Paradox of the particle's trajectory moving on a string

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Abstract

The paper deals with the paradoxical property of the solution of string vibration under a moving mass. Solutions published up to date do not exhibit sufficient simplicity and can not be applied to the investigation in a whole range of the mass speed, also in over-critical range. We propose the formulation of the problem that allows us to reduce the problem to the second order matrix differential equation. Its solution is characteristic of all features of the critical, sub-critical and over-critical motion. Results exhibit discontinuity of the mass trajectory at the end support point. This fact has not been previously reported in the literature. The closed solution in the case of massless string is analysed and the discontinuity is proved. Numerical results obtained for inertial string demonstrate similar features. Small vibrations are analysed and that is why the effect discussed in the paper is of pure mathematical interest. However, the phenomenon results in complexity in discrete solutions.

keywords: moving mass, vibrations of string, inertial load

1 Introduction

Inertial loads moving on strings and beams with the sub or super critical speed are of special interest. Theoretical solutions are applied to many practical problems: train-track interaction, vehicle-bridge interaction, pantograph collectors in railways, magnetic rails, guideways in robotic solutions, etc. The problem has been widely treated in literature. Attempts for a solution to the problem started in the middle of the 19th century. However, till now we do not have the complete and closed analytical solution. The term describing the concentrated mass motion is the reason for difficulties. Differential equations of variable coefficients, which, except for a few cases, do not have analytical solutions, are serious limits in closed solutions. These types of equations are finally solved by numerical means.

In literature numerous historical reviews concerning the moving loads problem exist (for example Panowko [1], Jakuszew [2], Dmitrijew [3]). In most cases the moving massless constant force was considered as a moving load. This type of problem results in closed solutions. Unfortunately, the problem of inertial loads is still open. Saller in [4] considered the moving mass for the first time. He proved, in spite of essential simplifications, the significant influence of the moving mass in beam dynamics. In 1930s two contributions appeared, important for the researchers working in the field of moving loads. Inglis [5] applied simplifications and the solution was expressed by only the first term of the trigonometric series. Time function fulfilled the second order differential equation of variable coefficients. This equation was derived considering the acceleration under the moving mass, expressed by the so-called Renaudot formula. In fact it is the derivative with the constant velocity, computed with the chain rule. The final solution of the differential equation of variable coefficients was proposed as an infinite series. It results in an approached solution.

Schallenkamp [6] proposed another approach to the problem of moving mass. However, his attempt allows us to describe the motion only under the moving mass. The method of separation of variables by the expansion of the unknown function into a sine Fourier series was applied. Boundary conditions in the beam were taken into account in a natural way. The ordinary differential equation, which describes the motion under the moving mass was expressed in generalized coordinates by using the second Lagrange equations. The generalized force was derived from the virtual work principle. Schallenkamp's consideration is relatively complex and slowly converged since the final solution is expressed in terms of the triple infinite series.

The works of Inglis and Schallenkamp can be considered as the base for the analysis of the problem of moving mass in successive works of Bolotin [7, 8], Morgajewskij [9] and others. The excellent and important monograph in this field was written by Szcześniak [10]. One can find there hundreds of references concerning moving load on beams and strings. In [11] the authors consider a simply supported beam modeled by Bernoulli-Euler theory. The equation of motion is written in the integral-differential form with Green function terms. In order to compute this equation a dual numerical scheme has been used. a backward difference technique was applied to treat the time parameter and numerical integration was used for the spatial parameter. This way of the solution though applied to higher velocities, still requires complex mathematical operations. Each solution enables us to determine displacements under the moving load only and does not give solutions in a wide range of parameters x and t. Only one closed analytical solution can be found in literature. Smith [12] proposed the purely analytical solution for the inertial moving load, however, in the case of the massless string only. The basic motion equation without the term which describes the string inertia was transformed to the hyper-geommetrical equation. It has the analytical solution in terms of infinite series. Frýba [13] applied the same approach and found the closed analytical solution for a particular case $\alpha' = 1$. However, the formula given in [13] exhibits mistakes.

Recent papers contribute analysis of complex problems of structures subjected to moving inertial load [14] or oscillator [15, 16, 17]. Variable speed was analysed in [18, 19, 20]. Equivalent mass influence is analysed in [21]. The infinitely long string subjected to a uniformly accelerated point mass was also treated [22] and analytical solution of the problem concerning the motion of an infinite string on Winkler foundation subjected to an inertial load moving at a constant speed was given [23].

In the paper we consider small vibrations of the massless and mass string subjected to a moving inertial load. We propose the analytical–numerical solution of the problem. The final solution has a form of a matrix differential equation of the second order. Numerical integration results in a solution in a full range of the velocity: under critical and over critical. It exhibits discontinuity of the mass trajectory at the end support point. This new feature was not reported in literature. The closed solution in the case of massless string is analysed and its discontinuity is proved mathematically. Fully numerical results obtained for inertial string demonstrate the similar feature. Since small vibrations are analysed, the discontinuity effect discussed in the paper is of pure mathematical interest.

Results are compared with approached numerical solutions obtained by the finite element method. The string is subjected to a moving oscillator. In the case of the rigid spring we approach to the analytical solution. However, in the case of higher speed ($v > 0.2c$) the accuracy of the FEM solution is poor.

2 Analytical formulation

Let us consider a string of the length l, cross-sectional area A, mass density ρ, tensile force N, subjected to a mass m accompanied by a force P (Fig. 1), moving with a constant speed v. The motion equation of the string under moving inertial load with a constant speed v has a form

$$-N\,\frac{\partial^2 u(x,t)}{\partial x^2} + \rho A\,\frac{\partial^2 u(x,t)}{\partial t^2} = \delta(x-vt)\,P - \delta(x-vt)\,m\,\frac{\partial^2 u(vt,t)}{\partial t^2}\,. \qquad (1)$$

We impose boundary conditions

$$u(0,t) = 0 \qquad u(l,t) = 0 \qquad (2)$$

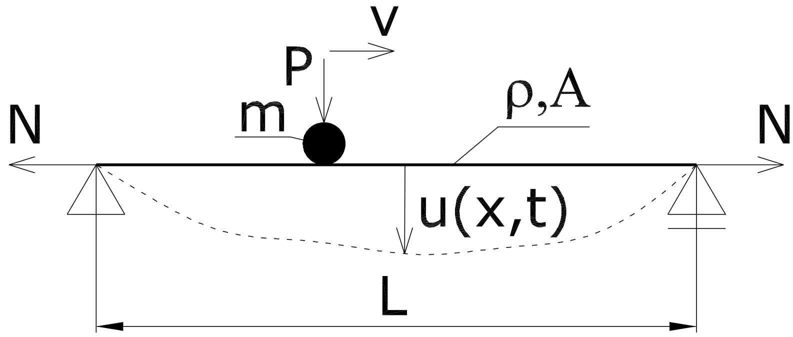


Figure 1: Moving inertial load.

and initial conditions

$$u(x,0) \;=\; 0 \qquad\qquad \left.\frac{\partial\, u(x,t)}{\partial\, t}\right|_{t\,=\,0} \;=\; 0 \quad . \tag{3}$$

In order to reduce partial differential equation to ordinary differential equation, we apply Fourier sine integral transformation in a finite range (i.e. finite length of the string) (4), (5)

$$V(j,t) \;=\; \int_0^l u(x,t)\,\sin\frac{j\pi x}{l}\,\mathrm{d}x \tag{4}$$

$$u(x,t) \;=\; \frac{2}{l}\,\sum_{j=1}^{\infty}\, V(j,t)\,\sin\frac{j\pi x}{l} \quad . \tag{5}$$

We can present each of the functions as a infinite sum of sine functions (5) with respective coefficients (4). Then the expansion of the moving mass acceleration in a series has a form

$$\frac{\partial^{\,2} u(vt,t)}{\partial\, t^{\,2}} \;=\; \frac{2}{l}\,\sum_{k=1}^{\infty}\left[\ddot V(k,t)\,\sin\frac{k\pi vt}{l} \;+\; \frac{2k\pi v}{l}\,\dot V(k,t)\,\cos\frac{k\pi vt}{l} \;-\; \frac{k^2\pi^2 v^2}{l^2}\,V(k,t)\,\sin\frac{k\pi vt}{l}\right] \quad . \tag{6}$$

The integral transformation (4) of the equation (1) with consideration of (6) can be performed

$$N\,\frac{j^2\,\pi^2}{l^2}\,V(j,t) \;+\; \rho A\,\ddot V(j,t) \;=\; P\,\sin\frac{j\pi ct}{l} \;-\; m\,\frac{\partial^{\,2} u(vt,t)}{\partial\, t^{\,2}}\,\int_0^l \delta(x-vt)\,\sin\frac{j\pi x}{l}\,\mathrm{d}x \;. \tag{7}$$

The integral with delta Dirac function in the above equation is as follows

$$\int_0^l \delta(x-vt)\,\sin\frac{j\pi x}{l}\,\mathrm{d}x \;=\; \sin\frac{j\pi vt}{l} \;. \tag{8}$$

Let us consider now (6) and (8):

$$
\begin{aligned}
N\,\frac{j^2\pi^2}{l^2}\,V(j,t) \;+\; \rho A\,\ddot{V}(j,t) \;=\; & P\,\sin\frac{j\pi vt}{l} \;-\; \frac{2m}{l}\sum_{k=1}^{\infty}\ddot{V}(k,t)\,\sin\frac{k\pi vt}{l}\,\sin\frac{j\pi vt}{l}\;- \\
& -\;\frac{2m}{l}\sum_{k=1}^{\infty}\frac{2k\pi v}{l}\,\dot{V}(k,t)\,\cos\frac{k\pi vt}{l}\,\sin\frac{j\pi vt}{l}\;+ \\
& +\;\frac{2m}{l}\sum_{k=1}^{\infty}\frac{k^2\pi^2v^2}{l^2}\,V(k,t)\,\sin\frac{k\pi vt}{l}\,\sin\frac{j\pi vt}{l}\;.
\end{aligned}
\tag{9}
$$

Finally, the motion equation after Fourier transformation can be written

$$
\begin{aligned}
\rho A\,\ddot{V}(j,t) \;+\; \alpha\sum_{k=1}^{\infty}\ddot{V}(k,t)\,\sin\omega_k t\,\sin\omega_j t \;+\; 2\alpha\sum_{k=1}^{\infty}\omega_k\,\dot{V}(k,t)\,\cos\omega_k t\,\sin\omega_j t\;+ \\
+\;\Omega^2\,V(j,t) \;-\; \alpha\sum_{k=1}^{\infty}\omega_k^2\,V(k,t)\,\sin\omega_k t\,\sin\omega_j t \;=\; P\,\sin\omega_j t\;,
\end{aligned}
\tag{10}
$$

where

$$
\omega_k \;=\; \frac{k\pi v}{l}\,, \qquad \omega_j \;=\; \frac{j\pi v}{l}\,, \qquad \Omega^2 \;=\; N\,\frac{j^2\pi^2}{l^2}\,, \qquad \alpha \;=\; \frac{2m}{l}\;. \tag{11}
$$

The analytical solution for this problem does not exist. We must solve this final equation numerically. Thus we obtain semi-analytical solution. The equation (10) is written in a matrix form, where matrix **M**, **C** and **K** are square matrices ($j, k = 1...n$).

$$
\mathbf{M}\begin{bmatrix}\ddot{V}(1,t)\\ \ddot{V}(2,t)\\ \vdots\\ \ddot{V}(n,t)\end{bmatrix} + \mathbf{C}\begin{bmatrix}\dot{V}(1,t)\\ \dot{V}(2,t)\\ \vdots\\ \dot{V}(n,t)\end{bmatrix} + \mathbf{K}\begin{bmatrix}V(1,t)\\ V(2,t)\\ \vdots\\ V(n,t)\end{bmatrix} = \mathbf{P} \tag{12}
$$

or

$$
\mathbf{M}\ddot{\mathbf{V}} + \mathbf{C}\dot{\mathbf{V}} + \mathbf{K}\mathbf{V} = \mathbf{P}\;, \tag{13}
$$

where

$$
\mathbf{M} = \begin{bmatrix}\rho A & 0 & \cdots & 0\\ 0 & \rho A & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \rho A\end{bmatrix} + \alpha\begin{bmatrix}\sin\frac{1\pi vt}{l}\sin\frac{1\pi vt}{l} & \sin\frac{1\pi vt}{l}\sin\frac{2\pi vt}{l} & \cdots & \sin\frac{1\pi vt}{l}\sin\frac{n\pi vt}{l}\\ \sin\frac{2\pi vt}{l}\sin\frac{1\pi vt}{l} & \sin\frac{2\pi vt}{l}\sin\frac{2\pi vt}{l} & \cdots & \sin\frac{2\pi vt}{l}\sin\frac{n\pi vt}{l}\\ \vdots & \vdots & \ddots & \vdots\\ \sin\frac{n\pi vt}{l}\sin\frac{1\pi vt}{l} & \sin\frac{n\pi vt}{l}\sin\frac{2\pi vt}{l} & \cdots & \sin\frac{n\pi vt}{l}\sin\frac{n\pi vt}{l}\end{bmatrix}, \tag{14}
$$

$$
\mathbf{C} = 2\alpha\begin{bmatrix}\frac{1\pi v}{l}\sin\frac{1\pi vt}{l}\cos\frac{1\pi vt}{l} & \frac{2\pi v}{l}\sin\frac{1\pi vt}{l}\cos\frac{2\pi vt}{l} & \cdots & \frac{n\pi v}{l}\sin\frac{1\pi vt}{l}\cos\frac{n\pi vt}{l}\\ \frac{1\pi v}{l}\sin\frac{2\pi vt}{l}\cos\frac{1\pi vt}{l} & \frac{2\pi v}{l}\sin\frac{2\pi vt}{l}\cos\frac{2\pi vt}{l} & \cdots & \frac{n\pi v}{l}\sin\frac{2\pi vt}{l}\cos\frac{n\pi vt}{l}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1\pi v}{l}\sin\frac{n\pi vt}{l}\cos\frac{1\pi vt}{l} & \frac{2\pi v}{l}\sin\frac{n\pi vt}{l}\cos\frac{2\pi vt}{l} & \cdots & \frac{n\pi v}{l}\sin\frac{n\pi vt}{l}\cos\frac{n\pi vt}{l}\;,\end{bmatrix} \tag{15}
$$

$$\mathbf{K} = \begin{bmatrix} \frac{1^2\pi^2}{l^2}N & 0 & \cdots & 0 \\ 0 & \frac{2^2\pi^2}{l^2}N & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{n^2\pi^2}{l^2}N \end{bmatrix} - \tag{16}$$

$$- \alpha \begin{bmatrix} \frac{1^2\pi^2v^2}{l^2}\sin\frac{1\pi vt}{l}\sin\frac{1\pi vt}{l} & \frac{2^2\pi^2v^2}{l^2}\sin\frac{1\pi vt}{l}\sin\frac{2\pi vt}{l} & \cdots & \frac{n^2\pi^2v^2}{l^2}\sin\frac{1\pi vt}{l}\sin\frac{n\pi vt}{l} \\ \frac{1^2\pi^2v^2}{l^2}\sin\frac{2\pi vt}{l}\sin\frac{1\pi vt}{l} & \frac{2^2\pi^2v^2}{l^2}\sin\frac{2\pi vt}{l}\sin\frac{2\pi vt}{l} & \cdots & \frac{n^2\pi^2v^2}{l^2}\sin\frac{2\pi vt}{l}\sin\frac{n\pi vt}{l} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1^2\pi^2v^2}{l^2}\sin\frac{n\pi vt}{l}\sin\frac{1\pi vt}{l} & \frac{2^2\pi^2v^2}{l^2}\sin\frac{n\pi vt}{l}\sin\frac{2\pi vt}{l} & \cdots & \frac{n^2\pi^2v^2}{l^2}\sin\frac{n\pi vt}{l}\sin\frac{n\pi vt}{l} \end{bmatrix},$$

$$\mathbf{P} = P \begin{bmatrix} \sin\frac{1\pi vt}{l} \\ \sin\frac{2\pi vt}{l} \\ \vdots \\ \sin\frac{n\pi vt}{l} \end{bmatrix} . \tag{17}$$

When coefficients $V(j,t)$ are computed, displacements of the string (5) can be appointed as a solution of (1). It is the solution in a full range. We can calculate displacement in each point of string and and for all values of v. We see that assuming $\rho = 0$ in (14) we have the formulation for the massless string.

3 Results

First we present the moderate convergence rate of the series which constitutes the solution (Fig. 2). We denote the wave speed in the unloaded string as c ($c^2 = N/\rho A$). Further diagrams exhibit vertical deflection of the string u related to the deflection in quasi-static mass motion in the middle of the span u_0. We can notice that the first term is already close to the exact solution. Three or five terms are sufficient for the accurate result in the engineering sense. We must emphasize here that higher speed of the mass, for example equal to $0.9c$ or c requires even hundred term and short time step for time integration of the differential equation, since the solution exhibits small jumps near the final support. The plot for different velocity v is given in Fig. 3.

Let us look at the diagrams of displacements of the string in the point under the mass. A diagram for various mass related to the string mass, for the speed $v = 0.2c$ is depicted in Fig. 4. More detailed presentation of the string motion is given in Fig. 5. We can notice the sharp edge of the wave and reflection from both supports. Moreover, the wave reflection from the travelling mass is clearly visible, especially for the case $v = 1.2c$. Both the mass trajectory and waves are depicted.

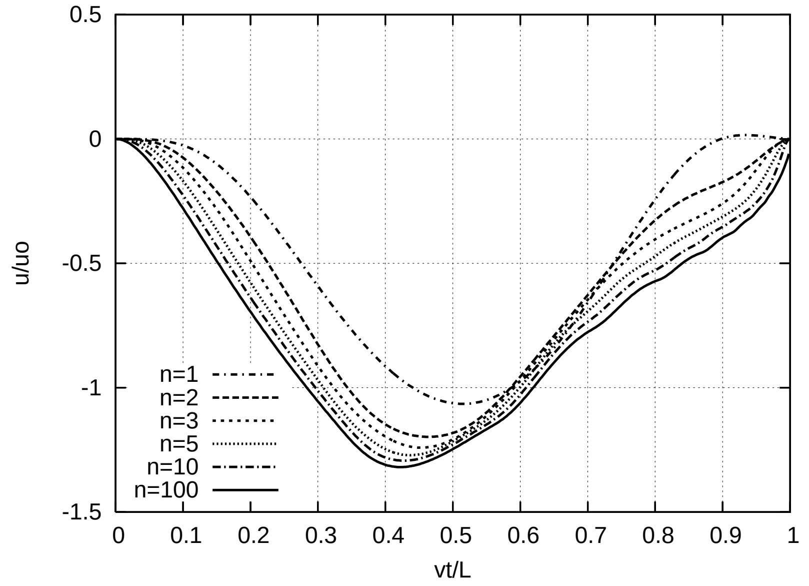


Figure 2: Trigonometric series convergence for $v = 0.2c$.

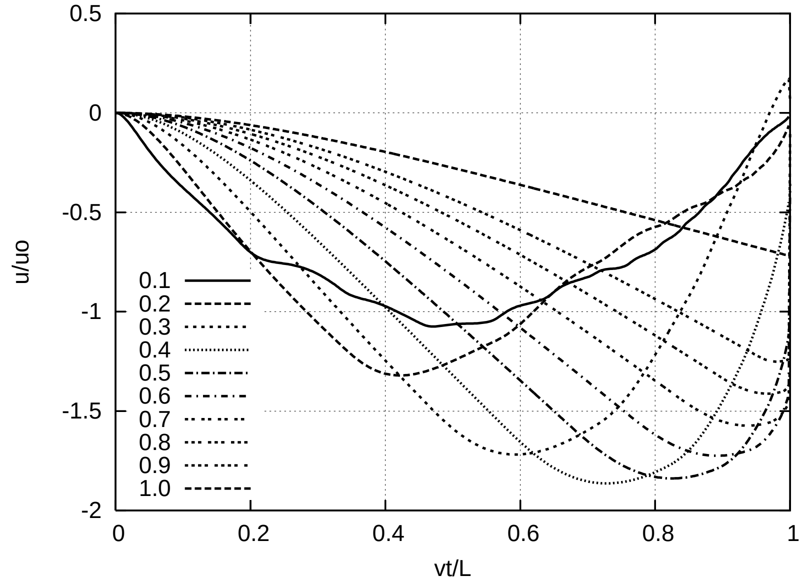


Figure 3: Inertial string — displacements computed semi-analytically.

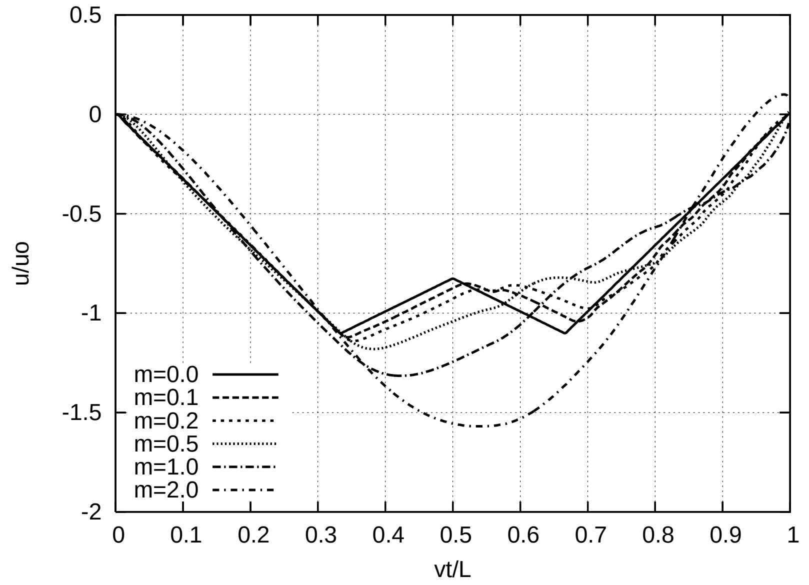


Figure 4: Displacements under the mass for different mass values at the speed $v = 0.2c$.

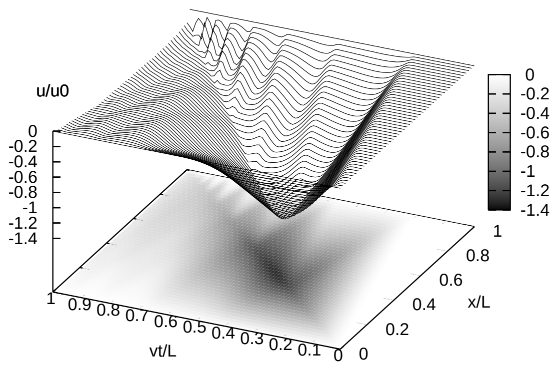
The convergence near the end point is depicted in Fig. 6. The mass trajectory is plotted for increasing number of term at the speed $v = 0.5c$. We notice that the function tends slowly to the jump at $x = l$. All characteristic lines are smooth. The convergence rate is low and especially near $x = l$ the taken number of term must be at least then 50. At high velocity range (in our case $v > 0.8$) sufficiently low time step of the integration of (13) must be applied (even 10^{-5}) to avoid small oscillations of the solution in the last stage.

Supersonic motion of the mass results in zero displacement. In the diagram obtained numerically this value oscillates with low amplitude. The amplitude decreases with the increase of the number of terms in a sum (Fig. 7).

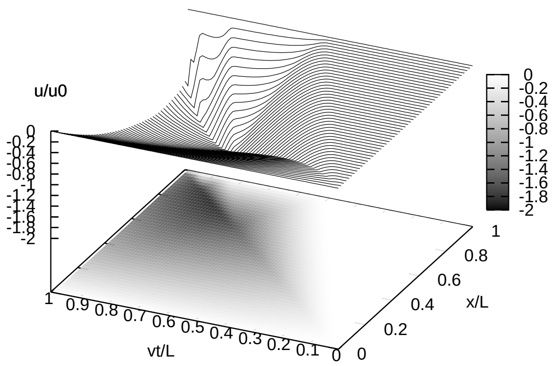
Analytical results are compared with numerical solutions obtained by the finite element method. The string was discretized by a set of 100 finite elements. It was subjected with an oscillator moving over the span. Two separate systems were considered: a string subjected to a contact force between the oscillator spring and the string, and the oscillator itself, subjected to a force P applied to a mass and displacements determined from the string motion, applied to a spring. The oscillator spring stiffness was assumed to be high enough, to simulate a rigid contact of the mass with the string. Results are depicted in Fig. 8.

4 Discontinuity of the solution

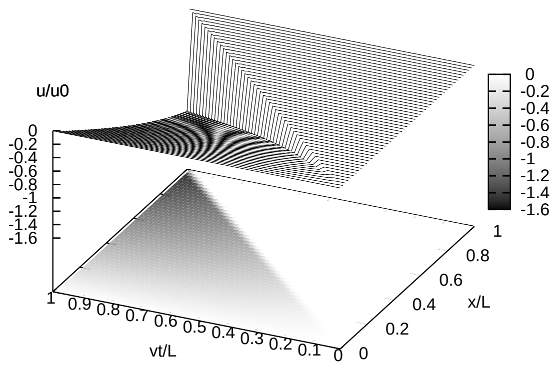
Advantages of the solution method presented in the paper allowed us to demonstrate an interesting feature of the solution near the end support. Resulting diagrams exhibit jumps of the mass displacement in time. Let us consider the physical nature of these jumps. The



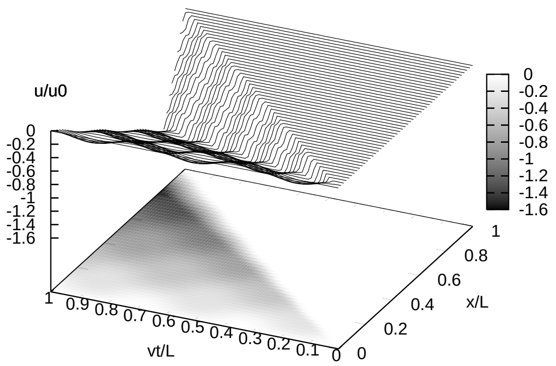
$v = 0.2c$



$v = 0.5c$



$v = 1.0c$



$v = 1.2c$

Figure 5: Simulation of the string motion under the mass moving at v=0.2c, 0.5c, 1.0c, and 1.2c.

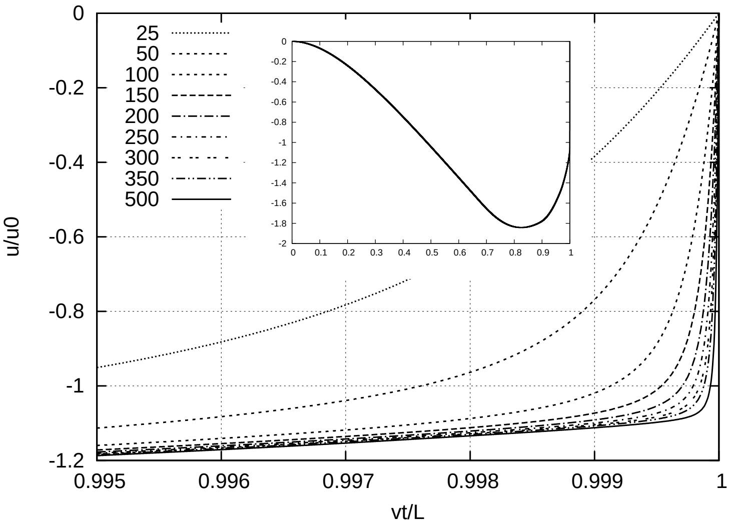


Figure 6: The convergence of the mass trajectory travelling with $v = 0.5c$ near the end point, for various number of term (25, 50,..., 500).

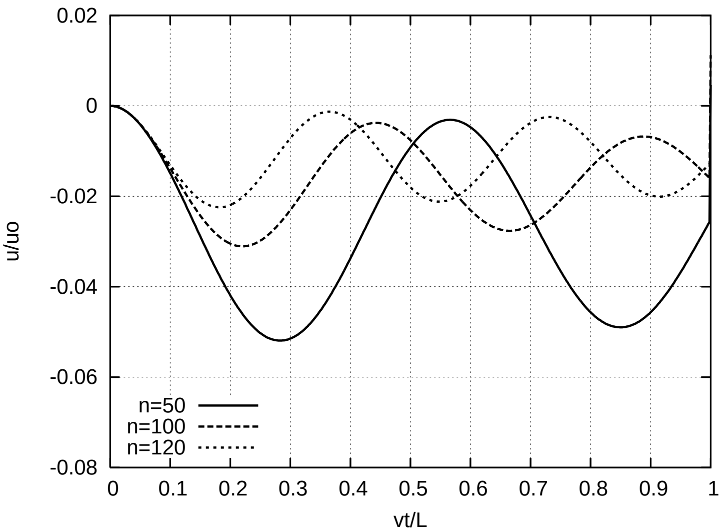


Figure 7: Convergence of displacements under the mass at the speed v=1.05c.

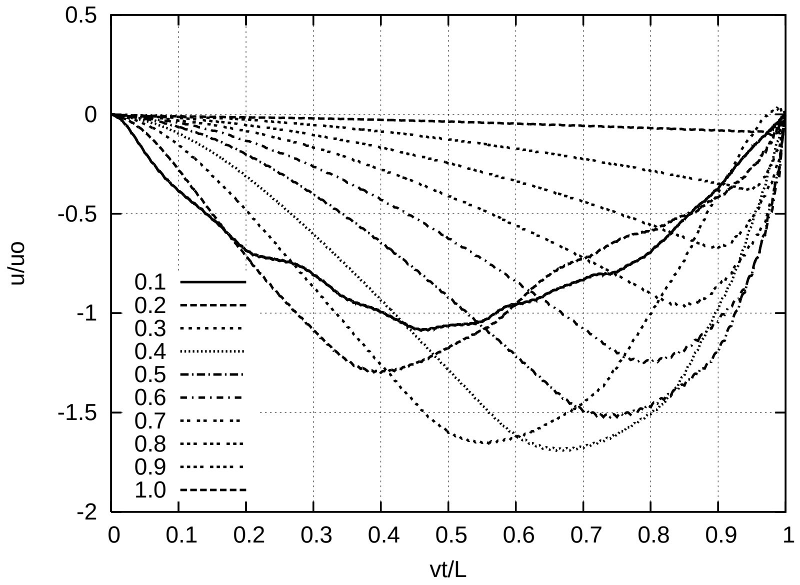


Figure 8: Finite element solution – displacements of the string under the oscillator.

simplest explanation can be based on the force equilibrium (Fig. 9). We must remember, that constant string tension N is the fundamental assumption in our problem. Moreover, in Fig. 9 the horizontal force pushing the mass to hold the speed v must be seen in the scheme. At the final stage (as depicted in Fig. 9) the remaining distance d will be passed in time d/v. In this period the mass m must be lifted from the position u_B to zero. If the deflection u_B is high enough, compared with other parameters, the necessary acceleration applied to the mass must result in high forces in the string $F \sim umv^2/d^2$. In such a case F can exceed N if m or v is sufficiently high. This fact violates our assumptions and the condition of applicability of the small vibrations equation $(\partial u/\partial x)^2 << 1$.

Let us consider a massless string, which is a particular case of our problem. The solution is given by a sum [13]

$$y(\tau) = \frac{4\,\alpha}{\alpha\,-\,1}\,\tau\,(\tau-1)\sum_{k=1}^{\infty}\prod_{i=1}^{k}\frac{(a+i-1)(b+i-1)}{c+i-1}\,\frac{\tau^k}{k!} \qquad (18)$$

where $\tau = vt/l > 0$ is time parameter, $\alpha = Nl/(2mv^2) > 0$ determines the dimensionless parameter. Parameters a, b and c are given below:

$$a_{1,2} \;=\; \frac{3\,\pm\,\sqrt{1\,+\,8\,\alpha}}{2} \qquad b_{1,2} \;=\; \frac{3\,\mp\,\sqrt{1\,+\,8\,\alpha}}{2} \qquad c = 2 \qquad (19)$$

In the case of $\alpha = 1$ the initial problem has a closed solution:

$$u(\tau) = \left[\frac{4}{3}\tau(1-\tau) - \frac{4}{3}\tau\,(1 + 2\tau\ln(1-\tau) + 2\ln(1-\tau))\right]\;. \qquad (20)$$

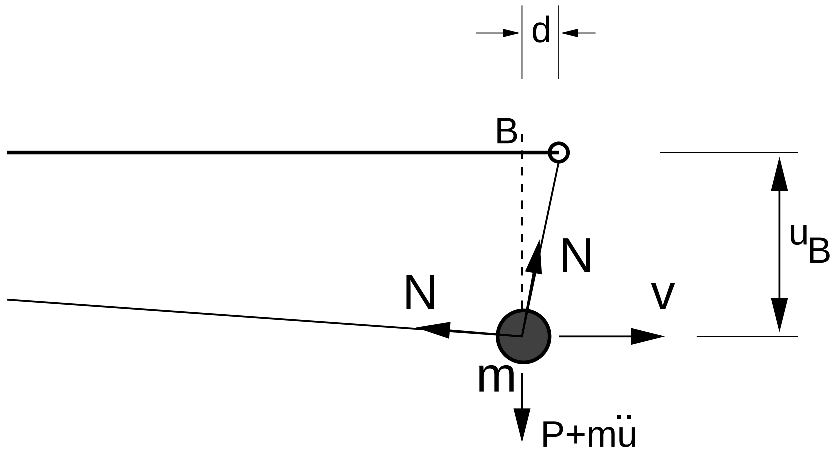


Figure 9: Final stage of the moving mass.

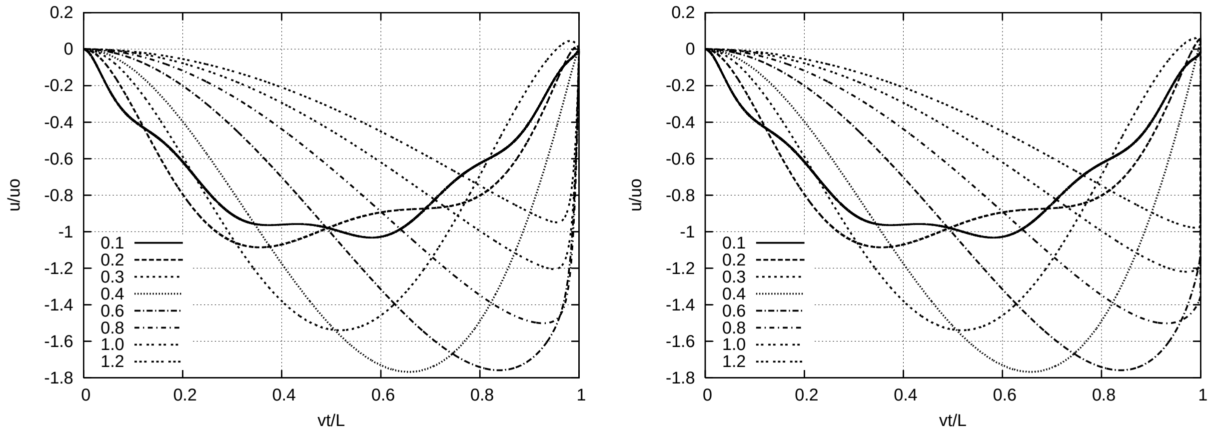


Figure 10: Trajectories of the mass moving on massless string: lower number of terms in a sum (left diagram) and higher number of terms (right diagram).

Here we consider the case of $\alpha \neq 1$. In Fig. 10 we can notice the strong influence of the precision on the solution near the end support. Let us consider the solution given by (18). The first term $\tau(\tau-1)$ results in zero at $\tau = 1$.

$$\sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{(a+i-1)(b+i-1)}{c+i-1} \, \frac{\tau^k}{k!} \tag{21}$$

tends to ∞ if $\tau \to 1$. We have indefinite solution at $\tau = 1^-$.

The same result can be obtained on the base of Abel theorem. The power series can be written in a form

$$\sum_{k=1}^{\infty} A_k \, \tau^k \; , \qquad A_k = \prod_{i=1}^{k} \frac{(a+i-1)(b+i-1)}{(c+i-1)\; i} \tag{22}$$

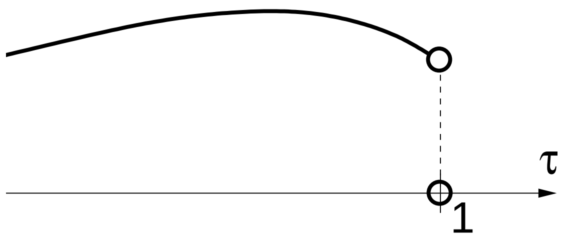


Figure 11: Discontinuity of the function (18) at $\tau = 1$.

In this case $\lim_{\tau \to 1^-} A_k \, \tau^k = \infty$ and $y(1^-) = 0 \cdot \infty$.

In the case $a+b < c$ the series (22) is convergent and there are no singularities. However, this is not our case. In the case of $a + b > c$ the series diverges (the sum tends to ∞). We have indefinite value $0 \cdot \infty$ while testing the function.

We can also perform another scheme of analysis. Below we will include the term $\tau(\tau - t)$ into the sum. Thus (18) can be reduced to the following form:

$$(1-\tau)\sum_{k=1}^{\infty} \frac{(a_k)\,(b_k)}{(c_k)}\,\frac{\tau^k}{k!} = \frac{a\,b\,\tau}{c} + \sum_{k=2}^{\infty} \frac{(a_{k-1})\,(b_{k-1})}{(c_{k-1})}\left(\frac{(a+k-1)(b+k-1)}{k(c+k-1)} - 1\right)\frac{\tau^k}{(k-1)!} \tag{23}$$

where

$$(a_k) = a\,(a\,+\,1)\ldots(a\,+\,k\,-\,1)$$

$$(b_k) = b\,(b\,+\,1)\ldots(b\,+\,k\,-\,1)$$

$$(c_k) = c\,(c\,+\,1)\ldots(c\,+\,k\,-\,1)$$

By using Rabbe criterion one can show that for $a + b < c + 2$ the limit

$$\lim_{\tau \to 1}\left[(1-\tau)\sum_{k=1}^{\infty}\prod_{i=1}^{k}\frac{(a+i-1)(b+i-1)}{c+i-1}\;\frac{\tau^k}{k!}\right]$$

is finite. Now we can estimate the value of the sum (23). The sum of the first two–three terms, depending on parameters, including $ab\tau/c$, is positive. Next terms are all positive. This proves that the sum (23) is finite and is greater than 0. The function (18) can be depicted in Fig. 11.
The case $a+b = c+1$ is particular (our set of parameters), for which the convergence is faster.

Let us look at the boundary condition at $\tau = 1$. We can say that it is fulfilled, however. We can imagine a symmetrical problem, with the mass moving from $\tau = 2$ towards $\tau = 1$

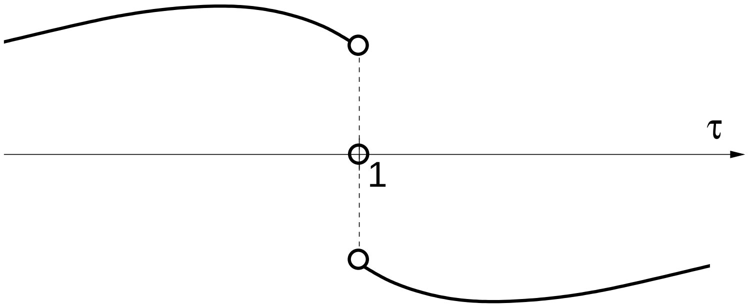


Figure 12: Left and right limits at $\tau = 1$.

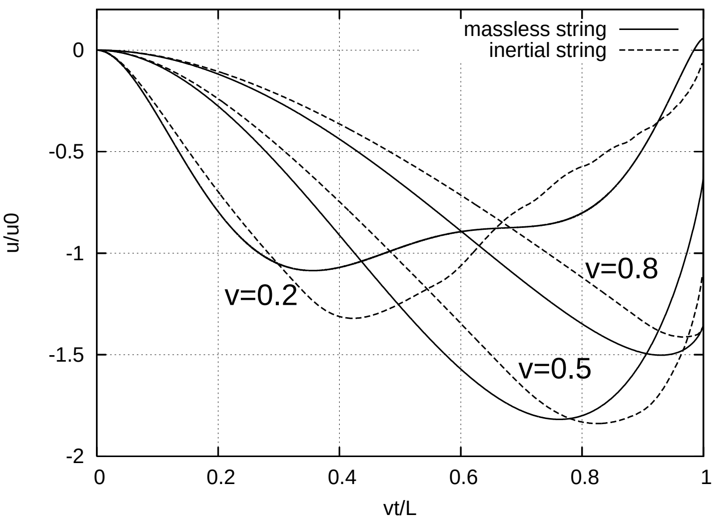


Figure 13: Comparison of particle's trajectory moving on massless and inertial string.

(with opposite direction of the force P). Then we have two analogous problems at $\tau = 1$. Both limits result in zero value at $\tau = 1$:

$$\frac{1}{2}\left(\lim_{\tau\to 1^-} y(\tau) + \lim_{\tau\to 1^+} y(\tau)\right) = 0 \ .$$

We can also consider the derivative $\mathrm{d}y/\mathrm{d}\tau$. The resulting formula can be derived. For negative P the result is

$$\lim_{\tau\to 1^-} \frac{\mathrm{d}y}{\mathrm{d}\tau} = \infty \ . \tag{24}$$

We can observe the same properties of the solution in the case of inertial string. Comparative plot is presented in Fig. 13. We can emphasize that in the case of lower $m/\rho A l$ ratio the coincidence of each pair of curves is higher. However, analytical proof of discontinuity in the case of inertial string is impossible to be obtained, because of the numerical integration stage.

5 Conclusions

In the paper we present a global analytical formulation of the vibration problem for the string, both massless and inertial, subjected to a moving mass. The numerical solution of the resulting matrix differential equation of the second order is relatively simple and is valid for the whole range of the speed v (sub-critical, critical and over-critical). The analysis of results exhibits a jump of the mass in the neighborhood of the end support. The force acting on the mass is, however, limited to the tensile force N. Discontinuity of the mass trajectory at $x = l$ exists in the case of $0 < v \leq c$. In the case of massless string this discontinuity is mathematically proved. In the case of $v > c$ there is no discontinuity, since for $x \geq vt$ the deflection $u(vt, t) = 0$.

Unfortunately, we can not give the answer to the question whether the string is continuous in case that the discontinuity of the particle's trajectory occurs. The massless or mass string shape is not determined in the analytical form. We can only expect such a discontinuity on the base of numerical results. The particle motion is continuous only in the trivial case of $m = 0$. The expression in parenthesis in (23) is equal to zero, and since $\alpha = 0$ in (18), finally $y(1^-) = 0$ and $y(1) = 0$.

We consider small vibrations. Discontinuity in this case is a feature of the mathematical interest rather then the practical one. However, in various analytical or numerical investigations of problems with travelling inertial load one can meet low convergence of solutions in places, where boundary conditions are imposed. Our analysis can explain anomalies in such cases.

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