Programming, numerics and optimization

Lecture C-6: Structural reanalysis in statics

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Outline

1. Basics
2. Combined approximations (CA)
3. Virtual distortion method (VDM)
4. VDM — sensitivity reanalysis
5. Homework 10

Outline

1. Basics
   - Sensitivity analysis vs. reanalysis of structures
   - Structural reanalysis and SHM
   - General formulation

Structural parameters and the response

Let $\mathbf{x}$ denote a vector of parameters that define a loaded structure, which can include

- material parameters: Young's moduli, cross-sections, densities, point masses, etc.
- geometric parameters: shape of the boundary, holes, etc. (nodal coordinates in FEM)
- sometimes also excitation parameters: forces, loading masses, etc.

Let $\mathbf{u}(\mathbf{x})$ be the corresponding structural response (static, dynamic, modal characteristics, etc.).

Sensitivity analysis

Sensitivity analysis

The process of computing the gradient and Hessian of the structural responses or of a response-based objective function,

$$\frac{d\mathbf{u}}{d\mathbf{x}}, \frac{d^2\mathbf{u}}{d\mathbf{x}^2}, \qquad \text{or} \qquad \frac{dF(\mathbf{u})}{d\mathbf{x}}, \frac{d^2F(\mathbf{u})}{d\mathbf{x}^2},$$

with respect to $\mathbf{x}$ is called respectively the first- and second-order sensitivity analysis.

Sensitivity analysis

Methods of sensitivity analysis

- finite differences
- direct differentiation
- adjoint variable
- automatic differentiation

Numerical cost of sensitivity analysis

Numerical costs are usually not smaller than

gradient $O(T)$

Hessian $O(nT)$

where

n is the number of parameters (length of $\mathbf{x}$) and

T is the time required to compute the response $\mathbf{u}(\mathbf{x})$.

Often, the costs are much higher: $O(nT)$ and $O(n^2T)$.

Reanalysis of structures

Structural reanalysis

The process of computing the structural response $\mathbf{u}(\mathbf{x} + \Delta\mathbf{x})$, where

- $\Delta\mathbf{x}$ is a given perturbation of the parameter vector, which
  - concerns only a small subset of all parameters and
  - is sometimes assumed to be small,
- the response of the original unmodified structure $\mathbf{u}(\mathbf{x})$ is known,

is called structural reanalysis.

Reanalysis of structures

Reanalysis methods

- combined approximations (Kirsch et al.)
- virtual distortion method (Holnicki-Szulc et al.)
- theorems of structural variations (Majid et al.)

In statics, they are all equivalent to the Sherman–Morrison–Woodbury formulas.

Numerical costs of reanalysis

Numerical cost of an effective reanalysis technique should be much smaller than the costs of a direct computation of the response of the modified structure.

Sensitivity reanalysis of structures

Structural reanalysis...

...is the process of computing the structural response $\mathbf{u}(\mathbf{x} + \Delta\mathbf{x})$, where $\Delta\mathbf{x}$ concerns only a small subset of all parameters and the response of the original unmodified structure $\mathbf{u}(\mathbf{x})$ is known.

Sensitivity reanalysis...

...is the process of computing the gradient or Hessian of $\mathbf{u}(\mathbf{x} + \Delta\mathbf{x})$ with respect to the few parameters being modified, which is usually done using the methods known from standard sensitivity analysis:

- direct differentiation method
- adjoint variable method
- finite differences

Structural reanalysis and health monitoring

Methods of structural reanalysis are often used for fast damage identification in model-based structural health monitoring (SHM):

1. A numerical model of the undamaged structure is obtained and validated.
2. The structure gets damaged. Measurements $\mathbf{u}^{\mathrm{M}}$ of the damaged structure are obtained.
3. A vector $\mathbf{x}$ of structural parameters is chosen that seem to properly capture all the important characteristics of the potential damages.
4. The actual damage is identified by minimizing an objective function of the following type:

$$F(\Delta\mathbf{x}) = \frac{\|\mathbf{u}^{\mathrm{M}} - \mathbf{u}(\mathbf{x} + \Delta\mathbf{x})\|^2}{\|\mathbf{u}^{\mathrm{M}}\|^2},$$

where the identified $\Delta\mathbf{x}_{\min}$ defines the identified damage.

General formulation

Let the unmodified structure obey

$$\mathbf{K}\mathbf{u}^{\mathrm{L}} = \mathbf{f}, \tag{1}$$

where $\mathbf{K}$, $\mathbf{u}^{\mathrm{L}}$ and $\mathbf{f}$ are all known.

Let the modifications be defined[a] by $\Delta\mathbf{K}$ and $\Delta\mathbf{f}$, so that

$$(\mathbf{K} + \Delta\mathbf{K})\,\mathbf{u} = \mathbf{f} + \Delta\mathbf{f}, \tag{2}$$

where $\mathbf{u}$ is the unknown response of the modified structure.

[a] Note that if some degrees of freedom are deleted or added, this formulation cannot be directly used.

The aim is to use (1) in order to quickly solve (2).

Reanalysis methods

$$\mathbf{K}\mathbf{u}^{\mathrm{L}} = \mathbf{f} \tag{1}$$

$$(\mathbf{K} + \Delta\mathbf{K})\,\mathbf{u} = \mathbf{f} + \Delta\mathbf{f} \tag{2}$$

Approximate reanalysis:

- Combined approximations (CA) is a method that computes an approximate solution of (2) using essentially a preconditioned conjugate gradient algorithm and a small number of iterations.

Exact reanalysis:

- Virtual distortion method (VDM) first reduces (2) to an equivalent equation with much fewer unknowns (virtual distortions or pseudo-loads) and then solves it exactly.
- Theorems of structural variations (TSV) seem to be almost directly equivalent to the VDM. However, they were developed independently and use a different terminology.

Outline

2 Combined approximations (CA)

Combined approximations (CA)

Kirsch et al.

$$\mathbf{K}\mathbf{u}^{\mathsf{L}} = \mathbf{f}, \qquad (\mathbf{K} + \Delta\mathbf{K})\,\mathbf{u} = \mathbf{f} + \Delta\mathbf{f}$$

1. Cholesky factorization is applied to the original stiffness matrix $\mathbf{K} = \mathbf{R}^{\mathsf{T}}\mathbf{R}$.
2. The basis is changed using a preconditioning matrix $\mathbf{P}$,

$$\mathbf{u} = \mathbf{P}\tilde{\mathbf{u}},$$

so that the modified equation takes the form

$$\mathbf{P}^{\mathsf{T}}(\mathbf{K} + \Delta\mathbf{K})\,\mathbf{P}\tilde{\mathbf{u}} = \mathbf{P}^{\mathsf{T}}(\mathbf{f} + \Delta\mathbf{f})$$

or, equivalently,

$$\tilde{\mathbf{K}}\tilde{\mathbf{u}} = \tilde{\mathbf{f}},$$

where $\tilde{\mathbf{K}} = \mathbf{P}^{\mathsf{T}}(\mathbf{K} + \Delta\mathbf{K})\,\mathbf{P}$ and $\tilde{\mathbf{f}} = \mathbf{P}^{\mathsf{T}}(\mathbf{f} + \Delta\mathbf{f})$.

Combined approximations (CA)

Kirsch et al.

3. The ideal preconditioning matrix $\mathbf{P}$ would result in the matrix of the modified equation $\tilde{\mathbf{K}} = \mathbf{P}^{\mathsf{T}} (\mathbf{K} + \Delta\mathbf{K})\, \mathbf{P}$ being the identity. However, this would require the Cholesky factorization of $\mathbf{K} + \Delta\mathbf{K}$, which cannot be done quickly. Hence, a reasonable approximation is used:

$$\mathbf{P} = \mathbf{R}^{-1},$$

   where $\mathbf{R}$ is the (upper triangular) Cholesky factor of $\mathbf{K}$.

4. The equation $\tilde{\mathbf{K}}\tilde{\mathbf{u}} = \tilde{\mathbf{f}}$ is solved iteratively by the conjugate gradient method (see Lecture B-2), which minimizes

$$F(\tilde{\mathbf{u}}) = \frac{1}{2}\tilde{\mathbf{u}}^{\mathsf{T}}\tilde{\mathbf{K}}\tilde{\mathbf{u}} - \tilde{\mathbf{u}}^{\mathsf{T}}\tilde{\mathbf{f}}.$$

   The starting point $\tilde{\mathbf{u}}^{(0)}$ (initial approximation) is $\mathbf{R}^{-\mathsf{T}} (\mathbf{f} + \Delta\mathbf{f})$, which corresponds to the solution to $\mathbf{K}\mathbf{u}^{(0)} = \mathbf{f} + \Delta\mathbf{f}$.

Outline

3. Virtual distortion method (VDM)
   - Pseudo-loads
   - Trusses
   - Local pseudo-loads and virtual distortions

Virtual distortion method (VDM) — pseudo-loads

Holnicki-Szulc et al.

Original structure

$$\mathbf{K}\mathbf{u}^{\mathsf{L}} = \mathbf{f}$$

Modified structure

$$(\mathbf{K} + \Delta\mathbf{K})\,\mathbf{u} = \mathbf{f} + \Delta\mathbf{f}$$

The modifications $\Delta\mathbf{K}$ and $\Delta\mathbf{f}$ are modeled with a pseudo-load $\mathbf{p}^0$, which acts in the original unmodified structure,

$$\mathbf{K}\mathbf{u} = \mathbf{f} + \mathbf{p}^0,$$

and which is coupled to the response $\mathbf{u}$,

$$\mathbf{p}^0 = \Delta\mathbf{f} - \Delta\mathbf{K}\mathbf{u}. \tag{3}$$

The original structure is linear, hence

$$\mathbf{u} = \mathbf{u}^{\mathsf{L}} + \mathbf{K}^{-1}\mathbf{p}^0. \tag{4}$$

Virtual distortion method (VDM) — pseudo-loads

Holnicki-Szulc et al.

Substitution of (4) into (3) yields a linear system with the pseudo-load $\mathbf{p}^0$ as the unknown vector,

$$\left(\mathbf{I} + \Delta\mathbf{K}\mathbf{K}^{-1}\right)\mathbf{p}^0 = \Delta\mathbf{f} - \Delta\mathbf{K}\mathbf{u}^{\mathrm{L}}. \qquad (\star)$$

- Elements of $\mathbf{p}^0$ are non-zero only in the degrees of freedom that are directly related to the modifications (involved DOFs).
- For limited modifications, $(\star)$ reduces thus to an equation that is much smaller in size than $\mathbf{K}\mathbf{u}^{\mathrm{L}} = \mathbf{f}$.
- As a result, instead of the full inverse matrix $\mathbf{K}^{-1}$, only a certain matrix $\mathbf{B}$ has to be computed, which
  - contains the responses in the involved DOFs to unit loads in each of the same DOFs (a small submatrix of $\mathbf{K}^{-1}$).
  - is called the influence matrix.

Virtual distortion method (VDM) — pseudo-loads

Holnicki-Szulc et al.

Reanalysis via the VDM:

1. Compute the influence matrix $\mathbf{B}$ (the submatrix of $\mathbf{K}^{-1}$ defined by the involved DOFs).
2. For each given modifications $\Delta\mathbf{K}$ and $\Delta\mathbf{f}$, solve the reduced equation (possibly using the conjugate gradient method),

$$(\mathbf{I} + \Delta\mathbf{K}\mathbf{B})\,\mathbf{p}^0 = \Delta\mathbf{f} - \Delta\mathbf{K}\mathbf{u}^\mathsf{L},$$

   and compute the response via

$$\mathbf{u} = \mathbf{u}^\mathsf{L} + \mathbf{B}\mathbf{p}^0.$$

Virtual distortion method (VDM) — pseudo-loads

Holnicki-Szulc et al.

$$(\mathbf{I} + \Delta\mathbf{K}\mathbf{B})\,\mathbf{p}^0 = \Delta\mathbf{f} - \Delta\mathbf{K}\mathbf{u}^\mathrm{L} \qquad \mathbf{u} = \mathbf{u}^\mathrm{L} + \mathbf{B}\mathbf{p}^0$$

The response of the modified structure is expressed in terms of

- certain local (limited to the involved DOFs) characteristics of the original unmodified structure,
  - the original response $\mathbf{u}^\mathrm{L}$,
  - the influence matrix $\mathbf{B}$ (responses to unit loads),
- and the modifications $\Delta\mathbf{K}$ and $\Delta\mathbf{f}$.

The VDM does not require the full model of the unmodified structure (unlike the CA method). Only a certain reduced model limited to the involved DOFs is used, which directly consists of experimentally obtainable characteristics.

Virtual distortion method (VDM) — pseudo-loads

Holnicki-Szulc et al.

$$(\mathbf{I} + \Delta\mathbf{K}\mathbf{B})\,\mathbf{p}^0 = \Delta\mathbf{f} - \Delta\mathbf{K}\mathbf{u}^{\mathrm{L}} \qquad \mathbf{u} = \mathbf{u}^{\mathrm{L}} + \mathbf{B}\mathbf{p}^0$$

Up to now, pseudo-load $\mathbf{p}^0$ is used to model the modifications.
You might ask: *Where are the distortions?*

Indeed, pseudo-loads are effective, if the modifications

- concern the load $\Delta\mathbf{f}$ (or, in dynamics, the mass) or
- are geometric in nature (nodal co-ordinates).

However, element distortions can be more natural

- in SHM, where
  - the geometry is known and
  - only material properties of finite elements are identified, or
- in modeling of plastic distortions in elastoplastic materials.

Virtual distortion method (VDM) — trusses

Holnicki-Szulc et al.

Stiffness reduction of a truss element is modeled by a single virtual strain distortion ε^0. The stress in the modified element is

$$\mu_i E_i A_i \varepsilon_i, \qquad \text{where} \qquad \mu_i = \frac{E_i + \Delta E_i}{E_i},$$

while in the original element with the distortion ε_i^0 the stress is

$$E_i A_i \left(\varepsilon_i - \varepsilon_i^0\right).$$

If the modeling is proper, both stresses are equal, hence

$$\varepsilon_i^0 = (1 - \mu_i)\varepsilon_i,$$

which is the counterpart of the general

$$\kappa_{ij}^0 = (1 - \mu_i)\kappa_{ij}.$$

Virtual distortion method (VDM) — trusses

Holnicki-Szulc et al.

Strains of the modified structure depend linearly on the distortions,

$$\varepsilon_i = \varepsilon_i^{\mathrm{L}} + \sum_j D_{ij}\varepsilon_j^0,$$

which, substituted into

$$\varepsilon_i^0 = (1-\mu_i)\varepsilon_i,$$

yields a system of linear equations

$$\sum_j \left[\delta_{ij} - (1-\mu_i)D_{ij}\right]\varepsilon_j^0 = \varepsilon_i^{\mathrm{L}},$$

which can be used to compute the distortions, given all μ_i.

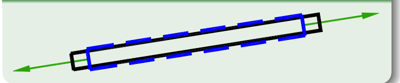
Virtual distortion method (VDM) — virtual distortions

Holnicki-Szulc et al.

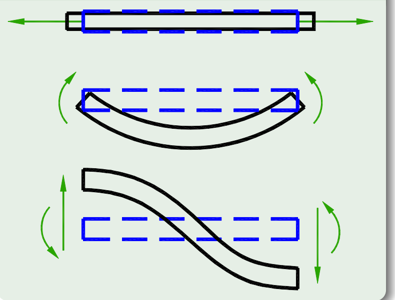
The pseudo-load that models stiffness reduction of a truss element is a pair of self-equilibrated axial forces. It corresponds to a single axial distortion state.

In elements of other types, more distortion states can occur.

Distortion of a truss element



Distortions of a beam element



Virtual distortion method (VDM) — local pseudo-loads

Holnicki-Szulc et al.

Let $\mathbf{L}_i$ be the transformation matrix from the global co-ordinates to the local co-ordinates of the ith finite element,

$$\mathbf{u}_i^{\text{elem}} = \mathbf{L}_i \mathbf{u}, \qquad \mathbf{K}_i = \mathbf{L}_i^{\mathsf{T}} \mathbf{K}_i^{\text{elem}} \mathbf{L}_i,$$

where $\mathbf{u}_i^{\text{elem}}$ are nodal displacements of the ith element in its local co-ordinates, while $\mathbf{K}_i$ and $\mathbf{K}_i^{\text{elem}}$ denote its stiffness matrix in the global and local co-ordinates, respectively.

The global modified stiffness matrix $\mathbf{K}$ and its modification $\Delta\mathbf{K}$ can be thus assembled as

$$\mathbf{K} = \sum_i \mathbf{L}_i^{\mathsf{T}} \mathbf{K}_i^{\text{elem}} \mathbf{L}_i, \qquad \Delta\mathbf{K} = \sum_i \mathbf{L}_i^{\mathsf{T}} \Delta\mathbf{K}_i^{\text{elem}} \mathbf{L}_i.$$

Virtual distortion method (VDM) — local pseudo-loads

Holnicki-Szulc et al.

The global pseudo-load

$$\mathbf{p}^0 = -\Delta\mathbf{K}\mathbf{u}$$

can be expressed as a sum of local nodal pseudo-loads

$$\mathbf{p}^0 = \sum_i \mathbf{L}_i^{\mathsf{T}} \mathbf{p}_i^{0,\mathrm{elem}},$$

where

$$\mathbf{p}_i^{0,\mathrm{elem}} = -\Delta\mathbf{K}_i^{\mathrm{elem}} \mathbf{u}_i^{\mathrm{elem}}$$

is the local nodal pseudo-load that act in the nodes of the ith element of the original unmodified structure in order to model its modification defined by $\Delta\mathbf{K}_i^{\mathrm{elem}}$.

Virtual distortion method (VDM) — local pseudo-loads

Holnicki-Szulc et al.

If the damage is defined as a reduction of stiffness,

$$\mathbf{K}_i^{\text{elem}} + \Delta\mathbf{K}_i^{\text{elem}} = \mu_i \mathbf{K}_i^{\text{elem}}, \qquad \Delta\mathbf{K}_i^{\text{elem}} = -(1-\mu_i)\mathbf{K}_i^{\text{elem}},$$

then the local pseudo-loads $\mathbf{p}_i^{0,\text{elem}}$, originally expressed as

$$\mathbf{p}_i^{0,\text{elem}} = -\Delta\mathbf{K}_i^{\text{elem}} \mathbf{u}_i^{\text{elem}},$$

are related to the original stiffness matrix:

$$\mathbf{p}_i^{0,\text{elem}} = (1-\mu_i)\mathbf{K}_i^{\text{elem}} \mathbf{u}_i^{\text{elem}}.$$

Virtual distortion method (VDM) — virtual distortions

Holnicki-Szulc et al.

The local stiffness matrix $\mathbf{K}_i^{\text{elem}}$ of the ith element has eigenvectors of two kinds only:

1. rigid motion vectors that correspond to zero eigenvalues,
2. local distortion vectors $\mathbf{v}_{ij}$ that correspond to positive eigenvalues λ_{ij}.

The matrix can be expressed in the terms of the distortion vectors,

$$\mathbf{K}_i^{\text{elem}} = \sum_j \lambda_{ij} \mathbf{v}_{ij} \mathbf{v}_{ij}^T.$$

The eigenvector $\mathbf{v}_{ij}$ represents the jth local unit distortion. The corresponding vector of the nodal loads is

$$\hat{\mathbf{p}}_{ij}^{\text{elem}} = \mathbf{K}_i^{\text{elem}} \mathbf{v}_{ij} = \lambda_{ij} \mathbf{v}_{ij}.$$

Virtual distortion method (VDM) — virtual distortions

Holnicki-Szulc et al.

Consider the local nodal pseudo-loads that model stiffness reduction of the $i$the element,

$$\mathbf{p}_i^{0,\mathrm{elem}} = (1-\mu_i)\mathbf{K}_i^{\mathrm{elem}}\mathbf{u}_i^{\mathrm{elem}},$$

and substitute the eigenvalue expansion of $\mathbf{K}_i^{\mathrm{elem}}$ to obtain

$$\begin{aligned}
\mathbf{p}_i^{0,\mathrm{elem}} &= (1-\mu_i)\left(\sum_j \lambda_{ij}\mathbf{v}_{ij}\mathbf{v}_{ij}^{\mathrm{T}}\right)\mathbf{u}_i^{\mathrm{elem}} \\
&= \sum_j \left[(1-\mu_i)\mathbf{v}_{ij}^{\mathrm{T}}\mathbf{u}_i^{\mathrm{elem}}\right](\lambda_{ij}\mathbf{v}_{ij}) \\
&= \sum_j \kappa_{ij}^0\hat{\mathbf{p}}_{ij}^{\mathrm{elem}},
\end{aligned}$$

which is a linear combination of the local nodal loads $\hat{\mathbf{p}}_{ij}^{\mathrm{elem}}$ that correspond to the local unit distortions $\mathbf{v}_{ij}$.

Virtual distortion method (VDM) — virtual distortions

Holnicki-Szulc et al.

Therefore, the local pseudo-load that models stiffness reduction of a finite element,

$$\mathbf{p}_i^{0,\text{elem}} = \sum_j \kappa_{ij}^0 \hat{\mathbf{p}}_{ij}^{\text{elem}},$$

is equivalent to a linear combination $\sum_j \kappa_{ij}^0 \mathbf{v}_{ij}$ of local unit distortions.

The combination coefficients κ_{ij}^0 are fractions of the total distortions κ_{ij},

$$\kappa_{ij}^0 = (1 - \mu_i)\kappa_{ij},$$

where

$$\kappa_{ij} = \mathbf{v}_{ij}^{\mathsf{T}} \mathbf{u}_i^{\text{elem}}.$$

Virtual distortion method (VDM) — virtual distortions

Holnicki-Szulc et al.

The response depends linearly on the virtual distortions κ_{ij}^0,

$$\kappa_{ij} = \kappa_{ij}^{\mathrm{L}} + \sum_{k,l} D_{ijkl}\kappa_{kl}^0,$$

which, if substituted in

$$\kappa_{ij}^0 = (1-\mu_i)\kappa_{ij},$$

yields the following linear system with the virtual distortions as the unknowns

$$\sum_{k,l} \left[\delta_{ik}\delta_{jl} - (1-\mu_i)D_{ijkl}\right]\kappa_{kl}^0 = (1-\mu_i)\kappa_{ij}^{\mathrm{L}}.$$

Virtual distortion method (VDM) — virtual distortions

Holnicki-Szulc et al.

Hence, two sets of equations are finally obtained

Stiffness reductions of elements are modeled by virtual distortions

$$\sum_{k,l} \left[\delta_{ik}\delta_{jl} - (1-\mu_i) D_{ijkl}\right] \kappa^0_{kl} = (1-\mu_i)\kappa^{\mathsf{L}}_{ij}$$

$$\kappa_{ij} = \kappa^{\mathsf{L}}_{ij} + \sum_{k,l} D_{ijkl}\kappa^0_{kl}$$

which are the counterparts of the general formulation

General modifications are modeled by pseudo-loads

$$(\mathbf{I} + \Delta\mathbf{K}\mathbf{B})\,\mathbf{p}^0 = \Delta\mathbf{f} - \Delta\mathbf{K}\mathbf{u}^{\mathsf{L}} \qquad \mathbf{u} = \mathbf{u}^{\mathsf{L}} + \mathbf{B}\mathbf{p}^0$$

Virtual distortion method (VDM) — virtual distortions

Holnicki-Szulc et al.

The advantage of virtual distortions over pseudo-loads lies in

1. a smaller number of the distortions of an element in comparison to the number of its DOFs
2. the intuitiveness of the relation between the stiffness modification and the corresponding virtual distortions (especially apparent for trusses)
3. natural gradation of importance of the virtual distortions (related to the order of the distortion and to the excitation). Simulation or common engineering sense can be often used to determine which distortions are dominant in the response and which can be neglected.

On the other hand, distortions correspond to complex local loads $\hat{\mathbf{p}}_{ij}^{\text{elem}}$, which are hard to be applied experimentally. Thus, distortion-based modeling is usually used in model-based analysis.

Outline

4 VDM — sensitivity reanalysis
- Direct differentiation method
- Adjoint variable method

VDM — sensitivity reanalysis

Holnicki-Szulc et al.

Original and modified structures obey

$$\mathbf{K}\mathbf{u}^{\mathrm{L}} = \mathbf{f}, \qquad (\mathbf{K} + \Delta\mathbf{K})\,\mathbf{u} = \mathbf{f} + \Delta\mathbf{f},$$

where $\Delta\mathbf{K}$ and $\Delta\mathbf{f}$ directly depend on the vector $\mathbf{x}$ of (few) modifications parameters.

The objective function F can depend on $\mathbf{x}$ through the response $\mathbf{u}$ and directly,

$$F\,(\mathbf{u}, \mathbf{x})\,.$$

Given the response $\mathbf{u}$, the aim of sensitivity reanalysis is to quickly compute the gradient (or Hessian) of F with respect to $\mathbf{x}$.

VDM — sensitivity reanalysis

Holnicki-Szulc et al.

The derivative of the objective function, $F(\mathbf{u}, \mathbf{x})$, with respect to a design variable x_i,

$$\frac{dF}{dx_i} = \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial \mathbf{u}} \frac{d\mathbf{u}}{dx_i}, \tag{5}$$

involves the derivatives of the response.

- Direct differentiation method (DDM) computes the derivatives directly and substitutes them in (5).
- Adjoint variable method (AVM) defines and uses an adjoint system to eliminate the derivatives from (5).

VDM — sensitivity reanalysis by the DDM

Holnicki-Szulc et al.

$$\mathbf{u} = \mathbf{u}^{\mathsf{L}} + \mathbf{B}\mathbf{p}^{0} \qquad (\mathbf{I} + \Delta\mathbf{K}\mathbf{B})\,\mathbf{p}^{0} = \Delta\mathbf{f} - \Delta\mathbf{K}\mathbf{u}^{\mathsf{L}}$$

Both equations are differentiated with respect to x_i,

$$\frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{B}\frac{\partial \mathbf{p}^0}{\partial x_i}, \qquad (\mathbf{I} + \Delta\mathbf{K}\mathbf{B})\frac{\partial \mathbf{p}^0}{\partial x_i} = \frac{\partial \Delta\mathbf{f}}{\partial x_i} - \frac{\partial \Delta\mathbf{K}}{\partial x_i}\mathbf{u}.$$

1. The second equation is repeatedly solved for each of the considered design parameters x_i.
2. The result is substituted in the first equation to obtain $\frac{\partial \mathbf{u}}{\partial x_i}$.
3. These derivatives are used to compute the gradient by

$$\frac{dF}{dx_i} = \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial \mathbf{u}}\frac{d\mathbf{u}}{dx_i}.$$

VDM — sensitivity reanalysis by the AVM

Holnicki-Szulc et al.

$$\mathbf{u} = \mathbf{u}^{\mathrm{L}} + \mathbf{B}\mathbf{p}^0 \qquad (\mathbf{I} + \Delta\mathbf{K}\mathbf{B})\,\mathbf{p}^0 = \Delta\mathbf{f} - \Delta\mathbf{K}\mathbf{u}^{\mathrm{L}}$$

Both equations are differentiated with respect to x_i,

$$\frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{B}\frac{\partial \mathbf{p}^0}{\partial x_i}, \qquad (\mathbf{I} + \Delta\mathbf{K}\mathbf{B})\,\frac{\partial \mathbf{p}^0}{\partial x_i} = \frac{\partial \Delta\mathbf{f}}{\partial x_i} - \frac{\partial \Delta\mathbf{K}}{\partial x_i}\mathbf{u}.$$

1. The first equation is substituted in the derivative of F,

$$\frac{dF}{dx_i} = \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial \mathbf{u}}\frac{d\mathbf{u}}{dx_i} = \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial \mathbf{u}}\mathbf{B}\frac{d\mathbf{p}^0}{dx_i}. \tag{6}$$

VDM — sensitivity reanalysis by the AVM

Holnicki-Szulc et al.

2. The second equation is scalarly multiplied by a vector $\boldsymbol{\lambda}$ of adjoint variables and added to (6)

$$\frac{dF}{dx_i} = \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial \mathbf{u}} \mathbf{B} \frac{d\mathbf{p}^0}{dx_i} + \boldsymbol{\lambda}^\mathsf{T} \left[-\left(\mathbf{I} + \Delta\mathbf{K}\mathbf{B}\right) \frac{\partial \mathbf{p}^0}{\partial x_i} + \frac{\partial \Delta\mathbf{f}}{\partial x_i} - \frac{\partial \Delta\mathbf{K}}{\partial x_i}\mathbf{u} \right].$$

3. The terms involving the derivatives of the pseudo-loads are collected together,

$$\frac{dF}{dx_i} = \frac{\partial F}{\partial x_i} + \boldsymbol{\lambda}^\mathsf{T} \left[\frac{\partial \Delta\mathbf{f}}{\partial x_i} - \frac{\partial \Delta\mathbf{K}}{\partial x_i}\mathbf{u} \right] + \left[\frac{\partial F}{\partial \mathbf{u}} \mathbf{B} - \boldsymbol{\lambda}^\mathsf{T} \left(\mathbf{I} + \Delta\mathbf{K}\mathbf{B}\right) \right] \frac{\partial \mathbf{p}^0}{\partial x_i}.$$

VDM — sensitivity reanalysis by the AVM

Holnicki-Szulc et al.

4. The adjoint variables are chosen in such a way that the multiplier vanishes. As a result

$$\frac{dF}{dx_i} = \frac{\partial F}{\partial x_i} + \boldsymbol{\lambda}^{\mathsf{T}} \left(\frac{\partial \Delta \mathbf{f}}{\partial x_i} - \frac{\partial \Delta \mathbf{K}}{\partial x_i} \mathbf{u} \right), \tag{7}$$

   where the adjoint variables satisfies the adjoint equation

$$(\mathbf{I} + \Delta \mathbf{K} \mathbf{B})^{\mathsf{T}} \boldsymbol{\lambda} = \left(\frac{\partial F}{\partial \mathbf{u}} \mathbf{B} \right)^{\mathsf{T}}. \tag{8}$$

Notice that

- Equation (8) has to be solved only once for each objective function, irrespective of the number of the design parameters.
- Equation (7) often simplifies to

$$\frac{dF}{dx_i} = -\boldsymbol{\lambda}^{\mathsf{T}} \frac{\partial \Delta \mathbf{K}}{\partial x_i} \mathbf{u}.$$

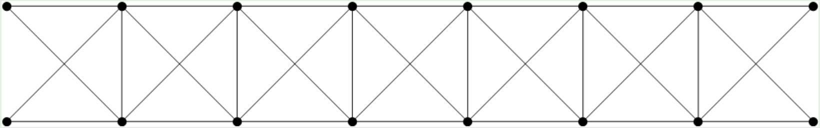
Outline

5 Homework 10

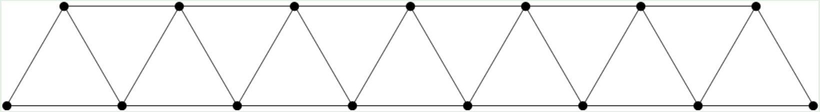
Homework 10 (25 points)

Optimization

Consider the following two 2D trusses:



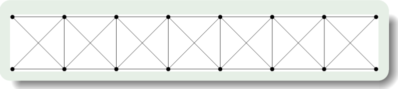
The four outermost nodes are fixed (no translations).

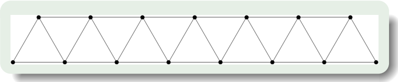


The leftmost bottom node is fixed (no translations). The rightmost node is fixed in vertical direction only (no vertical translations).

Homework 10 (25 points)

Optimization





Assume both trusses are statically loaded with unit vertical loads:

1. the 1st truss in all non-fixed nodes,
2. the 2nd truss in all non-fixed bottom nodes.

Minimize the objective function F (square sum of element stresses) with respect to:

(1) The 1st truss: vertical co-ordinates of all loaded nodes,

(2) The 2nd truss: vertical co-ordinates of all upper nodes,

(3) The 2nd truss: both co-ordinates of all upper nodes,

while the total volume of the structures is constant and all elements have the same cross-section.

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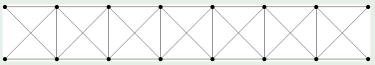
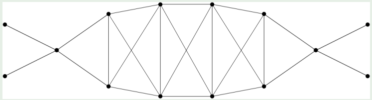
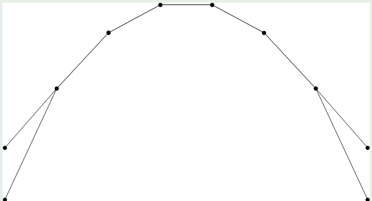
Optimization

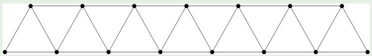


(1) $F_{\mathrm{min}}/F_0 = 73.4\%$

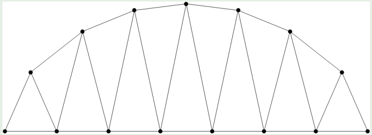


(1) $F_{\mathrm{min}}/F_0 = 19.6\%$

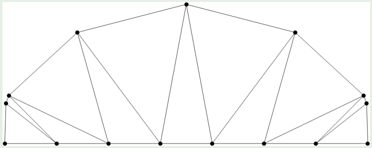




(2) $F_{\mathrm{min}}/F_0 = 51.9\%$



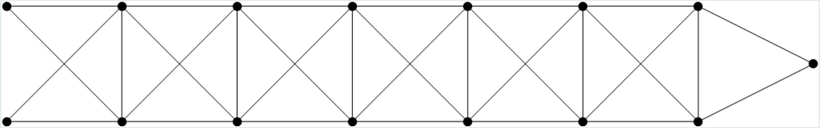
(3) $F_{\mathrm{min}}/F_0 = 41.7\%$



Homework 10 (25 points)

Optimization

Consider the following 2D truss:



The leftmost nodes are fixed.

Assume the truss is statically loaded with unit vertical load at the rightmost node. Minimize the objective function F (square sum of element stresses) with respect to vertical co-ordinates of all nodes besides the tip and the fixed nodes. Keep the total volume of the structures constant. Assume that all elements have the same cross-section. Use any method and software you like. E-mail the resulting truss and the source code to `ljank@ippt.pan.pl`.