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Semi-active control of 1D continuum vibrations under a travelling load

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Abstract

The paper presents a method for computing the response of a 1D elastic continuum supported by a set of semi-active viscous dampers and induced by a load travelling over it. The magnitude of the moving force has been assumed to be constant by neglect of the inertia forces. Full analytical solution is based on the power series method and is given in an arbitrary time interval. The time-marching scheme allows us to continue a solution in successive layers with initial conditions taken from the end of previous stages. The semi-active open loop control strategy is proposed. Shapes of damping functions are defined as a form of piecewise constant function. The control strategy is suboptimal and it outperforms the passive case. Numerical results are presented for the cases of a string and a Bernoulli-Euler beam.

Key words: travelling load, vibration control, semi-active control, vibrations of a string

1 Introduction

Problems of a load travelling along structures, such as strings, beams or plates at a higher range of speed, are of particular interest to practising engineers. A higher speed range means the speed at which successive passages of a moving load through the structure significantly increase amplitudes of displacements, up to infinity in the case of critical speed values. In the case of a string the considered speed can be within the range of 0.3 to 1.0 of the wave speed. Analytical and numerical solutions are applied to problems with a single or

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multi-point contact, such as train-track or vehicle-bridge interaction, pantograph collectors in railways, magnetic levitation railways, guideways in robotic technology, etc.

Increasing demand requires new technological solutions. Structures with external control of parameters can resist a load in a more efficient way. Structures with classical passive control are replaced by new, active or semi-active control systems. Old, weak structures can be reinforced by supplementary supports with magneto- or electro-rheological dampers controlled externally (Figures 1, 2). Active or semi-active control of structural vibrations plays an important role in the case of dynamic influence of external standing or travelling loads. Active methods of control are, unfortunately, energy-consuming and complicated in practical applications. Moreover, a poor control system can supply energy in the antiphase and in extreme cases can damage the structure. We will focus our research on semi-active systems composed of dampers, which require lower energetic effort.

Several evaluation criteria are subjects of interest. One of them describes the displacement in time of the midpoint. This criterion determines the resistance of the structure to deformations. Others describe displacement velocities or accelerations in time at the follower point under the travelling load. In those cases we can control the travel comfort, minimising the vertical dynamics of the vehicle. In our investigations we will concentrate on all the criteria mentioned above.

The semi-active control functions that represent evaluation of coefficients of viscous damping in time are continuous and bounded in a general case. In practical use they can be expressed by a piecewise constant function. The numerical optimisation of the control in the case of a higher number of those constants cannot be carried on efficiently. The variation of all the parameters of the damping control function in a discrete form will result in extremely lengthy computation time. Numerical analysis and classical methods of optimisation fail. We must elaborate a new efficient approach on the basis of the analysis of the differential equation or its solution.

In this paper we present the analytical solution of a semi-active control of vibrations in a string subjected to a travelling load. The string is supported by a set of viscous dampers. The method allows us to solve the problem analytically and express the continuous solution in a form useful for further analysis of the influence of damping functions on displacements and its derivatives. Whole time domain is split into time intervals. Full analytical solution in time interval in a form of power series is given. The time-marching scheme allows us to proceed to successive layers with initial conditions taken from the end of previous stages. The global solution can be written in a form with damping coefficients given as a vector. Thus the influence of a particular damper on the final global solution can be simply investigated. This fully analytical algorithm and the analytical form of the solution allow us to examine quantitatively the influence of a piecewise constant damping on vibrations. Further work could enable us to determine the general efficient strategy of the control instead of a particular numerical solution, useless for investigations.

Analysis of the moving load problem is commonly presented in the literature. The travelling load can be one of two types: non-inertial (massless) or inertial. The analysis of the moving massless force is relatively simple and is treated in numerous papers [1,2]. We include in this group all the papers devoted to the travelling oscillator, i.e. a mass particle joined to the base with a spring [9,13,14]. Some authors describe this type of a load as an inertial one. We consider it as a massless force generated only by the particle's inertia. The inertial load moving over the structure is less frequently reported in the literature [15,16,17]. The closed solution exists in the case of a mass moving on a massless string [2,18]. Otherwise the final results are obtained numerically, although the solution is preceded by complex analytical calculations. A new and important feature of discontinuity of the inertial particle trajectory is exhibited in [19]. In numerous references authors treat the problem in a very low range of the mass speed. In this case results are sufficient, even if the inertial term contributing to moving mass is not correctly treated by the time integration method. Simply, the moving mass influence is trivial compared with static displacements.

Purely numerical solutions of a group of engineering problems with travelling massless load are relatively simple and every particular case can be computed without significant computational effort. The numerical results in the case of inertial loads, however, are not sufficient [20,21]. Broad analysis of moving loads was given in [2,22]. In recent contributions complex problems of structures subjected to a moving inertial load [23] or oscillator [10,13,14] were also analysed.

Numerical algorithms implemented in commercial codes do not allow efficient analysis in the case of the moving massless load, nor is the inertial moving load implemented. Correct formulae for discrete analysis of moving mass problems were published recently [24,25] and implemented in the analysis of train/track interaction.

Numerous active and semi-active vibration control methods are widespread and some of them have been put into practice recently. Most of them are based on sky-hook or ground-hook concepts [7]. These approaches are used for semi-active control of the moving oscillator problem in [8]. Variable dampers are incorporated in seismic isolation in [11,12]. A theoretical approach to the problem of controlled beam vibration damping, based on the method of optimal Lyapunov functions, was presented in [6]. In [26] the authors assumed the semi-active control applied to a stiffness and to a damping. The control function led to maximum dissipation of the energy. Generally, the amplitude level decrease was to be achieved. Passive damping of a Euler beam under a moving load was presented in [27]. The load of cyclically travelling forces was considered as a periodic one. The decrease of the resonance peak was obtained by a gradient method. The beam subjected to a placed harmonic load was controlled by the active method [28]. The analysis of the frequency domain allowed the authors to reduce the maximum of amplitudes. In the next paper [29] the harmonic load at a fixed point was also applied. The control of stiffness parameters allowed the reduction of parametric vibrations. The structure elements were controlled by on/off state. The expected effect with reduced and shifted resonance curves was obtained. Active damping of structures under travelling load was described in [30,31].

Most of the semi-active methods identified lead to feedback controls determined by state-space measures. In the case of continuous systems such an observer design is often much too complicated. The alternative method is an open-loop control. It is of particular use in problems where the excitation is determined.

Preliminary investigation of the destination problem was published in [5]. The beam supported by two dampers exhibited lower amplitudes both in the midpoint and under the travelling load. Higher frequency modes, however, were dominant in the transient stage.

2 Mathematical formulation

Formulation and solution of the problem presented in this section are developed for a string, but the technique is not specific to an 1D continuum and can be applied to elements like Euler-Bernoulli or Timoshenko beams as well.

Let us consider the system shown in Figure 3. The string is stretched and simply supported by a set of control dampers. The moving load is passing along the string at a constant velocity. The mass accompanying the travelling load is small compared with the mass of the string and is neglected. Thus we assume a massless load. Reactions of dampers are proportional to the velocity of displacements in given points.

The transverse vibration of the string system shown in Figure 3 is governed by the partial differential equation

$$-N\,\frac{\partial^2 u(x,t)}{\partial x^2}+\mu\,\frac{\partial^2 u(x,t)}{\partial t^2}=-\sum_{i=1}^{Z} b_i(t)\,\frac{\partial u(a_i,t)}{\partial t}\,\delta(x-a_i)+P\,\delta(x-vt)\;, \quad (1)$$

where N is the force stretching the string, μ is the constant mass density per unit length, P is the concentrated force passing the string at the constant velocity v, $b_i(t)$ is the ith damping coefficient as a function of time, $u(x,t)$ is a transverse deflection of the string at the point (x,t), Z is the number of viscous supports, a_i is the ith fixed point of a damper and δ is the Dirac delta.

The boundary and initial conditions of the simply-supported and stretched string are as follows:

$$u(0,t)=0, \quad u(l,t)=0, \quad u(x,0)=0, \quad \dot{u}(x,0)=0\ . \tag{2}$$

Eqn. (1) with conditions (2) will be solved by the method of the sine Fourier transformation based on the following fundamental relations:

$$\begin{aligned} V(j,t) &= \int_0^l u(x,t)\,\sin\frac{j\pi x}{l}\,\mathrm{d}x\ , \\ u(x,t) &= \frac{2}{l}\sum_{j=1}^{\infty} V(j,t)\,\sin\frac{j\pi x}{l}\ . \end{aligned} \tag{3}$$

Each term of Eqn. (1) is multiplied by $\sin\frac{j\pi x}{l}$ and then integrated with respect to x in the interval $[0,l]$

$$\begin{aligned} &\int_0^l \left(-N\,\frac{\partial^2 u(x,t)}{\partial x^2}\,\sin\frac{j\pi x}{l} + \mu\,\frac{\partial^2 u(x,t)}{\partial t^2}\,\sin\frac{j\pi x}{l}\right)\mathrm{d}x = \\ &\int_0^l \left(-\sum_{i=1}^{Z} b_i(t)\,\frac{\partial u(a_i,t)}{\partial t}\,\sin\frac{j\pi x}{l}\delta(x-a_i) + P\,\sin\frac{j\pi x}{l}\delta(x-vt)\right)\mathrm{d}x\ . \end{aligned} \tag{4}$$

Thus,

$$\begin{aligned} &\frac{Nj^2\pi^2}{l^2}\,V(j,t) + \mu\,\ddot{V}(j,t) = \\ &\int_0^l \left(-\sum_{i=1}^{Z} b_i(t)\,\frac{\partial u(a_i,t)}{\partial t}\,\sin\frac{j\pi x}{l}\delta(x-a_i)\right)\mathrm{d}x + P\,\sin\frac{j\pi vt}{l}\ . \end{aligned} \tag{5}$$

The integral term can be rewritten as

$$\begin{aligned} &\int_0^l \left(-\sum_{i=1}^{Z} b_i(t)\,\frac{\partial u(a_i,t)}{\partial t}\,\sin\frac{j\pi x}{l}\delta(x-a_i)\right)\mathrm{d}x = \\ &-\sum_{i=1}^{Z} b_i(t)\int_0^l \frac{\partial u(a_i,t)}{\partial t}\,\sin\frac{j\pi x}{l}\delta(x-a_i)\mathrm{d}x = -\sum_{i=1}^{Z} b_i(t)\,\frac{\partial u(a_i,t)}{\partial t}\,\sin\frac{j\pi a_i}{l} = \\ &-\frac{2}{l}\sum_{i=1}^{Z}\sum_{k=1}^{\infty} b_i(t)\,\dot{V}(k,t)\,\sin\frac{k\pi a_i}{l}\,\sin\frac{j\pi a_i}{l}\ . \end{aligned} \tag{6}$$

Eqn. (5) is a system of ordinary differential equations

$$\mu\,\ddot{V}(j,t)+\frac{2}{l}\sum_{i=1}^{Z}\sum_{k=1}^{\infty}b_i(t)\,\dot{V}(k,t)\,\sin\frac{k\pi a_i}{l}\,\sin\frac{j\pi a_i}{l}+\frac{Nj^2\pi^2}{l^2}\,V(j,t)=P\,\sin\frac{j\pi vt}{l}\,. \tag{7}$$

Now we expect a solution of Eqn. (7) for an arbitrary shape of functions $b_i(t)$. It would be convenient to make the coefficients constant. For this purpose we define all $b(t)$ as step-shape functions depicted in Figure 4

$$b\colon\left[0,\frac{l}{v}\right]\longrightarrow[b_{min},b_{max}]\qquad b(t)=\begin{cases}b_p, & \forall t\in(t_{p-1},t_p],\quad p=1...s\\ 0, & t=0\end{cases}\,. \tag{8}$$

With the following notations

$$\frac{\pi v}{l}=\omega,\qquad \sin\frac{j\pi a_i}{l}\,\sin\frac{k\pi a_i}{l}=\alpha_{ijk}$$

Eqn. (7) is reduced to the form

$$\mu\,\ddot{V}(j,t)+\frac{2}{l}\sum_{i=1}^{Z}\sum_{k=1}^{\infty}b_{ip}\,\dot{V}(k,t)\,\alpha_{ijk}+\frac{Nj^2\pi^2}{l^2}\,V(j,t)=P\,\sin(j\omega t)\,, \tag{9}$$

where b_{ip} denotes the magnitude of the suspension of the ith damper in the pth time interval.

Eqn. (9) is linear and describes the nonhomogeneous system with constant coefficients. The solution sought is the general solution, where integration constants can be simply represented by initial values $C1_j=V(j,0),\quad C2_j=\dot{V}(j,0)$. The interval solutions can simply be combined to a global one. Investigations prove that the standard method for solving the linear system, i.e. by means of eigen-problems is not sufficient in this case. The solving procedure presented below is based on the power-series method. By denoting t_{p-1} by τ, the solution for $t\in(t_{p-1},t_p]$ is supposed to take the form

$$V(j,t)=\sum_{n=0}^{\infty}d_n(j)(t-\tau)^n\,, \tag{10}$$

where $d_n(j)$ are unknown sequences. Then

$$\dot{V}(j,t)=\sum_{n=0}^{\infty}nd_n(j)(t-\tau)^{n-1},\qquad \ddot{V}(j,t)=\sum_{n=0}^{\infty}(n-1)nd_n(j)(t-\tau)^{n-2}\,, \tag{11}$$

and Eqn. (9) can be written as

$$\mu \sum_{n=0}^{\infty}(n-1)nd_n(j)(t-\tau)^{n-2} + \frac{2}{l}\sum_{i=1}^{Z}\sum_{k=1}^{\infty}\sum_{n=0}^{\infty} b_{ip}\,\alpha_{ijk}\,nd_n(k)(t-\tau)^{n-1} + \\ + \frac{Nj^2\pi^2}{l^2}\sum_{n=0}^{\infty} d_n(j)(t-\tau)^n = P\,\sin(j\omega t)\ . \tag{12}$$

Representation of $\sin(j\omega t)$ in a power series gives

$$\sin(j\omega t) = \sin(j\omega(t-\tau+\tau)) = \\ \sin(j\omega(t-\tau))\cos(j\omega\tau) + \cos(j\omega(t-\tau))\sin(j\omega\tau) = \\ \cos(j\omega\tau)\sum_{n=0}^{\infty}\frac{(-1)^n(j\omega)^{2n+1}(t-\tau)^{2n+1}}{(2n+1)!} + \sin(j\omega\tau)\sum_{n=0}^{\infty}\frac{(-1)^n(j\omega)^{2n}(t-\tau)^{2n}}{(2n)!}\ . \tag{13}$$

Then we have

$$\mu \sum_{n=0}^{\infty}(n+1)(n+2)d_{n+2}(j)(t-\tau)^n + \\ + \frac{2}{l}\sum_{i=1}^{Z}\sum_{k=1}^{\infty}\sum_{n=0}^{\infty} b_{ip}\,\alpha_{ijk}\,(n+1)d_{n+1}(k)(t-\tau)^n + \frac{Nj^2\pi^2}{l^2}\sum_{n=0}^{\infty} d_n(j)(t-\tau)^n = \\ P\,\cos(j\omega\tau)\sum_{n=0}^{\infty}\frac{(-1)^n(j\omega)^{2n+1}(t-\tau)^{2n+1}}{(2n+1)!} + \\ + P\,\sin(j\omega\tau)\sum_{n=0}^{\infty}\frac{(-1)^n(j\omega)^{2n}(t-\tau)^{2n}}{(2n)!}\ . \tag{14}$$

It is commonly known that for every sequence γ_n, the following equation is satisfied:

$$\sum_{n=0}^{\infty}\gamma_n(t-\tau)^n = \sum_{n=0}^{\infty}\gamma_{2n}(t-\tau)^{2n} + \sum_{n=0}^{\infty}\gamma_{2n+1}(t-\tau)^{2n+1}\ . \tag{15}$$

Finally Eqn. (14) is rewritten in the form

$$
\begin{aligned}
&\mu \sum_{n=0}^{\infty}(2n+1)(2n+2)d_{2n+2}(j)(t-\tau)^{2n} + \frac{Nj^2\pi^2}{l^2} \sum_{n=0}^{\infty} d_{2n}(j)(t-\tau)^{2n} + \\
&+ \frac{2}{l}\sum_{i=1}^{Z}\sum_{k=1}^{\infty} b_{ip}\,\alpha_{ijk} \sum_{n=0}^{\infty}(2n+1)d_{2n+1}(k)(t-\tau)^{2n} + \\
&+ \mu \sum_{n=0}^{\infty}(2n+2)(2n+3)d_{2n+3}(j)(t-\tau)^{2n+1} + \frac{Nj^2\pi^2}{l^2} \sum_{n=0}^{\infty} d_{2n}(j)(t-\tau)^{2n} + \\
&+ \frac{2}{l}\sum_{i=1}^{Z}\sum_{k=1}^{\infty} b_{ip}\,\alpha_{ijk} \sum_{n=0}^{\infty}(2n+2)d_{2n+2}(k)(t-\tau)^{2n+1} = \\
&P\cos(j\omega\tau)\sum_{n=0}^{\infty}\frac{(-1)^n(j\omega)^{2n+1}(t-\tau)^{2n+1}}{(2n+1)!} + P\sin(j\omega\tau)\sum_{n=0}^{\infty}\frac{(-1)^n(j\omega)^{2n}(t-\tau)^{2n}}{(2n)!}\,.
\end{aligned}
\tag{16}
$$

Comparing equivalent terms, we obtain the system of recurrence equations

$$
\begin{aligned}
&\mu\,(2n+1)(2n+2)d_{2n+2}(j) = -\frac{2}{l}\sum_{i=1}^{Z}\sum_{k=1}^{\infty} b_{ip}\,\alpha_{ijk}(2n+1)d_{2n+1}(k) + \\
&-\frac{Nj^2\pi^2}{l^2}\,d_{2n}(j) + P\,\sin(j\omega\tau)\frac{(-1)^n(j\omega)^{2n}}{(2n)!}\,, \\
&\mu\,(2n+2)(2n+3)d_{2n+3}(j) = -\frac{2}{l}\sum_{i=1}^{Z}\sum_{k=1}^{\infty} b_{ip}\,\alpha_{ijk}(2n+2)d_{2n+2}(k) + \\
&-\frac{Nj^2\pi^2}{l^2}\,d_{2n+1}(j) + P\,\cos(j\omega\tau)\frac{(-1)^n(j\omega)^{2n+1}}{(2n+1)!}\,,
\end{aligned}
\tag{17}
$$

and $d_0(j) = V(j,\tau)$, $d_1(j) = \dot{V}(j,\tau)$.

Numerical results exhibiting the convergence rate of the obtained solution are presented next. In the analysis we use 60 modes and 40 terms in a power series. The following data were assumed: $\mu = 1$, $l = 1$, $N = 0.5$, $P = 0.1$, $v = 0.2\sqrt{\frac{N}{\mu}}$, $Z = 1$, $a_1 = 0.5l$. The suspension magnitude is assumed to be constant and equal to one ($b_{1p} = 1$, $\forall p = 1, ..., s$). Figure 5 presents the solution at $x = l/2$. Curves are plotted for various numbers of intervals $s =$ 59, 61 and 65. For a lower number of time intervals and greater time increments the solutions diverge.

To extend the radius of convergence, more terms in a power series have to be taken into account. Figure 6 shows the solution of the previous problem for $s = 25$ and the number of terms in a power series equal to 98 and 100. The dashed line represents the solution obtained by the finite element method.

3 Control strategy

In this section we present a control method based on the analysis and respective numerical results of the solution. Further, we investigate the efficiency of the proposed control strategy by means of values of defined payoffs. The advantage of the derived analytical solution is its continuity, which offers the possibility to define the performance index in the integral form.

The considered model is shown in Figure 7 and it is described by the equation

$$EJ\,\frac{\partial^4 u(x,t)}{\partial x^4} + \mu\,\frac{\partial^2 u(x,t)}{\partial t^2} = -\sum_{i=1}^{2} b_i(t)\,\frac{\partial u(a_i,t)}{\partial t}\,\delta(x-a_i) + P\,\delta(x-vt). \quad (18)$$

We consider the Bernoulli-Euler beam as a continuum with the following parameters: $l = 2m$, $\mu = 0.78kg/m$, $EJ = 10^4 Nm^2$. Active dampers are fixed to the beam at points a_1=0.25l and a_2=0.75l. The force $P = 1000N$ is travelling with the velocity v=0.7c, where c denotes so-called critical speed and $c = \pi/l\,\sqrt{EI/\mu}$.

The formulated system is classified as bilinear. Numerous techniques, which stem primarily from the calculus of variation, have been derived for the optimal control solution of such a system. Pontryagin's maximum principle uses Hamilton's equations and the Dynamic Programming method leads to the Bellman-Hamilton-Jacobi partial differential equation. Based on these theories numerous computational technics were developed in the 1960s and 1970s [3]. With the exception of the simplest cases, however, it is impossible to express controls in an explicit feedback form, owing to the complicated nature of the associated switching hypersurfaces in the state space. Difficulties increase in the case of the continuum that is transformed to a multidimensional discrete system.

We propose an open loop control strategy based on the concept presented in Figure 2. The assumption made the controls $b_1(t), b_2(t)$ piecewise constant and belonging to a closed set B. Numeric investigations proved that the *bang-bang* controls exerted the fairest efficiency. In this approach we do not pay attention to optimal solutions in the sense of minimising the performance index with respect to all admissible controls. We try rather to present cases where semi-active dampers may outperform passive ones. The goal is to design efficient control so that the practical realisation is the easiest way possible. For this purpose and for simplicity we take into account controls that are *bang-bang* and only one switching time for each of them is assumed so that

$$b_1(t) = b_{max}U_1(t) - b_{max}U_1(t-\tau_1), \quad b_2(t) = b_{max}U_1(t-\tau_2), \qquad (19)$$

where $U_1(t)$ is a unit step function and $b_{max} = sup(B)$. In fact, damper No. 1

is first switched *on* then in time $t = \tau_1$ it turns into *off* mode. The situation for damper No. 2 is reversed. Below we define the cost integrands such that they can determine travel comfort (cases 1, 3) or structural damage (case 2)

$$\begin{aligned} &(1) \quad \textsc{Payoff}_1 = \int_0^{l/v} |u(vt,t)| \, \mathrm{d}t \\ &(2) \quad \textsc{Payoff}_2 = \mathit{RMS}(\dot{u}(vt,t)) = \left(\frac{v}{l} \int_0^{l/v} (\dot{u}(vt,t))^2 \, \mathrm{d}t \right)^{1/2} \\ &(3) \quad \textsc{Payoff}_3 = \int_0^{l/v} |\ddot{u}(vt,t)| \, \mathrm{d}t \end{aligned} \tag{20}$$

The task is to find pairs (τ_1, τ_2) that minimise costs

$$(\tau_1, \tau_2) = \underset{\tau_1, \tau_2 \in [0, l/v]}{\arg\min} \; \textsc{Payoff}(u(t), b_1(t), b_2(t)), \tag{21}$$

where $b_1(t), b_2(t)$ are defined as before. In Figure 8 we present mappings $(\tau_1, \tau_2) \rightarrow \textsc{Payoff}_1$ and $(\tau_1, \tau_2) \rightarrow \textsc{Payoff}_3$. Numerical results exert the existence of unique solutions of (21) for all cases. Extremal trajectories for $u(t), \dot{u}(t), \ddot{u}(t)$ with their controls are shown in Figures 9, 10, and 11, respectively. By the passive case we mean constant damping $b_1(t) = b_{max}, b_2(t) = b_{max}, \forall t \in [0, l/v]$. In computations we assumed $b_{max} = 3 \cdot 10^4$ in all cases.

The best performance of the proposed strategy is observed in the first case, where the value of the cost functional is decreased by more than 30% compared with non-active damping. For cases 2 and 3 we expect much better performance by applying controls with more than one switching. Velocities and accelerations incorporated into these costs include high-frequency harmonics that can be reduced by high-frequency switching controls. Because of the significantly higher complexity of the optimisation problem, computing of such controls may be difficult. Appropriate gradient methods may, however, be useful [4]. The application of existing and the development of new methods for computing higher dimensional switching vectors are reserved for further work.

4 Conclusions

In this paper the analytical solution of the response of a semi-active controlled 1D continuum has been presented. The technique has been applied to exemplary control systems including string and Euler-Bernoulli beams. The open-loop control strategy has been proposed and its performance has been verified for three different cost integrands. Control strategy is simple for a practical design. Further optimisation is the ongoing research topic of the authors.

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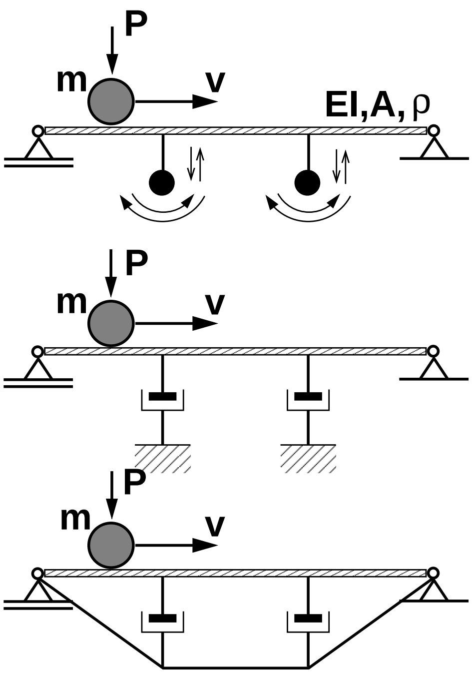

Figure 1. Examples of passive and semi-active control in a bridge span under a travelling load.

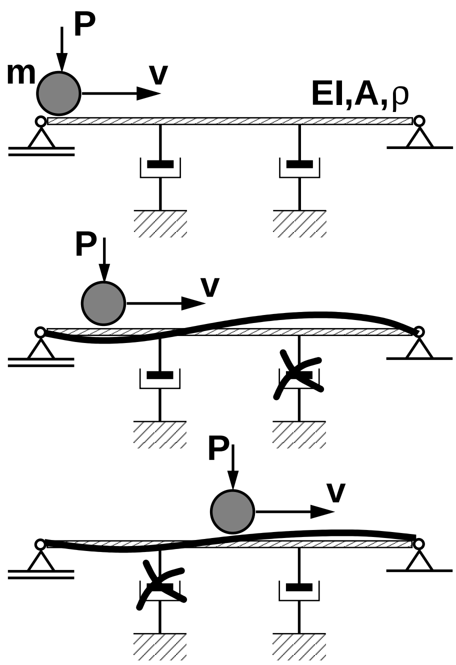

Figure 2. The idea of semi-active control of a beam deflection under a travelling load.

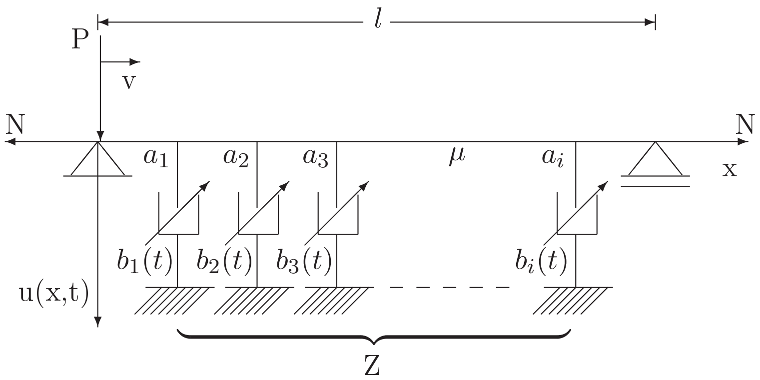

Figure 3. String system supported by active viscous dampers.

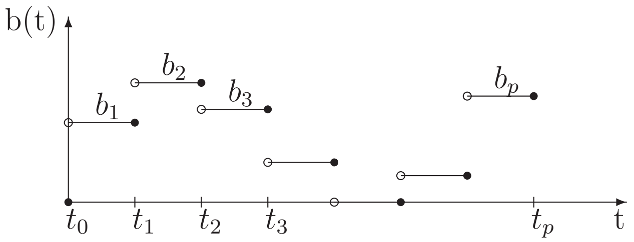

Figure 4. Piecewise constant damping function.

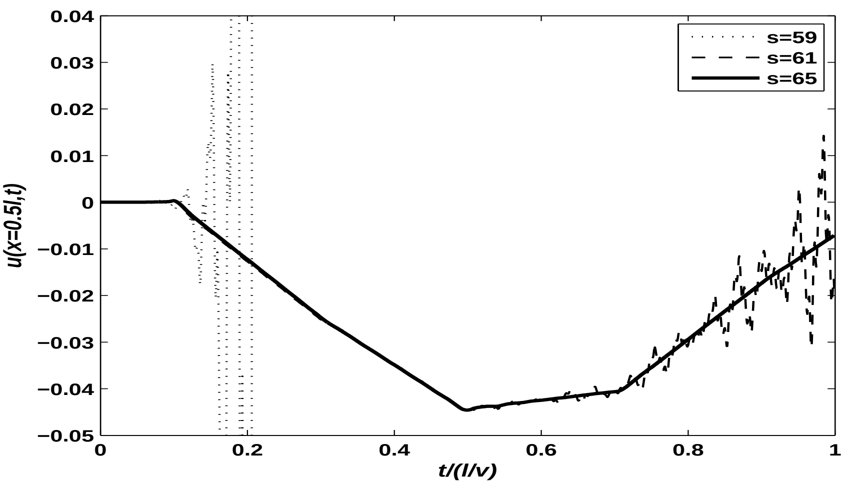

Figure 5. Solutions computed for different numbers of time intervals.

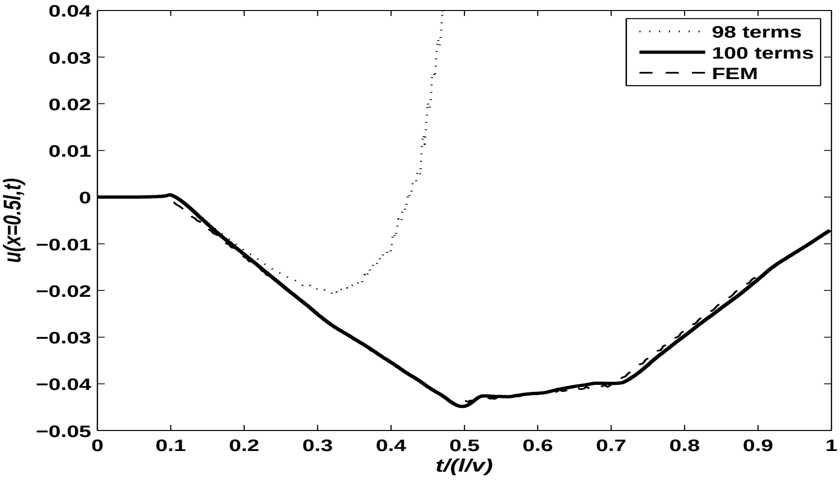

Figure 6. Solutions computed for different numbers of terms in a power series and compared with the FEM solution.

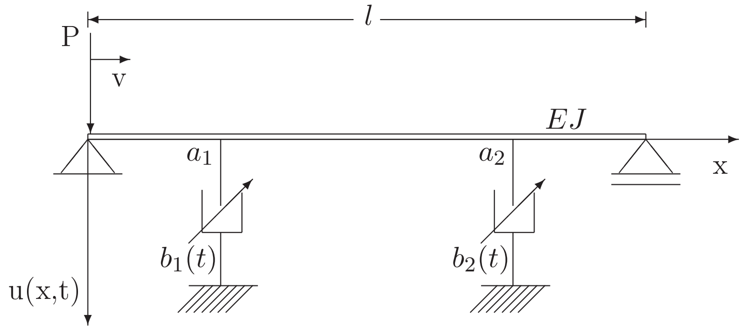

Figure 7. Euler-Bernoulli beam system supported with two viscous active dampers.

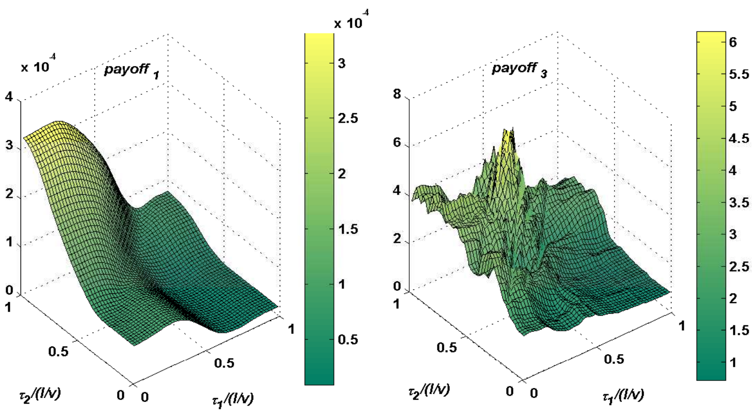

Figure 8. Cost functionals as functions of switching times (cases 1 and 3).

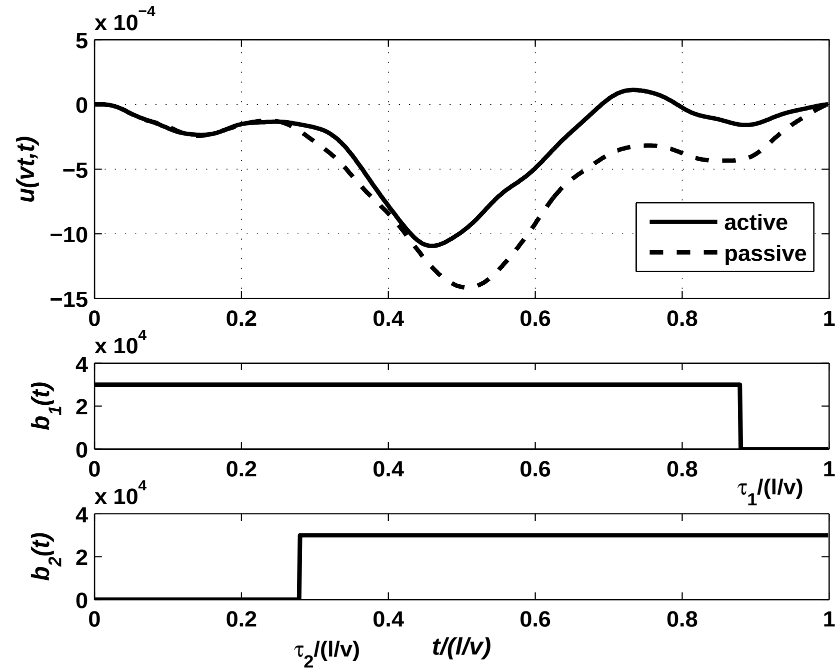

Figure 9. Extremal deflection trajectory and controls.

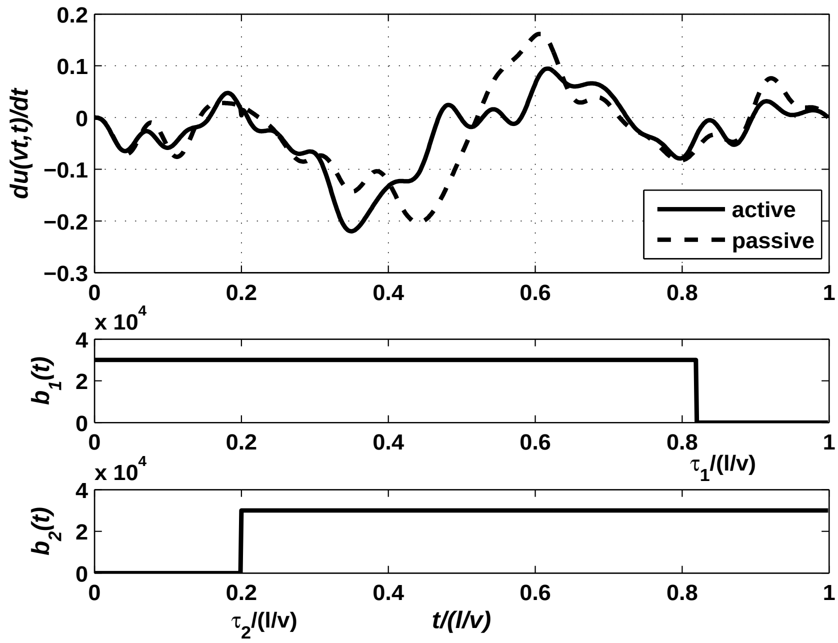

Figure 10. Extremal velocity trajectory and controls.

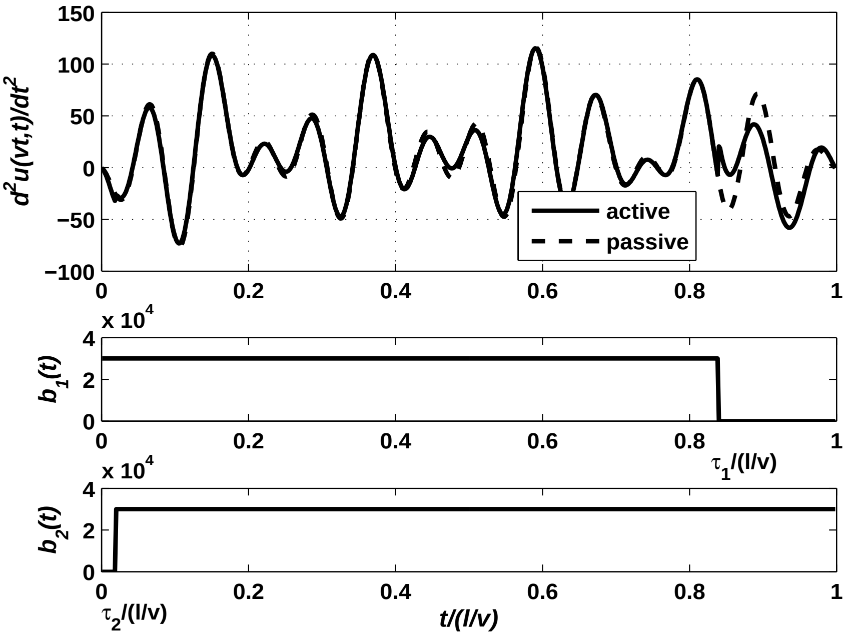

Figure 11. Extremal acceleration trajectory and controls.