Homework 7 (25 points)

Regularization and iterative linear solvers

March 3, 2026

1 The problem

In Lecture 8, the following deconvolution problem is considered:

$$y(t) = \int_0^t \cos(t-s)x(s)\,\mathrm{d}s, \tag{1}$$

where $y(t)$ can be interpreted as the velocity response of a single degree of freedom system to the force excitation $x(t)$.

Assume the response is observed for $t \in [0, 2\pi]$ and it is

$$y(t) = \begin{cases} \sin t & \text{for } 0 \le t \le \frac{\pi}{2} \\ \sin t + \cos t & \text{for } \frac{\pi}{2} < t \le 2\pi. \end{cases}$$

The corresponding unique exact solution is

$$x_{\text{exact}}(t) = \begin{cases} 1 & \text{for } 0 \le t \le \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} < t \le 2\pi. \end{cases} \tag{2}$$

2 Discretization

To solve the system numerically in time-domain, a simple discretization scheme is used of a time grid of T equally spaced time points,

$$t_i = 2\pi\frac{i}{T} = i\Delta t, \qquad i = 1, 2, \ldots, T,$$

where $\Delta t = 2\pi/T$. The system (1) takes then the following form:

$$\mathbf{y} = \mathbf{A}\mathbf{x},$$

where $\mathbf{y}$ is the discretized exact response and $\mathbf{A}$ is a $T \times T$ Toeplitz matrix,

$$\mathbf{y} = \left[\, y(t_i) \,\right]_{i=1}^{T}, \qquad \mathbf{A} = \left[a_{ij}\right], \qquad a_{ij} = \begin{cases} \cos(t_i - t_j)\Delta t & \text{for } i \ge j, \\ 0 & \text{otherwise.} \end{cases}$$

3 Your tasks

To solve this homework you can use any software you like (C/C++, Matlab, Mathematica, etc.), including built-in direct linear solvers, matrix and vector operations, etc. However, do not use built-in regularization capabilities or iterative solvers: you should program them yourself.

In practice, the right-hand side $\mathbf{y}$ is measured, and so it is inevitably contaminated with a measurement error. If the error is assumed to be at the level of p rms, the contaminated measurement data can be denoted by $\mathbf{y}_p$,

$$\mathbf{y}_p = \mathbf{y} + p\frac{\|\mathbf{y}\|}{\sqrt{T}}\mathbf{N},$$

where $\mathbf{N}$ is a vector of realizations of T independent Gaussian random variables $N(0, 1)$. The system takes then the form

$$\mathbf{y}_p = \mathbf{A}\mathbf{x}. \tag{3}$$

1. (10 points) Let $T = 200$, and compute $\mathbf{A}$ and $\mathbf{y}_{0.05}$. Solve the system (3) directly (without regularization) and using standard Tikhonov regularization with $\mathbf{D} = \mathbf{I}$.

   (a) Plot the L-curve for $\alpha \in [0.02, 1]$. Find approximately the optimum value α_{opt} according to the L-curve criterion, that is the value corresponding (more or less) to the corner point.

   (b) Plot and compare the five following solutions: the exact solution (2), the unregularized direct solution and the three regularized solutions corresponding to $\alpha \in \{0.02, \alpha_{\text{opt}}, 1\}$.

   (c) Plot the error of the regularized solution,

   $$\log_{10} \|\mathbf{x}_\alpha - \mathbf{x}_{\text{exact}}\|,$$

   as a function of α, where $\mathbf{x}_\alpha$ is the regularized solution computed for a given α and $\mathbf{x}_{\text{exact}}$ is the discretized exact solution (2). The actual optimum value of α corresponds to the minimum of the curve. However, note that in real cases the exact solution $\mathbf{x}_{\text{exact}}$ is unknown, and you have to trust the L-curve criterion (or others).

2. (10 points) Implement the standard CGLS iterative solver (Lecture 7). The number of iterations plays the role of the regularization parameter. In the exact arithmetic T iterations (the size of the discretized system) would be necessary to compute the exact solution.

   (a) Plot the L-curve, which in this case is defined by the points

   $$(\|\mathbf{y}_{0.05} - \mathbf{A}\mathbf{x}_i\|, \|\mathbf{x}_i\|)$$

   plotted in the log-log scale, where $\mathbf{x}_i$ is the ith iterate (the solution obtained in the ith iteration). Find the optimum number of the iterations according to the L-curve criterion (note that the optimally regularized solution appears very soon in comparison to the size of the system).

(b) Plot and compare the four following solutions: the discretized exact solution $\mathbf{x}_{\text{exact}}$, one of the first iterates, the optimum iterate (found based on the L-curve) and one of the later iterates.

(c) Plot the error of the successive iterates $\mathbf{x}_i$, that is

$$\log_{10} \|\mathbf{x}_i - \mathbf{x}_{\text{exact}}\|,$$

as a function of i. Again, in real cases the exact solution is unknown, so you have to trust the L-curve criterion (or any other).

Which method yields quicker the optimally regularized solution, Tikhonov or CGLS?

3. (5 points) Let $T = 1000$, and compute $\mathbf{A}$, $\mathbf{y}_{0.1}$, $\mathbf{y}_{0.01}$ and $\mathbf{y}_{0.001}$. Compute the corresponding CGLS-based regularized solutions using the L-curve criterion, and plot them together with the exact solution on a single plot. What is the optimum number of the iterations you found for each of the three considered error levels? Try to explain the apparent trend.

E-mail your answers to `ljank@ippt.pan.pl`.