Jan Holnicki-Szulc and Tomasz Bielecki
Institute of Fundamental Technological Research, Warsaw, Poland

Design of Adaptive Structures

1. Introduction

Adaptive structures (structures equipped with controllable semi-active dissipaters, so called structural fuses) with highest ability of adaptation to extremal overloading are discussed. The quasistatic formulation of this problem allows developing effective numerical tools necessary for farther considerations concerning dynamic problem of optimal design for the best structural crash-worthiness (see [2]). The structures with the highest impact absorption properties can be designed in this way. The proposed optimal design method combines sensitivity analysis with remodelling process, allowing approach (with material distribution as well as stress limits controlled) to an optimally redesigned structure. So called Virtual Distortion Method (see [1]), leading to analytical formulas for gradient calculations, has been used in numerically efficient algorithm.

2. VDM Based Structural Remodelling and Sensitivity Analysis

Let us concentrate on the sensitivity analysis for the truss structure under progressive collapse process due to extremal load applied. The superposition of virtual, plastic-like distortions \( \beta^o_i \), simulating non-linear member behaviour, with distortions \( \varepsilon^o_i \), modelling modifications of design variables (e.g. \( A_i \)), turns out to be productive in this case.

The strains and stresses, calculated with respect to initial cross-sections, can be expressed as follows (see [1], [3]):

\[
\sigma'_i = E_i (\varepsilon'_i - \varepsilon^o_i - \beta^o_i) = E_i \left( \varepsilon'_i + \sum_j (D_{ij} - \delta_y) \varepsilon^o_j + \sum_k (D_{ik} - \delta_k) \beta^o_k \right)
\]

\[
\varepsilon_i = \varepsilon'_i + \sum_j D_{ij} \varepsilon^o_j + \sum_k D_{ik} \beta^o_k
\]

(1)

where \( D_{ij} \) denote deformations caused in the members \( i \) by the unit virtual distortions \( \varepsilon^o_j \) generated in members \( j \). The corresponding derivatives take the following form:

\[
\frac{d\sigma'_i}{d\varepsilon^o_j} = E_i (D_{ij} - \delta_y) \quad \frac{d\sigma'_i}{d\beta^o_k} = E_i (D_{ik} - \delta_k)
\]

\[
\frac{d\varepsilon_i}{d\varepsilon^o_j} = D_{ij} \quad \frac{d\varepsilon_i}{d\beta^o_k} = D_{ik}
\]

(2)

The subscripts \( j \) and \( k \) in the above formulas run through all modified and plastified members, respectively. Taking advantage of two expressions for the internal forces applied to so called distorted (with modification of material distribution modelled through virtual distortions) and modified (with redesigned cross-sections from \( A'_i \) to \( A_i \), without imposing virtual distortions) structure:

\[
P'_i = E_i A'_i (\varepsilon'_i - \varepsilon^o_i - \beta^o_i)
\]

\[
P_i = E_i A_i (\varepsilon'_i - \beta^o_i)
\]

(3)

(where components of \( \varepsilon^o_i \), \( \beta^o_i \) are non-zero only in distorted or plastified members, respectively), the following formula can be derived:

\[
A \left( \varepsilon'_i + \sum_j D_{ij} \varepsilon^o_j + \sum_k (D_{ik} - \delta_k) \beta^o_k \right) = A \left( \varepsilon'_i + \sum_j (D_{ij} - \delta_y) \varepsilon^o_j + \sum_k (D_{ik} - \delta_k) \beta^o_k \right)
\]

(4)
what can be expressed alternatively:

\[
\sum_j \left[ A'(D_j - \delta_j) - A_i D_j \right] \varepsilon_j^o + \sum_k \left[ (A' - A)(D_a - \delta_a) \right] \beta_k^o = (A_i - A^o) \varepsilon_i^o
\]  

(5)

Calculation of the derivative with respect to \( A_m \) leads to:

\[
\delta_m \left[ \varepsilon_i^o + \sum_j D_j \varepsilon_j^o + \sum_k (D_a - \delta_a) \beta_k^o \right] + A \left( \sum_j D_j \frac{\partial \varepsilon_j^o}{\partial A_m} + \sum_k (D_a - \delta_a) \frac{\partial \beta_k^o}{\partial A_m} \right) = 0
\]

\[
= A' \left( \sum_j (D_j - \delta_j) \frac{\partial \varepsilon_j^o}{\partial A_m} + \sum_k (D_a - \delta_a) \frac{\partial \beta_k^o}{\partial A_m} \right)
\]

(6)

After rearranging the above formula, we have:

\[
\sum_j \left[ A'(D_j - \delta_j) - A_i D_j \right] \frac{\partial \varepsilon_j^o}{\partial A_m} + \sum_k \left[ (A' - A)(D_a - \delta_a) \right] \frac{\partial \beta_k^o}{\partial A_m} = (\varepsilon_i^o - \beta_i^o) \delta_m
\]

(7)

The associated conditions for derivatives \( \frac{\partial \beta_i^o}{\partial A_i} \) and \( \frac{\partial \varepsilon_j^o}{\partial A_m} \) can be determined from the yield criterion (cf. Fig.1), written for the modified structure (with modified cross-sections \( A_i \)):

\[
\sigma_i - \sigma^* = \gamma_i E_i \left( \varepsilon_i^o - \varepsilon^* \right)
\]

(8)

\[\text{Figure 1. Yield criterion for the modified structure}\]

For the modified structure, where \( \varepsilon^o \) affects the stress formula in an implicit way through modified deformations (cf. Eqs. (1) for distorted structure), we get the following strains and stresses, with respect to remodelled cross-sections \( A_i \):

\[
\sigma_i = E_i \left( \varepsilon_i^o - \beta_i^o \right) = E_i \left( \varepsilon_i^o + \sum_j D_j \varepsilon_j^o + \sum_k (D_a - \delta_a) \beta_k^o \right)
\]

\[
\varepsilon_i = \varepsilon_i^o + \sum_j D_j \varepsilon_j^o + \sum_k D_a \beta_k^o
\]

(9)

Substituting (9) to (8) we obtain:

\[
\sum_k B_{ak} \beta_k^o + \sum_j (1 - \gamma_j) D_j \varepsilon_j^o = -(1 - \gamma_i) \left( \varepsilon_i^o - \varepsilon^* \right)
\]

where \( B_{ak} = (1 - \gamma_j) D_a - \delta_a \)

(10)

Indices \( l \) and \( k \) run through plastified members and \( j \) runs through distorted members. The matrix B (so-called simulation matrix in collapse analysis) is non-positive definite. The
mechanical interpretation of VDM simulation requires that all diagonal elements of B are non-positive. Therefore the following constraint imposed on the softening parameters:

$$\gamma_k \geq -\frac{1-D_{kk}}{D_{kk}}$$  \hfill (11)

for all members $k$ has to be satisfied, to get correct solution through the VDM approach. If a member does not satisfy the above constraint, its contribution to the stress redistribution drops to zero and we have to apply the equation of line BC (Fig. 1) to model the corresponding local stress-strain characteristic. Now, calculating derivatives with respect to $\varepsilon^0_m$ we can get the following set of $l'$ equations:

$$\sum_k [(1-\gamma_k)D_{kk} - \delta_{kk}] \frac{\partial \beta^e_k}{\partial A_n} + \sum_j (1-\gamma_j)D_{kj} \frac{\partial \varepsilon^e_j}{\partial A_n} = 0$$  \hfill (12)

(where $l'$ denotes the number of plastified members and $l, k=1, 2... l'$).

Finally, to calculate the sensitivities (for example, with respect to modifications of material distribution) for elasto-plastic structure:

$$\frac{\partial \sigma'_i}{\partial A_n} = \sum_j \frac{\partial \sigma'_i}{\partial e^e_j} \frac{\partial e^e_j}{\partial A_n} + \sum_k \frac{\partial \sigma'_i}{\partial \beta^e_k} \frac{\partial \beta^e_k}{\partial A_n} = \sum_j E_j D_{ij} \frac{\partial e^e_j}{\partial A_n} + \sum_k E_k (D_{ij} - \delta_{ij}) \frac{\partial \beta^e_k}{\partial A_n}$$

$$\frac{\partial \varepsilon'_i}{\partial A_n} = \sum_j \frac{\partial \varepsilon'_i}{\partial e^e_j} \frac{\partial e^e_j}{\partial A_n} + \sum_k \frac{\partial \delta^e_k}{\partial A_n} = \sum_j D_{ij} \frac{\partial e^e_j}{\partial A_n} + \sum_k D_k \frac{\partial \beta^e_k}{\partial A_n}$$  \hfill (13)

the partial derivatives determined by $l'$ equations (12) and $m$ equations (7) (for each chosen design variable $\mu_m = A_{m'/A_m}$) have to be determined from the following set (15) of equations:

$$\begin{bmatrix}
\sum_{l=1}^m \left[ \begin{array}{c}
(1-\gamma_l)D_{ll} - \delta_{ll}
\end{array} \right]
\sum_{j=1}^l \left[ \begin{array}{c}
(1-\gamma_j)D_{lj}
\end{array} \right]
\vdots
\sum_{k=1}^l \left[ \begin{array}{c}
(1-\gamma_k)D_{lk}
\end{array} \right]
\end{bmatrix}
\begin{bmatrix}
\varepsilon^e_j
\end{bmatrix}
= \begin{bmatrix}
(1-\gamma_l)(\varepsilon^e_l - \varepsilon^e_l^*)
\end{bmatrix}$$  \hfill (14)

$$\begin{bmatrix}
\sum_{l=1}^m \left[ \begin{array}{c}
(1-\gamma_l)D_{ll} - \delta_{ll}
\end{array} \right]
\sum_{j=1}^l \left[ \begin{array}{c}
(1-\gamma_j)D_{lj}
\end{array} \right]
\vdots
\sum_{k=1}^l \left[ \begin{array}{c}
(1-\gamma_k)D_{lk}
\end{array} \right]
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \varepsilon^e_j}{\partial A_n}
\frac{\partial \beta^e_k}{\partial A_n}
\end{bmatrix}
= \begin{bmatrix}
(1-\gamma_l)(\beta^e_l - \beta^e_l^*) & 0
\end{bmatrix}$$  \hfill (15)

On the other hand, the set (14), with the same main matrix, describes the virtual distortion fields simulating modified structure.

The above formulas can be, for example, applied to the gradient based optimal remodelling processes of adaptive structures. The plastic-like behaviour (simulated through $\beta^e$) corresponds to the performance of actuators, while the material redistribution modified during the redesign process is modelled through virtual distortions $\varepsilon^0$. The gradients computed from Eqs.15 allow calculation of gradients (13) and finally, the gradient of an objective function. Then, corrections for the material distribution leading to reduction of the objective function can be performed, the corresponding modifications of virtual distortions can be determined from (14) and again new gradients can be computed from (15). Following this algorithm we can approach step by step the minimum of the objective function.
If stress limits $\sigma^* \leq \sigma^u$ will be considered as design variables, rather than material redistribution, the gradient formulas (14) will take the following form:

$$
\begin{bmatrix}
\frac{m}{(1-\mu_i)D_{x_i} - \delta_y} \\
\vdots \\
\frac{l}{(1-\mu_i)D_{x_l} - \delta_y}
\end{bmatrix} \cdot 
\begin{bmatrix}
\frac{d\epsilon^\gamma_{ij}}{d\sigma_{ij}} \\
\vdots \\
\frac{d\beta^\gamma_{ij}}{d\sigma_{ij}}
\end{bmatrix} = 
\begin{bmatrix}
0 \\
\vdots \\
(1-\gamma_i)\delta_{ij}
\end{bmatrix}
$$

(15a)

3. Design of Adaptive Structures

Described above VDM technique for structural remodelling and sensitivity analysis will be applied for design of adaptive structures. The material distribution as well as stress limits triggering plastic-like flow in controlled dissipaters will be considered as design variables. Postulating maximisation of the plastic-like energy dissipation, the best effect of structural adaptation to overloading can be achieved. The general problem formulation as well as particular cases’ discussion will be presented. Finally, several numerical examples demonstrating efficiency of the VDM method will be analysed.

4. References