{2}$$

with boundary conditions $\widetilde{T}|_\Gamma = h(\boldsymbol{x}_\Gamma, t)$ and $T'|_\Gamma = 0$ and initial conditions $\widetilde{T}(\boldsymbol{x},0) = T_0(\boldsymbol{x})$ and $T'(\boldsymbol{x},0) = \boldsymbol{0}$ [5]. The definition of T' implies $\lim_{Pe\to 0} T' = 0$, meaning that T' represents the effect of fluid motion – and thereby of convective heat transfer – upon the temperature distribution. This decomposition admits expression in the transport form

$$\frac{\partial \widetilde{T}}{\partial t} + \boldsymbol{\nabla}\cdot\widetilde{\boldsymbol{q}} = 0, \qquad \frac{\partial T'}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{Q}' = F, \qquad \widetilde{\boldsymbol{q}} = -\frac{1}{Pe}\boldsymbol{\nabla}\widetilde{T}, \qquad \boldsymbol{Q}' = \boldsymbol{q}_c + \boldsymbol{q}', \qquad \boldsymbol{q}_c = \boldsymbol{u}T', \qquad \boldsymbol{q}' = -\frac{1}{Pe}\boldsymbol{\nabla}T', \tag{3}$$

exposing $F = -\boldsymbol{u}\cdot\boldsymbol{\nabla}\widetilde{T}$ as source for contribution T' and thus as "driving force" behind convective heat transfer. Flux $\boldsymbol{q}_c$ represents the convective heat flux solely by fluid transport into regions at different temperature ("direct convective flux"). Transport of warmer fluid into a colder region ($T' > 0$) causes supply of thermal energy and a convective heat flux in the flow direction (direct convective *heating*). Conversely, transport of colder fluid into a warmer region ($T' < 0$) extracts thermal energy and effectively sets up a convective heat flux *against* the flow direction (direct convective *cooling*). Flux $\boldsymbol{Q}'$, similarly, represents the net heat transfer induced by the fluid motion ("net convective flux") relative to the conductive background $\widetilde{T}$ and comprises the direct convective flux $\boldsymbol{q}_c$ and the additional conductive contribution $\boldsymbol{q}'$ by non-zero T'. This renders $\boldsymbol{Q}'$ the key quantity in analyses on heat-transfer enhancement by convection. Distinction can, similar as for $\boldsymbol{q}_c$, again be made between net convective heating ($T' > 0$) and net convective cooling ($T' < 0$).

Flux $\boldsymbol{Q}'$ delineates the convective thermal transport routes in an analogous way as the velocity $\boldsymbol{u}$ delineates the transport routes of fluid parcels. These "thermal paths" are formally defined as

$$\frac{d\boldsymbol{x}'}{dt} = \frac{\boldsymbol{Q}'}{T'} \quad \Rightarrow \quad \boldsymbol{x}'(t) = \boldsymbol{\Phi}'\left(\boldsymbol{x}_0'\right), \tag{4}$$

constituting the thermal counterpart to the kinematic equation $d\boldsymbol{x}/dt = \boldsymbol{u} = \boldsymbol{M}/\rho$, with $\boldsymbol{M}$ and ρ respectively mass flux and fluid density, and corresponding flow $\boldsymbol{x}(t) = \boldsymbol{\Phi}\left(\boldsymbol{x}_0\right)$ for fluid motion.

The proposed formalism of convective heat flux and thermal paths expands on that according to [3, 4] which, in turn, is a generalisation of earlier versions [1]. Primary difference with the latter is that here the non-uniform conductive temperature contribution $\widetilde{T}$ serves as reference, which is a unique and physically-meaningful state. The original ansatz, on the other hand, adopts a uniform reference state, which introduces a degree of arbitrariness that, though conceptually immaterial, hampers physical interpretation of heat fluxes and thermal paths [2, 5].

AN ILLUSTRATIVE EXAMPLE

The Lagrangian representation is demonstrated for the laminar heat transfer in a 2D non-dimensional channel confined by a hot bottom wall and cold top wall. The flow under steady conditions consists of two adjacent counter-rotating

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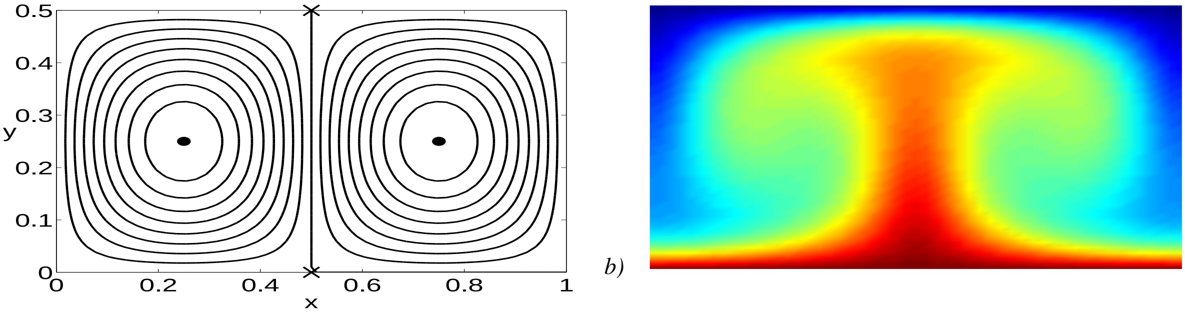
vortices (Figure 1a). A typical temperature distribution T is shown in Figure 1b and clearly reveals the heat transfer from hot bottom to cold top wall via a thermal plume. The corresponding convective temperature contribution T' and net convective flux Q' are given in Figure 2a. Distribution T' exposes the convective heating ($T' > 0$; red) and cooling ($T' < 0$; blue) zones. Location of former and latter in the proximity of cold top and hot bottom wall, respectively, reflects the upward transport of warmer fluid in the centre and downward transport of colder fluid near the side walls by Q'.

a) 

b)

Figure 1. *Laminar heat transfer in a 2D channel confined by a hot bottom wall and cold top wall: (a) adjacent counter-rotating vortices; (b) temperature distribution (red/blue indicate highest/lowest temperatures).*

Net convective heat transfer is shown in Figure 2b in terms of Q' (vectors) and its thermal paths (black curves), where blue/red indicate convective cooling/heating zones separated by isothermals $T' = 0$ (grey curves). Flux $\boldsymbol{Q}'$ running counter to the circulation in the convective-cooling zone (blue) signifies transport of colder fluid from the upper region towards the warmer bottom wall. This effectuates heating of the colder fluid and, inherently, results in an effective thermal transport *against* the flow direction. The predominantly inward and outward orientation of $\boldsymbol{Q}'$ on bottom and top walls, respectively, signifies augmented fluid-wall heat exchange and, given $\boldsymbol{Q}'$ is inextricably linked to fluid motion, directly represents heat-transfer enhancement by convection. The thermal paths nicely visualise the net convective heat transfer from hot bottom to cold top wall and their twisting exposes a strong effect of the fluid circulation. Their geometry and arrangement is subject to similar mechanisms as those dictating the organisation of fluid trajectories. This is a fundamental analogy of great importance in that it admits generalisation of well-established Lagrangian concepts from mixing studies to thermal analyses [3, 4, 5]. Furthermore, the present Lagrangian approach extends to 3D unsteady systems [5].

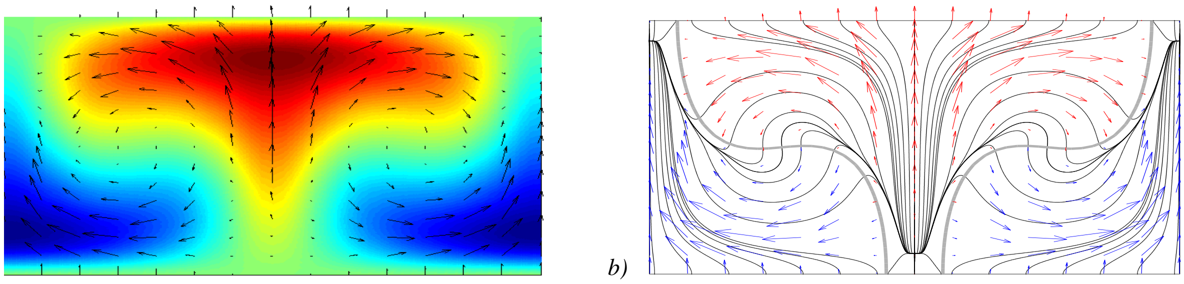
a) 
b)

Figure 2. *Net convective heat transfer: (a) convective temperature contribution T' (red/blue indicate highest/lowest temperatures) and corresponding net convective flux $\boldsymbol{Q}'$ (vectors); (b) thermal paths (black curves) of net convective heat transfer. Blue/red vectors indicate convective cooling/heating zones; heavy grey curves indicate isothermals $T' = 0$.*

CONCLUSIONS AND OUTLOOK

The proposed Lagrangian representation for convective heat transfer affords insight into thermal transport that is complementary to the Eulerian picture. It enables demarcation of the internal redistribution routes of thermal energy due to fluid motion and thus *directly* visualises convective heat transfer. This exposed the formation of convective-cooling and convective-heating zones by the transport of colder fluid into warmer regions and *vice versa*. Such zones steepen temperature gradients at non-adiabatic walls and, in consequence, are a key mechanism behind convective heat-transfer enhancement. The present approach further develops Lagrangian heat-transfer formalisms in literature by facilitating accessible physical interpretation of thermal fluxes and thermal paths. Its application to realistic 3D flows is in progress.

References

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