MOVING INERTIAL LOAD AND NUMERICAL MODELLING

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Abstract

The paper presents the numerical approach to the moving mass problem. We consider the string and beam discrete element carrying a mass particle. In the literature efficient computational methods can not be found. The same disadvantage can be observed in commercial codes for dynamic simulations. Classical finite element solution fails. The space-time finite element approach is the only method which now results in convergent solutions and can be successfully applied in practice. Characteristic matrices and resulting solution scheme are briefly described. Examples prove the efficiency of the approach.
Keywords: moving mass, time integration, space-time finite element

Introduction

Classical problems of structures subjected to a moving force were intensively treated in recent years. Closed analytical solutions can be found for example in [1]. We must mention here that numerous publications deal with the problem, starting from the 18th century. Numerical application of the moving force is also relatively simple. The force for example can be distributed between neighbouring nodes in the mesh with the ratio varying in time and depending on the position of the particle. The problem of inertial moving load applied to discrete systems and efficiently solved unfortunately is practically not reported. Inertial force, which should be considered as a couple of a force and a mass is usually replaced by a spring-mass system. Finally the problem is solved as a problem with a massless force. This approach is characteristic of significant error, which raises to the ratio 1:3 comparing with the accurate solution, in the case of the speed between 0.8 and 1.0 of the wave speed (Fig. 1). We must also emphasize that the ad-hoc mass distribution between neighbouring nodes simply fails. In the case of the beam at low speed ranges and low ratio of the moving mass to the beam mass results exhibit errors. Unfortunately, Such formulations exist in spite of a wrong formulation and analysis.

In the paper we present the numerical approach to the moving inertial load problem. Classical finite element method with Newmark time integration scheme mentioned in [2, 3] fails. The space-time finite element method is the only method which enables us to describe the mass passing through the spatial finite element in a continuous way. We present the solution in the case of a string and a Bernoulli-Euler beam. The reader should be familiar with the basics of the space-time finite element approach described in velocities [4, 5].

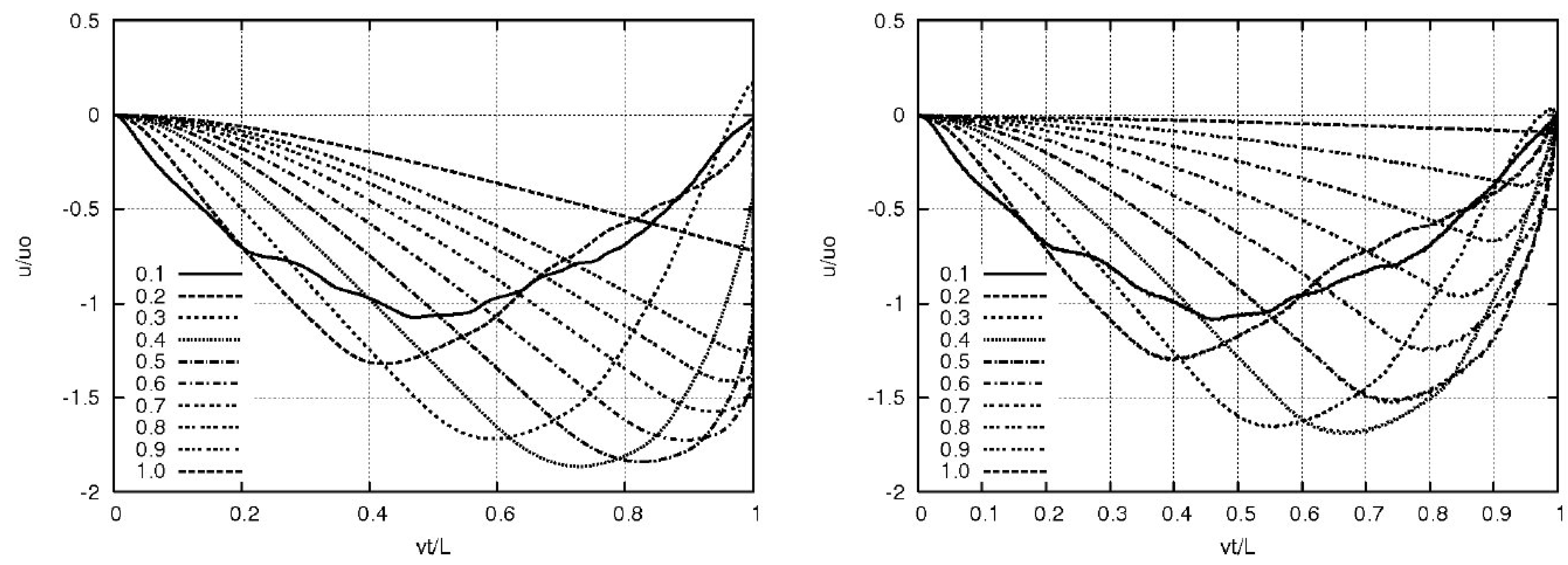

Figure 1: Displacements under the moving mass from the semi-analytical solution (left) and under the rigid oscillator (right).

The motion equation of a string under a moving inertial load with a constant speed v has a form

$$-N\,\frac{\partial^2 u(x,t)}{\partial x^2} + \rho A\,\frac{\partial^2 u(x,t)}{\partial t^2} = \delta(x-vt)\,P - \delta(x-vt)\,m\,\frac{\partial^2 u(vt,t)}{\partial t^2}\,. \quad (1)$$

We assume initial conditions $u(x,0) = 0$, $\dot{u}(x,0) = 0$ and boundary conditions $u(0,t) = u(l,t) = 0$.

1. String element carrying moving mass

The last term $\delta(x-vt)\,m\,\partial^2 u(vt,t)/\partial t^2$ in the motion equation (1) describes the inertial moving mass. $\partial^2 u(vt,t)/\partial t^2$ is the vertical acceleration of the moving mass and at the same time the acceleration of the point of the string in which the mass is temporarily placed (it is $x = x_0 + vt$). The acceleration of the mass $\partial^2 u(vt,t)/\partial t^2$ moving with a constant velocity v, according to the Renaudot formula (which in fact is the chain rule of differentiation), results in three terms:

$$\frac{\partial^2 u(vt,t)}{\partial t^2} = \left.\frac{\partial^2 u(x,t)}{\partial t^2}\right|_{x=vt} + 2v\left.\frac{\partial^2 u(x,t)}{\partial x \partial t}\right|_{x=vt} + v^2\left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{x=vt}. \quad (2)$$

Thus we can separate the transverse acceleration, the Coriolis acceleration, and the centrifugal acceleration, respectively. This is the so-called Renaudot notation for the constant speed v. Another one, the so-called Jakushev notation (or approach) finally gives the same result in our case of the constant mass m.

In our space-time finite element method we formulate equations in terms of velocities. The mass acceleration $\partial^2 u(vt,t)/\partial t^2$ is expressed in terms of velocities as well:

$$\frac{\partial^2 u(vt,t)}{\partial t^2} = \frac{\partial v(vt,t)}{\partial t} = \left.\frac{\partial v(x,t)}{\partial t}\right|_{x=vt} + v\left.\frac{\partial v(x,t)}{\partial x}\right|_{x=vt}. \quad (3)$$

The first term on the right-hand side of (2) states the real inertia (when multiplied by m) and the second term (also multiplied by m) expresses forces similar to damping forces.

In the final stage three resulting matrices are responsible for transverse inertia (the matrix has the form of the inertia matrix), damping forces (the matrix multiplied by the velocity vector has a form similar to the Coriolis forces) and stiffness (potential) forces (the matrix, if multiplied by the velocity vector, has a form similar to the centrifugal forces). The third matrix appears as a result of initial displacements in the time interval.

Let us now follow this idea and treat numerically the right-hand side inertial term of (1). The same mathematical steps as in the case of pure string enables us to integrate the inertial term

$$\int_0^h \int_0^b \mathbf{N}^* m\, \delta(x - vt)\, \frac{\partial^2 u(x_0 + vt, t)}{\partial t^2}\, \mathrm{d}x, \mathrm{d}t\ . \qquad (4)$$

First we must formulate the virtual power equation. Then it is integrated in the space-time domain. The resulting virtual work equations allows us to derive required metrices in the time stepping scheme. We use the linear interpolation of the velocity in space and in time. The virtual velocity v^*:

$$v^*(x,t) = \mathbf{N}^*\, \dot{\mathbf{q}}_p = \delta(t - \alpha h) \begin{bmatrix} 1 - \frac{x}{b} \\ \frac{x}{b} \end{bmatrix} \dot{\mathbf{q}}_p \qquad (5)$$

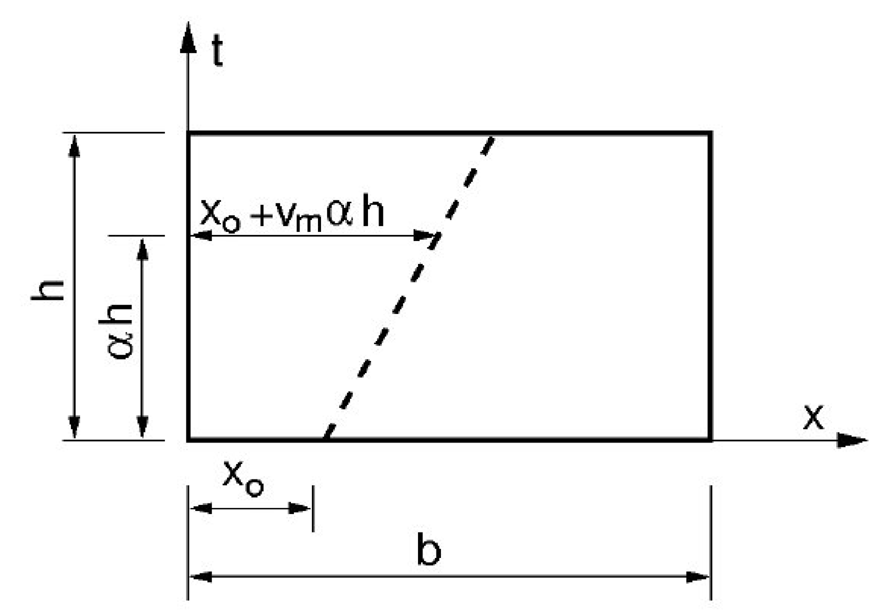

Figure 2: Mass path in the space-time finite element domain.

Consequent integration results in two matrices: the moving mass inertia matrix $\mathbf{K}_m$

$$\mathbf{M}_m = \frac{m}{h} \left[\begin{array}{cc|cc} -(1-\kappa)^2 & -\kappa(1-\kappa) & (1-\kappa)^2 & \kappa(1-\kappa) \\ -\kappa(1-\kappa) & -\kappa^2 & \kappa(1-\kappa) & \kappa^2 \end{array} \right], \qquad (6)$$

where $\kappa = (x_0 + v\alpha h)/b$, x_0 is a starting position of the mass in the space-time element (at $t = t_0$) (see Fig. 2), and the moving mass damping matrix $\mathbf{C}_m$

$$\mathbf{C}_m = \frac{mv}{b} \left[\begin{array}{cc|cc} -(1-\kappa)(1-\beta) & (1-\kappa)(1-\beta) & -(1-\kappa)\beta & (1-\kappa)\beta \\ -\kappa(1-\beta) & \kappa(1-\beta) & -\kappa\beta & \kappa\beta \end{array} \right]. \qquad (7)$$

Let us now consider the contribution of $u(x,0)$ being the constant term of the integration in time. We integrate by parts the virtual work

$$v^2 \int_0^h \int_0^b v^* \frac{\partial^2 u_0}{\partial x^2} \mathrm{d}x\mathrm{d}t = -v^2 \int_0^h \int_0^b \frac{\partial v^*}{\partial x} \frac{\partial u_0}{\partial x} \mathrm{d}x\mathrm{d}t \qquad (8)$$

Since displacements of the left and right node of the element are expressed by $u_L = u_L^0 + h[\beta v_1 + (1-\beta)v_3]$ and $u_R = u_R^0 + h[\beta v_2 + (1-\beta)v_4]$, we can derive the required $\mathrm{d}u_0/\mathrm{d}x$

$$\frac{\mathrm{d}u_0}{\mathrm{d}x} = \frac{u_R - u_L}{b} = \frac{u_R^0 - u_L^0}{b} + \frac{h}{b}[-\beta v_1 + \beta v_2 - (1-\beta)v_3 + (1-\beta)v_4] \tag{9}$$

Matrix $\mathbf{K}_m$ is the stiffness mass matrix

$$\mathbf{K}_m = \frac{hmv^2}{b^2}\left[\begin{array}{cc|cc} \beta & -\beta & 1-\beta & -(1-\beta) \\ -\beta & \beta & -(1-\beta) & 1-\beta \end{array}\right] \tag{10}$$

The term $(u_R^0 - u_L^0)/b$ in (9) multiplied by mv^2/b results in initial nodal forces $\mathbf{e}$ in the space-time layer.

2. Beam element carrying moving mass

We remember that virtual time function v^* in the hat shape is constant in time and in the case of the Bernoulli-Euler beam has the following form

$$v_m^*(x,t) = \left(1 - 3\frac{x^2}{b^2} + 2\frac{x^3}{b^3}\right) v_3 + \ldots\ \dot{\varphi}_3 + \ldots\ v_4 + \ldots\ \dot{\varphi}_4 \tag{11}$$

We recognize here the well known shape functions that describe displacements (or velocities) in terms of nodal displacement and nodal rotations. The same interpolation formulas are applied as real spatial shape functions. Then the the elements of the matrix $\mathbf{M}_m$ can be computed. We present here the analysis in the case of the first element $(\cdot)_{11}$ of the inertia matrix only.

$$\begin{aligned} (\mathbf{M}_m)_{11} &= -\frac{m}{h}\int_0^h\int_0^b \delta(x - x_0 - vt)\left(1 - 3\frac{x^2}{b^2} + 2\frac{x^3}{b^3}\right)^2 \mathrm{d}x\mathrm{d}t = \\ &= -\frac{m}{h}\int_0^h\int_0^b \left[1 - 3\frac{(x_0+vt)^2}{b^2} + 2\frac{(x_0+vt)^3}{b^3}\right]^2 \mathrm{d}x\mathrm{d}t \end{aligned} . \tag{12}$$

We introduce the substitution:

$$s = \frac{x_0 + vt}{b} \quad \text{and } \mathrm{d}s = \frac{v}{b}\mathrm{d}t\ . \tag{13}$$

The coefficient $(\mathbf{M}_m)_{11}$ can be written then

$$(\mathbf{M}_m)_{11} = -\frac{m}{h}\int_0^h \left(1 - 3s^2 + 2s^3\right)^2 \mathrm{d}s = -\frac{m}{h}\frac{b}{v}\left(\frac{4}{7}s^7 - 2s^6 + \frac{9}{5}s^5 + s^4 - 2s^3 + s\right)\Bigg|_0^h \tag{14}$$

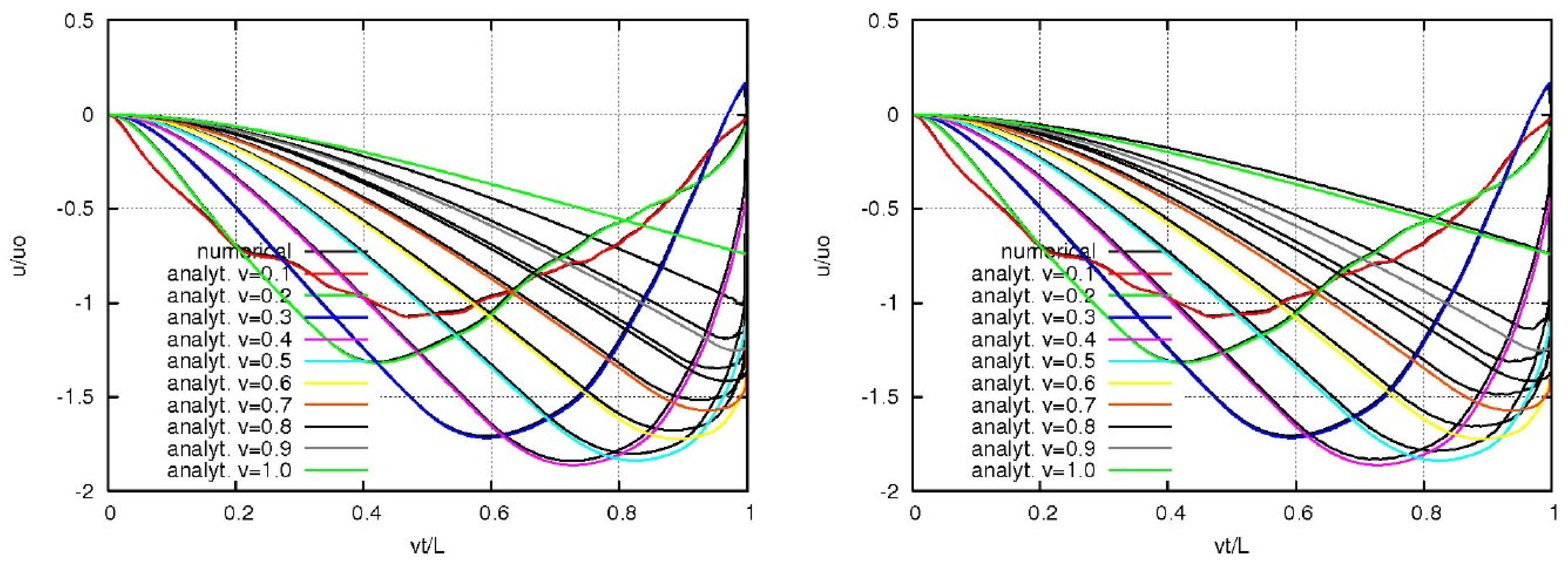

Figure 3: Displacements under moving mass – space-time finite element solution for $\alpha = 0.5$ (upper) and $\alpha = 1.0$ (lower) compared with analytical solution.

3. Numerical results

In our tests the string was discretized by a set of 200 finite elements. The time step h was equal to $b/40v$. It means that the mass passes from joint to joint in 40 time steps. Results obtained by the space-time finite element method are presented in Fig. 3.

Higher velocity can also be considered. Fig. 4 presents displacements in time of the particle for $0.9 \leq v/c \leq 1.2$. We notice a good coincidence of the plot with the expected zero line. We can recall only for information the plot of oscillator displacements moving over the span. The oscillator spring stiffness was assumed to be high enough, to simulate a rigid contact of the mass with the string. Results are depicted in Fig. 1. The solution is significantly worse than results obtained with the method presented in this paper.

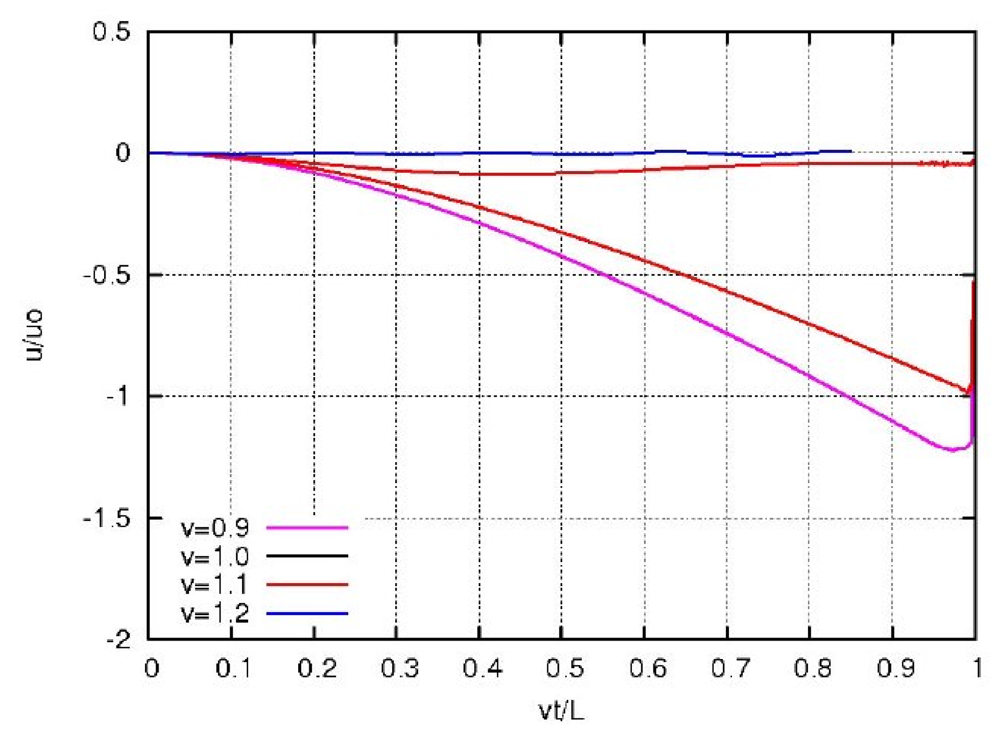

Figure 4: Displacements under the mass moving on a string for v equal to 0.9, 1.0, 1.1 and 1.2 c.

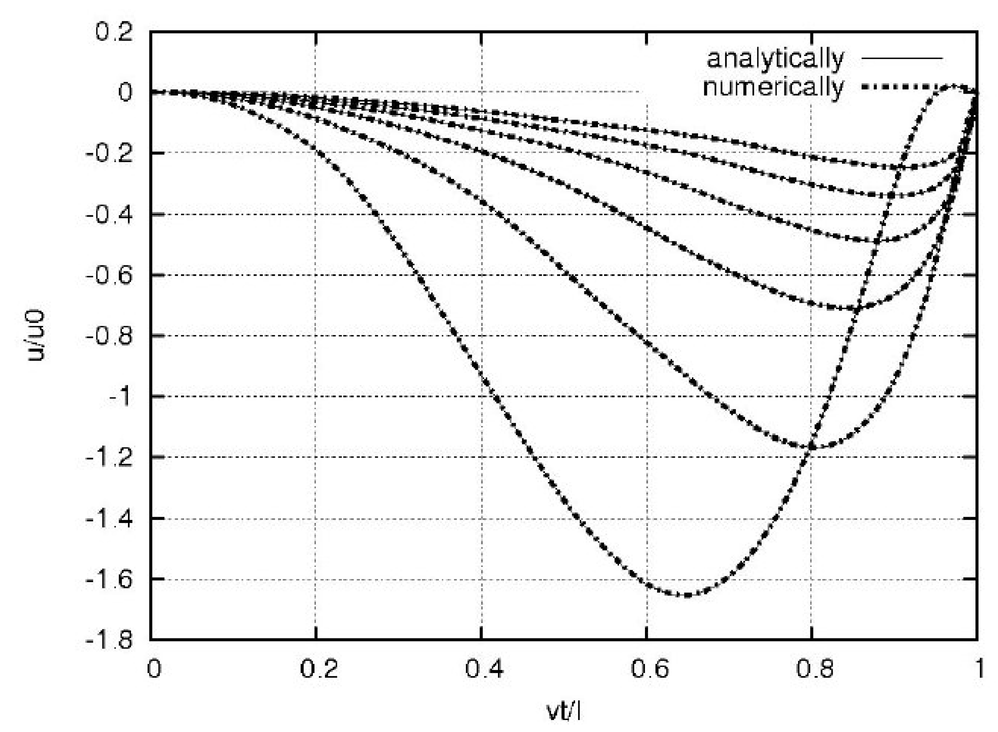

Figure 5: Displacements under the mass moving on the Bernoulli-Euler beam at the speed v=0.1, 0.2, ..., 0.6 (numerical and semi-analytical results).

Numerical results of displacements in time of the Bernoulli-Euler beam are presented in Fig. 5. The following data were applied: E=1.0, A=1.0, I=0.01, l=1.0, ρ=1.0, m=1.0, and P=1.0. We emphasize here that numerical results perfectly coincide with semi-analytical solution in a wide range of the mass velocity. We applied non-dimensional speed v up to 0.6, which corresponds with the 0.4 of the critical speed. The critical speed means the speed of the force travelling in a cyclic way through a beam and increases the vertical deflection to infinity. In the case of the moving mass the critical speed has considerably lower value and in our example we approaches to it.

4. Conclusions

We deal with the problem of the numerical treatment of the moving mass problem. The solution presented in the paper shows the way of mathematical analysis which results in a universal time stepping procedure. It enables us to solve the problem with the arbitrary speed. The solution in the case of the string exhibits discontinuous mass trajectory [6, 7] at the end support. This fact influences high gradients of the solution at the final stage of the motion. This phenomenon is the paradoxical property of the differential equation (1) since considering boundary conditions we intuitively expect smooth curves. Numerical results of the string vibrations exhibit good accuracy, comparing with semi-analytical solution. In the case of the beam the coincidence of both curves is perfect.

The solution presented in the paper is the only correct end efficient numerical solution of the moving mass problem in the literature.

References

1. L. Frýba. *Vibrations of solids and structures under moving loads.* Academia, Prague, 1972.
2. F.V. Filho. Finite element analysis of structures under moving loads. *The Shock and Vibration Digest*, 10(8):27–35, 1978.
3. E.C. Ting, J. Genin, and J.H. Ginsberg. A general algorithm for moving mass problems. *J. Sound Vib.*, 33(1):49–58, 1974.
4. C. Bajer. Space–time finite element formulation for the dynamical evolutionary process. *Appl. Math. and Comp. Sci.*, 3(2):251–268, 1993.
5. C.I. Bajer and C. Bohatier. The soft way method and the velocity formulation. *Comput. and Struct.*, 55(6):1015–1025, 1995.
6. B. Dyniewicz and C.I. Bajer. Paradox of the particle's trajectory moving on a string. *Arch. Appl. Mech.*, 2008. in print.
7. C.I. Bajer and B. Dyniewicz. Space-time approach to numerical analysis of a string with a moving mass. *Int. J. Numer. Meth. Engng.*, 2008. in print.