Programming, numerics and optimization

Lecture B-5: Linear integral equations

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Outline

Outline

1. Classification

Fredholm integral equations

Fredholm integral equation of the first kind

$$\int_a^b K(x,y)\phi(y)\,dy = f(x), \qquad x \in [a,b]$$

Fredholm integral equation of the second kind

$$\phi(x) - \int_a^b K(x,y)\phi(y)\,dy = f(x), \qquad x \in [a,b]$$

In the above, ϕ is the unknown function, while the kernel K and the right-hand side f are given functions that are usually assumed to be continuous[2]. The interval $[a,b]$ can be substituted with any non-empty compact Jordan measurable subset of $\mathbb{R}^n$.

[2] Or "reasonably" piecewise continuous, weakly singular, etc.

Volterra integral equations

Integral equations of the form

$$\int_a^x K(x,y)\phi(y)\,dy = f(x), \qquad x \in [a,b],$$

$$\phi(x) - \int_a^x K(x,y)\phi(y)\,dy = f(x), \qquad x \in [a,b],$$

with variable limits of integrations are called Volterra integral equations of the first and second kind.

Volterra integral equations can be treated as special cases of Fredholm equations with $K(x,y)=0$ for $y>x$, but they have special properties[3].

[3] For example, Volterra integral equations of the second kind with a continuous (weakly continuous) kernel are always uniquely solvable.

Outline

2 Integral operators
- Operator equations
- Basic notions
- Compact linear operator
- Singular value expansion
- Finite vs. infinite-dimensional case

Integral equations as operator equations

Integral equations can be stated in the form of *operator equations*:

$$\mathcal{K}\phi = f,$$
$$\phi - \mathcal{K}\phi = (\mathcal{I} - \mathcal{K})\,\phi = f,$$

in appropriate normed spaces.

The operator $\mathcal{K}\colon \mathbf{X} \to \mathbf{Y}$ is the integral operator

$$(\mathcal{K}\phi)\,(x) = \int_a^b K(x,y)\phi(y)\,dy$$

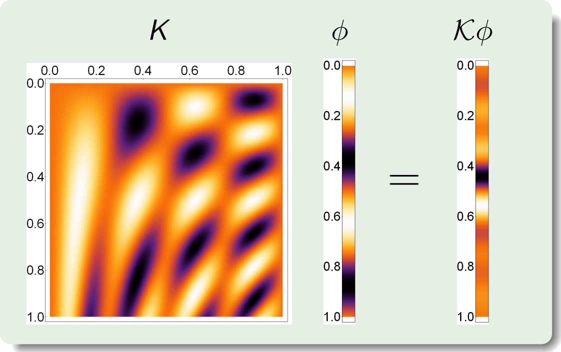
whose domain is an appropriate normed space $\mathbf{X}$ and whose range is contained in an appropriate normed space $\mathbf{Y}$.

Integral operator

At first glance, an integral operator $\mathcal{K}$,

$$(\mathcal{K}\phi)(x) = \int_a^b K(x,y)\phi(y)\,dy,$$

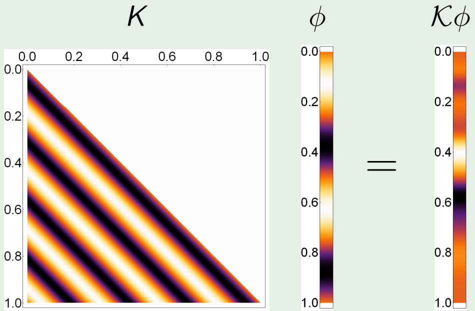
can be understood as an infinite-dimensional generalization of a finite-dimensional linear operator (a matrix).



Difference kernel

The kernel $K(x, y)$ is called the difference kernel, if $K(x, y) = K(x - y)$. Difference kernels occur in time-invariant systems (if x and y represent time) and in space-invariant systems, in case x and y represent space.

Difference kernels can be intuitively understood as infinite-dimensional counterparts of Toeplitz matrices (lower-triangular for Volterra equations).



Basic notions

Integral equations can be analysed directly, but rendering in the form of operator equations allows to use the much more general machinery of functional analysis. However, a lot of related general notions are necessary:

- vector spaces, normed spaces
- open and closed sets
- convergence of a sequence of elements of a normed space
- Cauchy convergence and completeness of normed spaces (which makes them Banach spaces)
- pointwise, norm and uniform convergence of functions
- continuity of operators
- boundedness and compactness of sets and operators
- scalar products and the induced norms (Hilbert spaces)
- dual systems, adjoint operators

Normed space

A norm $\|\cdot\|$ on a vector space $\mathbf{X}$

Let $\mathbf{X}$ be a real/complex vector space. A function $\|\cdot\| : \mathbf{X} \to \mathbb{R}$ with the properties

$$\|\mathbf{x}\| \geq 0 \qquad \text{(positivity)}$$

$$\|\mathbf{x}\| = 0 \text{ iff } \mathbf{x} = \mathbf{0} \qquad \text{(definitness)}$$

$$\|k\mathbf{x}\| = |k|\|\mathbf{x}\| \qquad \text{(homogeneity)}$$

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| \qquad \text{(triangle inequality)}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and all $k \in \mathbb{R}$ (or $\mathbb{C}$) is called a *norm* on $\mathbf{X}$.

Normed space

A vector space with a norm is called a *normed space*.

Normed function space

An integral operator $\mathcal{K}\colon \mathbf{X} \to \mathbf{Y}$ is defined on a normed space (*domain*) $\mathbf{X}$ and has a certain range $\mathcal{K}(\mathbf{X})$, which is contained in the normed space $\mathbf{Y}$. These are often the linear space $C[a, b]$ of continuous real (or complex) valued functions defined on the interval $[a, b]$ and furnished with either the maximum or the mean square norms:

$$\|\phi\|_\infty = \max_{x\in[a,b]} |\phi(x)|, \qquad \|\phi\|_2 = \left(\int_a^b |\phi(x)|^2 dx \right)^{\frac{1}{2}}.$$

In the case of the mean square norm, the requirement of continuity is sometimes dropped, which yields the $L^2[a, b]$ space.

Notice that these function spaces are infinite-dimensional.

Linear operator

Linear operator

An operator $\mathcal{K}\colon \mathbf{X} \to \mathbf{Y}$ from a vector space $\mathbf{X}$ into a vector space $\mathbf{Y}$ is called linear if for all $\phi, \psi \in \mathbf{X}$ and all $a, b \in \mathbb{R}$ (or $\mathbb{C}$)

$$\mathcal{K}(a\phi + b\psi) = a\mathcal{K}\phi + b\mathcal{K}\psi.$$

All integral operators of the form

$$(\mathcal{K}\phi)(x) = \int_a^b K(x,y)\phi(y)\,dy$$

are obviously linear.

Bounded set and bounded linear operator

Bounded set

A subset of a normed space is called bounded if it is contained in a ball of a finite radius.

Bounded linear operator

A linear operator $\mathcal{K}: \mathbf{X} \to \mathbf{Y}$ from a normed space $\mathbf{X}$ into a normed space $\mathbf{Y}$ is called bounded if

- there exists a positive number C such that for all $\phi \in \mathbf{X}$

$$\|\mathcal{K}\phi\| \leq C\|\phi\|.$$

- or, equivalently, if it maps bounded sets in $\mathbf{X}$ into bounded sets in $\mathbf{Y}$.

Compact set

Compact set

A subset $A \subset \mathbf{X}$ of a normed space $\mathbf{X}$ is called compact, if

- every open covering of A contains a finite subcovering.
- every sequence of elements from A contains a subsequence converging to an element in A.

Bounded, closed and finite-dimensional subsets of normed spaces are compact.

> In other words, a finite-dimensional subset of a normed space is compact if and only if it is bounded and closed.

A subset of a normed space is called relatively compact, if its closure is compact.

- A finite-dimensional subsets of a normed space is relatively compact if and only if it is bounded.

Compact linear operator

Compact linear operator

A linear operator $\mathcal{K}\colon \mathbf{X} \to \mathbf{Y}$ from a normed space $\mathbf{X}$ into a normed space $\mathbf{Y}$ is called compact, if

- for each bounded sequence ϕ_n in $\mathbf{X}$ the sequence $\mathcal{K}\phi_n$ contains a convergent subsequence in $\mathbf{Y}$ or, alternatively,
- each sequence from the set $\{\mathcal{K}\phi\colon \phi \in \mathbf{X}, \|\phi\| \leq 1\}$ contains a convergent subsequence or, alternatively,
- it maps bounded sets in $\mathbf{X}$ into relatively compact sets in $\mathbf{Y}$.

- Compact linear operators are bounded and continuous.
- Products of two bounded linear operators are compact if at least one of them is compact.
- Linear combinations of compact linear operators are compact.

Compactness of integral operators

A very important fact is that *almost all integral operators used in practice are compact*.

If the kernel $K(x, y)$ is a continuous[a] function $K : [a, b] \times [a, b] \to \mathbb{R}$ (or $\mathbb{C}$), then the corresponding integral operator $\mathcal{K}$ defined by

$$(\mathcal{K}\phi)(x) = \int_a^b K(x, y)\phi(y)\, dy,$$

is compact in $C[a, b]$ and/or $L^2[a, b]$.

[a]The kernel may be also "reasonably" piecewise continuous, weakly singular, square integrable, etc.

Compact integral operators

Intuitively (or in physical terms), a compact integral operator $\mathcal{K}$ has a smoothing effect, that is $\mathcal{K}$ damps the high-frequency components in ϕ, so that $\mathcal{K}\phi$ is substantially more smooth than ϕ.

This smooting effect is illustrated by the Riemann-Lebesgue lemma: if k is integrable on $[a, b]$, then

$$\int_a^b k(x) \sin(\lambda x)\, dx \xrightarrow{\lambda \to \infty} 0.$$

Eigenvalues and spectrum of a compact integral operator

Eigenvalue and spectrum

Let $\mathcal{K}\colon \mathbf{X} \to \mathbf{X}$ be a compact linear operator mapping a normed space $\mathbf{X}$ into itself. A complex number λ is called an eigenvalue of $\mathcal{K}$, if there exists a non-zero $\phi \in \mathbf{X}$, such that

$$\mathcal{K}\phi = \lambda\phi.$$

The spectrum $\mu(\mathcal{K})$ of $\mathcal{K}$ is defined as a set of its all eigenvalues.

Spectral radius

The spectral radius $r(\mathcal{K})$ of $\mathcal{K}$ is defined as

$$r(\mathcal{K}) = \sup_{\lambda\in\mu(\mathcal{K})} |\lambda|.$$

Singular value expansion

A continuous counterpart of the singular value decomposition (SVD) of a matrix, $\mathbf{A} = \sum_i \sigma_i \mathbf{u}_i \mathbf{v}_i^T$, is the singular value expansion (SVE) of the kernel of a compact linear integral operator[4].

Singular Value Expansion

For any square integrable kernel $K\colon [a,b] \times [a,b] \to \mathbb{R}$ (or $\mathbb{C}$),

$$K(x,y) = \sum_{i=1}^{\operatorname{rank} K} \sigma_i u_i(x) v_i(y),$$

where σ_i are the singular values of K, and u_i and v_i are the singular functions of K. The singular values are all positive, ordered in the nonincreasing order and decay to zero, while the singular functions form orthonormal systems, that is $\langle u_i, u_j \rangle = \langle v_i, v_j \rangle = \delta_{ij}$. For non-degenerate kernels, rank $K = \infty$.

[4] In fact, the SVE exists for all compact linear operators on Hilbert spaces.

Finite vs. infinite-dimensional normed spaces

In finite-dimensional normed spaces

- all norms are equivalent (w.r.t. convergence of sequences).
- a set is compact if and only if it is bounded and closed.
- all linear operators are bounded, continuous and compact (compactness is equivalent to boundedness).
- in particular, the identity operator $\mathcal{I}$ is compact.
- each linear operator can be represented as a multiplication by a certain matrix.
- a linear operator from $\mathbf{X}$ into itself is surjective if and only if it is injective (the dimensions must match).
- the SVD exists for all matrices (linear operators). There is a finite number of singular values.

Finite vs. infinite-dimensional normed spaces

In infinite-dimensional normed function spaces

- two norms need not be equivalent (like $\|\cdot\|_\infty$ and $\|\cdot\|_2$).
- a set can be bounded and closed, but not compact.
- a linear operator need not be bounded, continuous or compact (however, it is continuous if and only if it is bounded).
- in particular, the identity operator $\mathcal{I}$ is linear, continuous and bounded, but not compact.
- not all linear operators can be represented in the form of a typical integral operator.
- the properties of surjectivity and injectivity of a linear operator are unrelated to each other.
- the SVE exists for compact operators. There can be infinitely many singular values, which decay to zero.

Infinite-dimensional normed spaces

In particular, the facts that

- an integral operator with a continuous[5] kernel is compact,
- the identity operator is not compact,

justify the difference between the integral equations of the first and second kind.

$$\mathcal{K}\phi = f,$$
$$\phi - \mathcal{K}\phi = (\mathcal{I} - \mathcal{K})\,\phi = f,$$

If the operator $\mathcal{K}$ is compact, then the operator $\mathcal{I} - \mathcal{K}$ is not compact, and so they have very different properties.

[5] "reasonably" piecewise continuous, weakly singular, square integrable, etc.

Outline

3 Integral equations of the second kind
- Riesz theory
- Volterra integral equations of the second kind
- Fredholm alternative

Riesz theorem

Let $\mathcal{K}\colon \mathbf{X} \to \mathbf{X}$ be a compact linear operator and $\mathbf{X}$ a normed space. Consider the homogenous equation

$$\phi - \mathcal{K}\phi = 0. \tag{$\star$}$$

Then either

Equation ($\star$) has only the trivial solution $\phi = 0$ and the inhomogeneous equation

$$\phi - \mathcal{K}\phi = f$$

has a unique solution $\phi \in \mathbf{X}$, which depends continuously on $f \in \mathbf{X}$. That is, the inverse operator $(\mathcal{I} - \mathcal{K})^{-1}$ exists and is bounded and continuous, so that the inhomogeneous equation is well-posed.

Or...

Riesz theorem

Or

Equation $(\star)$ has a finite number of linearly independent nontrivial solutions and the inhomogeneous equation

$$\phi - \mathcal{K}\phi = f$$

is either unsolvable or its general solution is of the form

$$\phi = \hat{\phi} + \sum_{i=1}^{m} \alpha_i \phi_i,$$

which is a sum of a particular solution $\hat{\phi}$ and a linear combination of the finite number of the linearly independent solutions to $(\star)$.

Riesz theory

As in the case of a square matrix $\mathbf{A}$ and the related finite-dimensional mapping $\mathbf{A}\colon \mathbb{R}^n \to \mathbf{R}^n$, the Riesz theory states that the operator $\mathcal{I} - \mathcal{K}\colon \mathbf{X} \to \mathbf{X}$ with a compact $\mathcal{K}$ is surjective if and only if it is injective.

In other words, the Riesz theory allows to deduce existence from uniqueness of the solution to the operator equation of the second kind

$$\phi - \mathcal{K}\phi = f$$

with a compact $\mathcal{K}$, which makes it similar to finite-dimensional equations $\mathbf{Ax} = \mathbf{b}$ with a square matrix $\mathbf{A}$.

Volterra integral equations of the second kind

In particular, the Riesz theorem allows to deduce that

The Volterra integral equation of the second kind with a continuous (weakly continuous, etc.) kernel K,

$$\phi(x) - \int_a^x K(x,y)\phi(y)\,dy = f(x), \qquad x \in [a,b]\,,$$

has a unique solution $\phi \in C[a,b]$ for each right-hand side $f \in C[a,b]$.

Fredholm alternative

Let K be a continuous[a] integral kernel.

[a] "reasonably" piecewise continuous, weakly singular, square integrable, etc.

Consider the inhomogeneous integral equations

$$\phi(x) - \int_a^b K(x,y)\phi(y)\,dy = f(x), \qquad x \in [a,b],$$

$$\psi(x) - \int_a^b \overline{K(y,x)}\psi(y)\,dy = g(x), \qquad x \in [a,b],$$

and the related homogeneous integral equations

$$\phi(x) - \int_a^b K(x,y)\phi(y)\,dy = 0, \qquad x \in [a,b],$$

$$\psi(x) - \int_a^b \overline{K(y,x)}\psi(y)\,dy = 0, \qquad x \in [a,b].$$

Fredholm alternative

Either

The homogeneous integral equations have only the trivial solutions $\phi = \psi = 0$ and the inhomogeneous integral equations have unique solutions $\phi, \psi \in C[a, b]$ for each right-hand side $f \in C[a, b]$.

or

The homogeneous integral equations have the same finite number of linearly independent solutions and the inhomogeneous integral equations have solutions if and only if the right-hand sides $f, g \in C[a, b]$ satisfy

$$\int_a^b f(x)\overline{\psi(x)}\, dx = 0, \qquad \int_a^b \phi(x)\overline{g(x)}\, dx = 0,$$

for all solutions ϕ, ψ of the homogeneous equations.

Outline

4. Integral equations of the first kind
   - Inherent ill-posedness
   - On some Volterra integral equations of the first kind
   - Singular value expansion

Inherent ill-posedness

Compact linear operators on infinite-dimensional normed spaces cannot have a bounded inverse.

Let $\mathcal{K}$ be a compact operator and assume that its inverse $\mathcal{K}^{-1}$ exists and is bounded. Then the product of the two operators, $\mathcal{K}\mathcal{K}^{-1}$ would be compact (as a product of a compact and a bounded operator). But it is the identity operator $\mathcal{I}$, which is not compact in an infinite-dimensional space.

Inherent ill-posedness

Inverse of a compact linear operator on an infinite-dimensional normed space is linear, unbounded and non-continuous.

As a result, integral equations of the first kind are inherently ill-posed (extremely ill-conditioned).

Well-posed problem (J. Hadamard 1902)

A problem is well-posed, if its solution

1. exists,
2. is unique,
3. continuously depends on the input data.

A well-posed problem can be ill-conditioned. Many practical (inverse) problems are not only ill-conditioned, but also ill-posed.

On some Volterra integral equations of the first kind

In some cases, Volterra integral equations of the first kind,

$$\int_a^x K(x,y)\phi(y)\,dy = f(x), \qquad x \in [a,b], \qquad (\star)$$

can be handled by converting them into an equivalent integral equation of the second kind. Assume that $f(a) = 0$ and $K(x,x) \neq 0$ for $x \in [a,b]$.

1. If the derivatives f' and $\partial K/\partial x$ exist and are continuous, then differentiation of $(\star)$ with respect to x yields

$$\phi(x) + \int_a^x \frac{\frac{\partial K(x,y)}{\partial x}}{K(x,x)}\phi(y)\,dy = \frac{f'(x)}{K(x,x)}, \qquad x \in [a,b].$$

On some Volterra integral equations of the first kind

2. If the derivative $\partial K/\partial y$ exists and is continuous, then an integration by parts of $(\star)$ yields

$$\psi(x) - \int_a^x \frac{\frac{\partial K(x,y)}{\partial y}}{K(x,x)}\psi(y)\,dy = \frac{f(x)}{K(x,x)}, \qquad x \in [a,b],$$

where

$$\psi(x) = \int_a^x \phi(y)\,dy.$$

Singular value expansion

The ill-posedness of a compact linear operator is revealed by its SVE and an analysis of the decay rate of the singular values.

For any square integrable kernel $K\colon [a,b]\times[a,b]\to\mathbb{R}$ (or $\mathbb{C}$),

$$K(x,y) = \sum_{i=1}^{\operatorname{rank} K} \sigma_i u_i(x) v_i(y),$$

where σ_i are the *singular values* of K, and u_i and v_i are the *singular functions* of K. The singular values are all positive, ordered in the nonincreasing order and decay to zero, while the singular functions form orthonormal systems, that is

$$\langle u_i, u_j\rangle = \langle v_i, v_j\rangle = \delta_{ij}.$$

For degenerate kernels, the upper summation limit (rank K) is finite.

Singular value expansion

Just as in the case of the SVD, $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\mathsf{T} = \sum_i \sigma_i \mathbf{u}_i \mathbf{v}_i^\mathsf{T}$, where

$$\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i,$$

in the case of the SVE of the kernel K

$$\mathcal{K} v_i = \sigma_i u_i.$$

The unique solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$, if exists, can be expressed as

$$\mathbf{x} = \sum_{i=1}^{\operatorname{rank} \mathbf{A}} \frac{\mathbf{v}_i \mathbf{u}_i^\mathsf{T}}{\sigma_i} \mathbf{y}.$$

Similarly, the solution of the integral equation of the first kind $\mathcal{K}\phi = f$, if exists, can be expressed as

$$\phi = \sum_{i=1}^{\operatorname{rank} K} \frac{v_i \langle u_i, f \rangle}{\sigma_i}.$$

Singular value expansion

- In the SVD, the ratio between the largest and the smallest singular value is a numerical measure of ill-conditioning of the matrix.
- In case of a compact integral operator, there are usually infinitely many singular values, which decay to zero. As a result, the related integral equation of the first kind, $\mathcal{K}\phi = f$, is unbounded and ill-posed.
- As $\mathcal{K}v_i = \sigma_i u_i$, the operator $\mathcal{K}$ (or its "smoothing effect") can be characterized by the decay rate of its singular values:
  - The faster the singular values decay to zero, the more "smoothing" is the kernel.
  - In practice, the smaller singular value σ_i, the more oscillatory are the singular functions u_i and v_i.

Singular value expansion

The formula for the solution to $\mathcal{K}\phi = f$,

$$\phi = \sum_{i=1}^{\text{rank}\, K} \frac{v_i \langle u_i, f \rangle}{\sigma_i},$$

clearly illustrates the ill-posed nature of the equation:

- The larger i, the more amplified is the corresponding spectral component $\langle u_i, f \rangle \, u_i$ of the right-hand side f.
- If the solution exists, the series is convergent, and so $\sigma_i^{-1} \langle u_i, f \rangle \xrightarrow{i \to \infty} 0$. In line with the Riemann-Lebesgue lemma, and as the singular functions are increasingly more oscillatory, the right-hand side f must be "well-behaved" for large i (smooth enough).
- However, the noise overlaid on f is often less "well-behaved" (smooth) than f. As a result, for large enough i, the noise can dominate in the corresponding components of the solution.

Outline

5. Selected methods of numerical solution
   - Successive approximations (+example)
   - Quadrature/Nyström methods (+example)
   - Collocation methods
   - Kernel approximations

Successive approximations

Let $\mathcal{K}\colon \mathbf{X} \to \mathbf{X}$ be a bounded linear operator mapping a Banach[a] space $\mathbf{X}$ into itself with the spectral radius[b] $r(\mathcal{K}) < 1$. Then for all $f \in \mathbf{X}$ the *successive approximations*

$$\phi_{i+1} = \mathcal{K}\phi_i + f, \qquad i = 0, 1, 2, \ldots,$$

with arbitrary[c] $\phi_0 \in \mathbf{X}$ converge to the unique solution ϕ of

$$\phi - \mathcal{K}\phi = f.$$

[a] A complete normed space is called a Banach space (for example, $C[a, b]$ with $\|\cdot\|_\infty$ or $L^2[a, b]$ with $\|\cdot\|_2$). Successive approximations ϕ_i are a Cauchy sequence and completeness ensures its convergence. Every incomplete normed space can be uniquely completed to a Banach space.

[b] A weaker condition is $\|\mathcal{K}\| < 1$.

[c] Often $\phi_0 = f$.

Successive approximations

In other words, $\mathcal{I} - \mathcal{K}$ has a bounded inverse operator on $\mathbf{X}$, which can be expressed by the *von Neumann series*:

$$(\mathcal{I} - \mathcal{K})^{-1} = \sum_{i=0}^{\infty} \mathcal{K}^i.$$

This is similar to the convergence of a geometric series $\sum_i s^i$ with $|s| < 1$.

Successive approximations

In particular, each Volterra integral operator,

$$\mathcal{K}\phi(x) = \int_a^x K(x,y)\phi(y)\,dy, \qquad x \in [a,b],$$

has a spectral radius $r(\mathcal{K}) = \{0\}$. Thus,

Volterra integral equations of the second kind,

$$\phi(x) - \int_a^x K(x,y)\phi(y)\,dy = f(x), \qquad x \in [a,b],$$

can be always solved by successive approximations:

$$\phi_{i+1}(x) = f(x) + \int_a^x K(x,y)\phi_i(y)\,dy, \qquad i = 0, 1, 2, \ldots$$

Successive approximations

(A simplistic) example

Consider equation of motion of a simple undamped 1-DOF system,

$$\ddot{x}(t) + x(t) = f(t). \qquad (\star)$$

The acceleration $a(t) := \ddot{x}(t)$ is measured for $t \in [0, 10]$.

$$a(t) = f(t) - \int_0^t f(\tau) \sin(t - \tau) \, \mathrm{d}\tau, \qquad (\star\star)$$

where the first term is due to Newton's second law, and the second term is the convolution with the impulse-response.

The inverse problem is to compute the excitation $f(t)$, given the measured acceleration: solve $(\star\star)$, given $a(t)$[a].

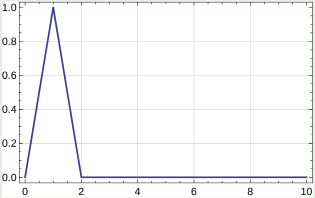
[a] Here, it is easier to compute the excitation directly from $(\star)$. However, consider a multi-DOF problem with a, perhaps, measured impulse-response...

Successive approximations

Example

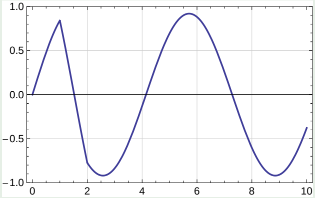
Exact excitation (assumed to be unknown)

$$f(t) = \begin{cases} t & t \in [0,1), \\ 2-t & t \in [1,2), \\ 0 & t \in [2,10]. \end{cases}$$



Exact measurement (given analytically)

$$a(t) = \begin{cases} \sin t & t \in [0,1), \\ 2\sin(1-t) + \sin t & t \in [1,2), \\ 4(\sin 0.5)^2 \sin(1-t) & t \in [2,10]. \end{cases}$$



Successive approximations

Example

Successive approximations

By a rearrangement of the original equation,

$$a(t) = f(t) - \int_0^t f(\tau)\sin(t-\tau)\,\mathrm{d}\tau,$$

the formula for successive approximations is obtained:

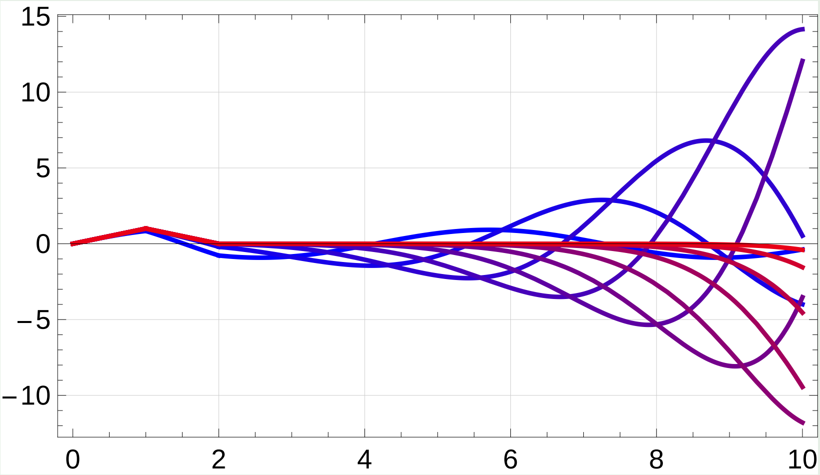
$$\begin{aligned} f_0(t) &:= 0, \\ f_{n+1}(t) &:= a(t) + \int_0^t f_n(\tau)\sin(t-\tau)\,\mathrm{d}\tau, \end{aligned}$$

Notice that $f_1(t) = a(t)$.

Successive approximations

Example

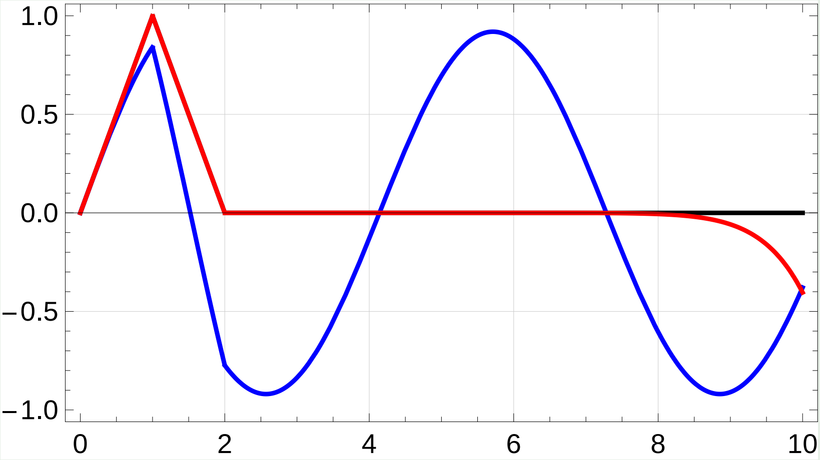
successive approximations: $f_1(t)$, ..., $f_{11}(t)$



Successive approximations

Example

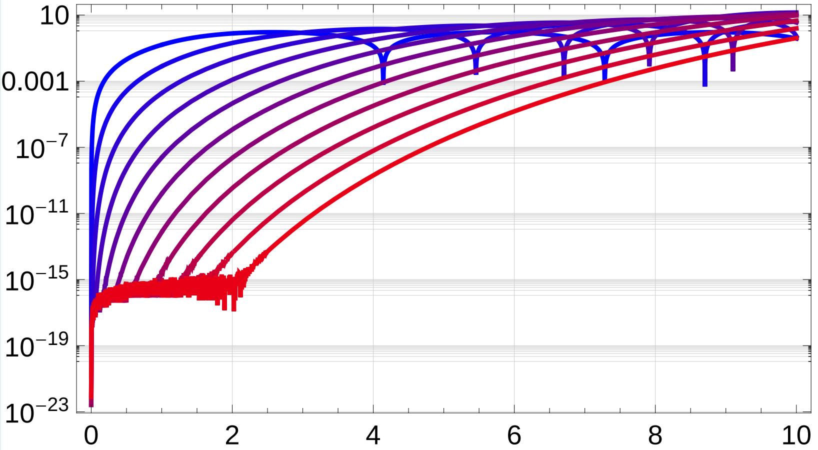
$f_1(t)$, $f_{11}(t)$ and $f(t)$



Successive approximations

Example

errors $|f_n(t) - f(t)|$ in log scale



Quadrature/Nyström methods

Quadrature or Nyström methods are used for the approximate solutions of integral equations of the *second kind* with continuous or weakly singular kernels. They are based on the following standard formula for numerical integration:

$$\int_a^b h(x)\,dx \approx \sum_{j=0}^{n} \alpha_{n,j} h(x_{n,j}),$$

where $x_{n,j} \in [a, b]$ are the quadrature points and $\alpha_{n,j}$ are the quadrature weights. Different quadrature rules give rise to different versions of the method.

Quadrature/Nyström methods

In the integral equation of the second kind,

$$\phi(x) - \int_a^b K(x,y)\phi(y)\,dy = f(x), \qquad x \in [a,b],$$

the integration is approximated numerically:

$$\phi_n(x) - \sum_{j=0}^{n} \alpha_{n,j} K(x, x_{n,j})\phi_n(x_{n,j}) = f(x), \qquad x \in [a,b].$$

Quadrature/Nyström methods

Let ϕ_n be a solution of

$$\phi_n(x) - \sum_{j=0}^{n} \alpha_{n,j} K(x, x_{n,j}) \phi_n(x_{n,j}) = f(x), \qquad x \in [a, b]. \qquad (\star)$$

The values $\phi_n(x_{n,i})$ at the quadrature points can be obtained by solving the following finite-dimensional linear system:

$$\phi_n(x_{n,i}) - \sum_{j=0}^{n} \alpha_{n,j} K(x_{n,i}, x_{n,j}) \phi_n(x_{n,j}) = f(x_{n,i}).$$

Conversely, given the values $\phi_n(x_{n,i})$, equation $(\star)$ is satisfied by

$$\phi_n(x) = f(x) + \sum_{j=0}^{n} \alpha_{n,j} K(x, x_{n,j}) \phi_n(x_{n,j}), \qquad x \in [a, b].$$

Quadrature/Nyström methods

Example

Divide the time interval $[0, 10]$ evenly into $n+1$ quadrature points,

$$t_{j,n} := 10\frac{j}{n}, \qquad j = 0, 1, \ldots, n,$$

and use the quadrature weights

$$\alpha_{j,n} = \begin{cases} 0.5\frac{10}{n} & \text{for } j = 0, \\ 1.0\frac{10}{n} & \text{for } j = 1, 2, \ldots, n-1 \\ 0.5\frac{10}{n} & \text{for } j = n, \end{cases}$$

which amount to the linear interpolation between the quadrature points.

Quadrature/Nyström methods

Example

The original equation,

$$a(t) = f(t) - \int_0^t f(\tau)\sin(t-\tau)\,\mathrm{d}\tau,$$

is transformed into the following discrete system

$$a\left(10\frac{i}{n}\right) = f_n\left(10\frac{i}{n}\right) - \sum_{j=0}^{n} \alpha_{n,j} K\left(10\frac{i}{n}, 10\frac{j}{n}\right) f_n\left(10\frac{j}{n}\right) \qquad (\star)$$

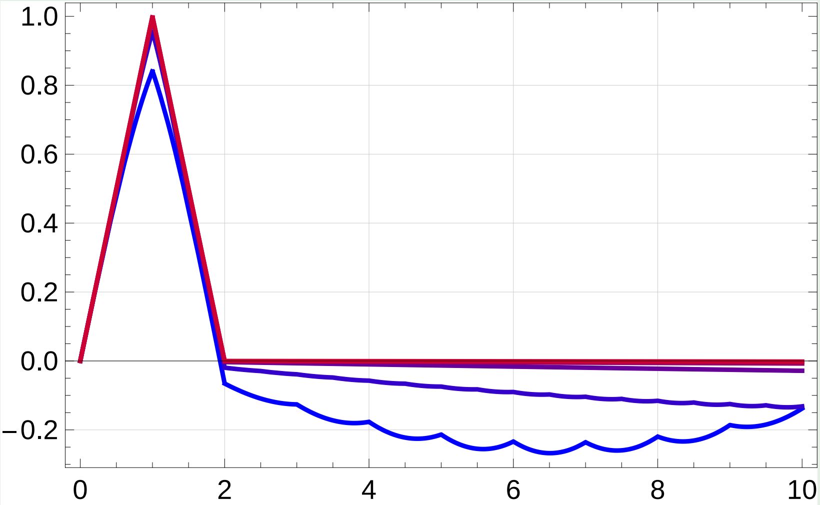
with $n+1$ unknowns $f_n(10\frac{i}{n})$. The kernel $K(t,\tau) = \sin(t-\tau)$ for $t \geq \tau$ and 0 otherwise. Solved $(\star)$, the unknown excitation is

$$f_n(t) = a(t) + \sum_{j=0}^{n} \alpha_{n,j} K\left(t, 10\frac{j}{n}\right) f_n\left(10\frac{j}{n}\right).$$

Quadrature/Nyström methods

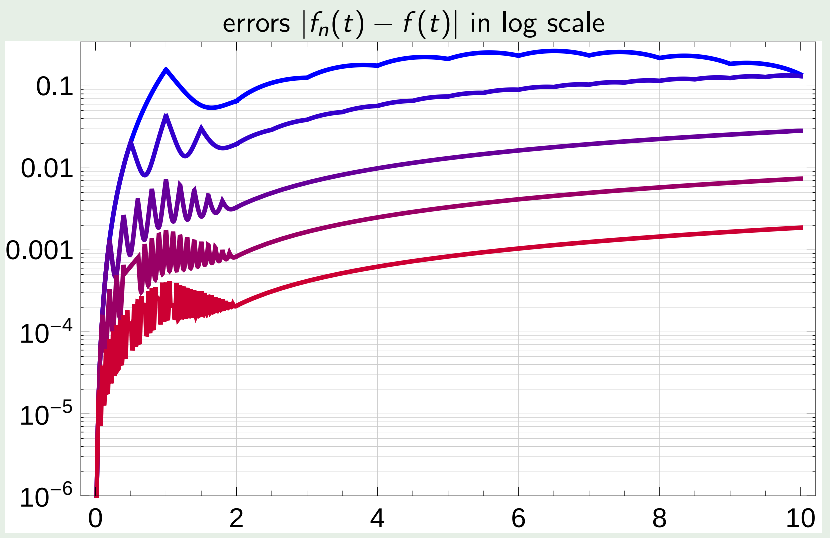
Example

$f_n(t)$ for $n = 10, 20, 50, 100, 200$



Quadrature/Nyström methods

Example



Collocation methods

Projection methods

Collocation method belong to the wider class of projection methods, which solve operator equation[6]

$$\mathcal{L}\phi = f,$$

by projecting it onto finite n-dimensional subspaces $\mathbf{Y}_n$,

$$\mathcal{P}_n \mathcal{L} \phi_n = \mathcal{P}_n f,$$

where $\mathcal{P}_n \colon \mathbf{Y} \to \mathbf{Y}_n$ is a projection operator.

[6] which can be of the first or second form

Collocation methods

In collocation methods,

- the finite n-dimensional subspaces are often generated via splines or trigonometric interpolations,
- the equation is required to be satisfied only at a finite number n of collocation points,

$$\mathcal{L}\phi_n(x_{n,i}) = f(x_{n,i}), \qquad i = 1, 2, \ldots \qquad (\star)$$

If ϕ_n is expressed as a linear combination,

$$\phi_n(x) = \sum_{j=1}^{n} \beta_j u_j(x),$$

equation $(\star)$ is equivalent to the linear system

$$\sum_{j=1}^{n} \beta_j \mathcal{L} u_j(x_{n,i}) = f(x_{n,i}), \qquad i = 1, 2, \ldots$$

Kernel approximations

Kernel approximation methods approximate the original kernel K of an integral equation of the second kind with a degenerate kernel K_n of a finite rank n:

$$K(x,y) \approx K_n(x,y) = \sum_{j=1}^{n} u_j(x)v_j(y).$$

Substitution into an integral equation of the second kind,

$$\phi(x) - \int_a^b K(x,y)\phi(y)\,dy = f(x), \qquad x \in [a,b],$$

yields

$$\phi_n(x) - \sum_{j=1}^{n} u_j(x) \int_a^b v_j(y)\phi_n(y)\,dy = f(x), \qquad x \in [a,b],$$

Kernel approximations

...that is

$$\phi_n(x) - \sum_{j=1}^{n} u_j(x) V_{n,j} = f(x), \qquad x \in [a, b], \qquad (\star)$$

where

$$V_{n,j} = \int_a^b v_j(y)\phi_n(y)\, dy.$$

Equation $(\star)$ is multiplied by $v_i(x)$ and integrated, which yields the following linear n-dimensional system

$$V_{n,i} - \sum_{i=1}^{n} \left[\int_a^b u_j(x) v_i(x)\, dx \right] V_{n,j} = \int_a^b f(x) v_i(x)\, dx \qquad (\star\star)$$

with the unknowns V_i, $i = 1, 2, \ldots, n$. Given the solution to $(\star\star)$, the approximate solution ϕ_n is obtained from $(\star)$ as

$$\phi_n(x) = f(x) + \sum_{j=1}^{n} V_{n,j} u_j(x).$$

Outline

Further reading

- P. Ch. Hansen, Discrete Inverse Problems: Insights and Algorithms, SIAM 2010.
- P. Ch. Hansen, Rank-deficient and discrete ill-posed problems: numerical aspects of linear inversion, SIAM 1998.
- R. Kress, Linear integral equations, 2nd ed., Springer 1999.