Spherical cloud of point particles falling in a viscous fluid

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Statistical mechanics is applied to calculate ensemble-averaged particle and fluid velocity fields of a spherical cloud of point particles sedimenting at a low Reynolds number. The analogy with the fall of a liquid drop in another lighter fluid is discussed. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2186692]

This Brief Communication focuses on the motion of a spherical drop of particles sedimenting at low Reynolds numbers. This problem has been tackled by different authors,[1–3] using a point-particle model as it contains the minimum physics needed to describe the interactions between particles. The cloud has also been interpreted as an effective medium of excess mass[2] and its fall has been related to the sedimentation of a spherical drop of fluid in an otherwise lighter fluid, solved by Hadamard and Rybczyński.[4] Here, we analytically derive particle and fluid motions by describing statistically a discrete system of particles. We link our results to the literature and compare with the hydrodynamic continuum approach.

We consider N identical point forces $\mathbf{F}$ at positions $\mathbf{r}_i$, $i=1,\ldots,N$, and immersed in an unbounded fluid of viscosity η. The fluid is at rest at infinity (this defines the reference frame of the study). The velocity $\mathbf{u}(\mathbf{r})$ and pressure $p(\mathbf{r})$ of the fluid-flow satisfy the Stokes equations.[4] The solution is the sum of N Stokeslets,

$$\mathbf{u}(\mathbf{r}) = \sum_{j=1}^{N} \mathbf{T}(\mathbf{r}-\mathbf{r}_j)\cdot\mathbf{F}, \tag{1}$$

where $\mathbf{T}(\mathbf{r})=(\mathbf{I}+\hat{\mathbf{r}}\hat{\mathbf{r}})/(8\pi\eta r)$ stands for the Oseen tensor,[4] $r=|\mathbf{r}|$, $\hat{\mathbf{r}}=\mathbf{r}/r$ and $\mathbf{I}$ is the unit tensor. The velocity $\mathbf{v}_i$ of a point particle i located at $\mathbf{r}_i$ is the sum of its velocity $\mathbf{U}_0$ when settling in isolation and of the Stokeslets generated by the other point particles,

$$\mathbf{v}_i = \mathbf{U}_0 + \sum_{j\neq i}^{N} \mathbf{T}(\mathbf{r}_i-\mathbf{r}_j)\cdot\mathbf{F}. \tag{2}$$

We introduce a standard statistical description[5] by considering an ensemble of configurations of N point particles distributed randomly inside a sphere of radius R. Using the N-particle probability distribution function $P(\mathbf{r}_1,\ldots,\mathbf{r}_N)$, we average the velocities (1) and (2),

$$\bar{\mathbf{u}}(\mathbf{r}) = \sum_{i=1}^{N} \int d\mathbf{r}_1 \cdots \int d\mathbf{r}_N P(\mathbf{r}_1,\ldots,\mathbf{r}_N)\mathbf{T}(\mathbf{r}-\mathbf{r}_i)\cdot\mathbf{F}, \tag{3}$$

$$\bar{\mathbf{v}}(\mathbf{r}) = \mathbf{U}_0 + \frac{\displaystyle\sum_{i=2}^{N} \int d\mathbf{r}_2 \cdots \int d\mathbf{r}_N P(\mathbf{r},\mathbf{r}_2,\ldots,\mathbf{r}_N)\mathbf{T}(\mathbf{r}-\mathbf{r}_i)\cdot\mathbf{F}}{\displaystyle\int d\mathbf{r}_2 \cdots \int d\mathbf{r}_N P(\mathbf{r},\mathbf{r}_2,\ldots,\mathbf{r}_N)}. \tag{4}$$

We assume that P is uniform inside the sphere, $P(\mathbf{r}_1,\ldots,\mathbf{r}_N)=\mathcal{V}^{-N}$, if $r_i \leqslant R$ for all $i=1,\ldots,N$. Otherwise $P(\mathbf{r}_1,\ldots,\mathbf{r}_N)=0$. The cloud volume is denoted by $\mathcal{V}=4\pi R^3/3$ and $r_i \equiv |\mathbf{r}_i|$. Since the interactions between particles are pairwise,

$$\bar{\mathbf{u}}(\mathbf{r}) = \frac{N}{\mathcal{V}}\int_{\mathcal{V}} d\mathbf{r}_1 \mathbf{T}(\mathbf{r}-\mathbf{r}_1)\cdot\mathbf{F}, \tag{5}$$

$$\bar{\mathbf{v}}(\mathbf{r}) = \begin{cases} \mathbf{U}_0 + \dfrac{N-1}{\mathcal{V}}\displaystyle\int_{\mathcal{V}} d\mathbf{r}_2 \mathbf{T}(\mathbf{r}-\mathbf{r}_2)\cdot\mathbf{F} & \text{if } r \leqslant R, \\ \mathbf{0} & \text{if } r > R. \end{cases} \tag{6}$$

Using a similar statistical approach, Feuillebois[6] found the same expressions for the averaged fluid velocity $\bar{\mathbf{u}}(\mathbf{r})$ and pointed out that it satisfies the same equations as the fluid flow of a liquid drop of excess weight.

We now evaluate the ensemble-averaged flow fields,

$$\bar{\mathbf{u}}(\mathbf{r}) = \begin{cases} \dfrac{N\mathbf{F}}{4\pi\eta R}\cdot\left[\mathbf{I}-\dfrac{2}{5}\dfrac{r^2}{R^2}\left(\mathbf{I}-\dfrac{1}{2}\hat{\mathbf{r}}\hat{\mathbf{r}}\right)\right] & \text{if } r \leqslant R, \\ \dfrac{N\mathbf{F}}{8\pi\eta R}\cdot\left[\dfrac{R}{r}(\mathbf{I}+\hat{\mathbf{r}}\hat{\mathbf{r}})+\dfrac{1}{5}\dfrac{R^3}{r^3}(\mathbf{I}-3\hat{\mathbf{r}}\hat{\mathbf{r}})\right] & \text{if } r > R, \end{cases} \tag{7}$$

$$\bar{\mathbf{v}}(\mathbf{r}) = \begin{cases} \mathbf{U}_0 + \dfrac{N-1}{N}\bar{\mathbf{u}}(\mathbf{r}) & \text{if } r \leqslant R, \\ \mathbf{0} & \text{if } r > R. \end{cases} \tag{8}$$

We recognize the Hadamard-Rybczyński expressions.[4]

The crucial feature in the previous equations is that particles fall relative to the fluid in which they are immersed. The mean particle-velocity of the ensemble-averaged cloud is determined as $\bar{\mathbf{V}}=\int_{\mathcal{V}}\bar{\mathbf{v}}(\mathbf{r})d\mathbf{r}/\mathcal{V}=\mathbf{U}_0+(N-1)\mathbf{F}/5\pi\eta R$. The mean fluid-velocity inside the cloud is $\bar{\mathbf{U}}=\int_{\mathcal{V}}d\mathbf{r}\,\bar{\mathbf{u}}(\mathbf{r})/\mathcal{V}=N\mathbf{F}/(5\pi\eta R)$. If $\mathbf{U}_0$ is identified with the Stokes settling-velocity $\mathbf{F}/(6\pi\eta a)$ of a single sphere of radius a, the relative slip in the mean is

$$\bar{\mathrm{S}}=\frac{\bar{V}-\bar{U}}{\bar{U}}=\left(\frac{5R}{6a}-1\right)\frac{1}{N}, \tag{9}$$

with $\bar{V}=|\bar{V}|$ and $\bar{U}=|\bar{\mathbf{U}}|$. Note that the slip becomes irrelevant when $N \gg R/a$.

In the reference frame moving with $\bar{\mathbf{V}}$, particles circulate along toroidal closed trajectories inside the whole volume of the ensemble-averaged cloud. The fluid also performs closed toroidal circulations but the consequence of the gravitational slip is that it only occurs inside a core $\mathcal{V}'$ having a smaller radius $r_0=R\sqrt{1-4\bar{\mathrm{S}}}$, provided that $\bar{\mathrm{S}}<1/4$. The core as a whole moves with the same velocity as the ensemble-averaged cloud, since the mean velocity $\int_{\mathcal{V}'}d\mathbf{r}\,\bar{\mathbf{u}}(\mathbf{r})/\mathcal{V}'$ of the fluid inside the core volume $\mathcal{V}'=4\pi r_0^3/3$ is just equal to $\bar{\mathbf{V}}$. In the region between the core and cloud boundary, the fluid lags behind the cloud.

Such core region of closed fluid-streamlines lying inside the boundary of an individual cloud was also found by Nitsche and Batchelor[2] within a hydrodynamic continuum approach (see their Fig. 2). This result is in fact following from Batchelor's calculation of the low-Reynolds-number flow generated by a pure-fluid drop rising in a suspension where the drop was found to carry with it a circulating "halo" of pure fluid.[7] In the present context, Nitsche and Batchelor[2] approximated the sphere containing both particles and fluid by a drop of effective fluid of excess weight settling in the pure fluid. They recovered the Hadamard-Rybczyński solution for the motion of both effective and pure fluids. To obtain the particle motion, they considered that, inside the drop, particles fall relative to the effective fluid (the mixture of fluid and particles), with a slip equal to U_0 (i.e., the relative slip $\bar{\mathrm{S}}$ equal to $5R/6aN$). Their estimate is improved by the present result (9). Our statistical description applies to an arbitrary number of particles. Both our and Batchelor-Nitsche results hold only in the dilute limit as we use the point-particle model.

We would like also to comment that modification of the Oseen tensor would lead to modification of the motion of the system in an arbitrary way. When the distance between point particles approaches zero, their velocities tend to infinity, owing to the divergent nature of the Oseen tensor. Obviously, this does not happen for spheres and this is why some authors have modified the model for close interparticle distances. Nitsche and Batchelor[2] introduced an artificial short-range repulsive pairwise force. Machu *et al.*[3] discussed a model with a cutoff length λ below which interactions are switched off: $\mathbf{T}_\lambda(\mathbf{r})=\mathbf{T}(\mathbf{r})$ if $r\geq\lambda$, and $\mathbf{T}_\lambda(\mathbf{r})=\mathbf{0}$ for $r<\lambda$. Integration of Eq. (6) with the modified Oseen tensor results in $\bar{\mathbf{V}}_\lambda-\mathbf{U}_0=[1-5/8\lambda^2(\lambda^3/20-\lambda+2)](\bar{\mathbf{V}}-\mathbf{U}_0)$. As expected, elimination of close interactions between point particles leads to a decrease of their mean velocity. However, a correct description of hard spheres would also require a different statistical description.

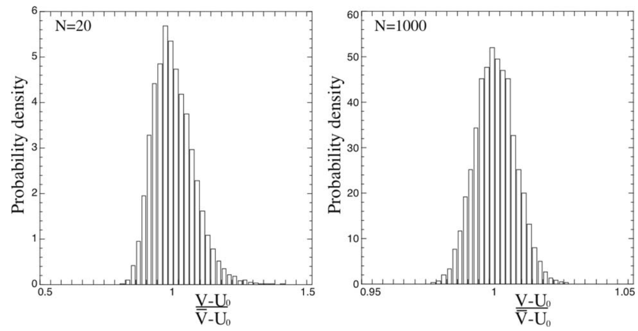

FIG. 1. Distribution of normalized velocities for 4500 clouds made of $N=20$ (left) and $N=1000$ (right) point particles.

We now examine how accurately the ensemble-averaged velocity $\bar{V}$ is approximated by the mean velocity of an individual spherical cloud $\mathbf{V}=\Sigma_{i=1}^N\mathbf{v}_i/N$, with $\mathbf{v}_i$ given by (2). In this numerical computation, N point particles are uniformly distributed inside a sphere of radius $R=1$ owing to a random number generator. Velocity distributions for $N=20$ and $N=1000$ are presented in Fig. 1. Mean velocities are computed in the reference frame moving with $\mathbf{U}_0$, and normalized by the ensemble-averaged cloud velocity in this frame; $(V-U_0)/(\bar{V}-U_0)$ equals to 1.0009 ± 0.0011 for $N=20$ and to 1.00018 ± 0.00011 for $N=1000$, in agreement with the analytical result. For $N=20$, the factor $N-1$ is reproduced as 19.017 ± 0.020. For small N, the distribution of velocities is positively skewed. This is caused by configurations which contain very close pairs having large velocities. For small N, this greatly affects the mean velocity as the weight of a single pair is significant. Note that the dispersion of velocities around the mean is very small even for numbers of particles as small as $N=20$.

In this work, random static distributions of particles were assumed. The important issues are whether this static distribution persists and whether the cloud boundary remains spherical as the cloud sediments and flows. These issues have been discussed in Refs. 2 and 3, but they are not settled definitively and will be examined in the future.

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