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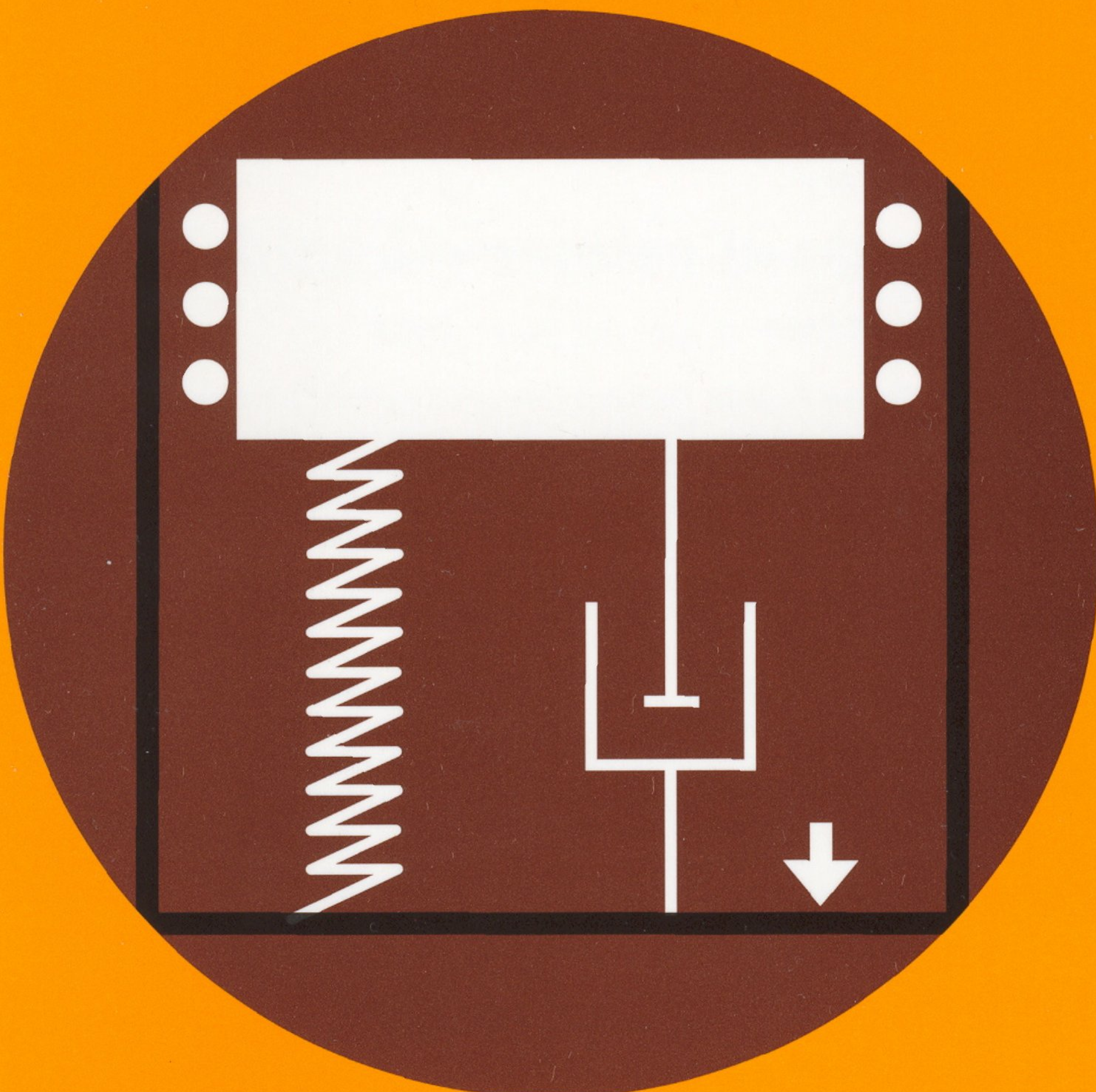

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Discontinuous Trajectory of the Mass Particle Moving on a String or a Beam

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Abstract

The paper deals with a new solution of the string or beam vibrating under a moving mass. Numerous solutions published up to date exhibit incorrect solutions. Moreover, they are not sufficiently simple and can not be applied to a whole range of the mass speed, also in over-critical range. We propose the solution of the problem that allows us to reduce the problem to the second order matrix differential equation. Its solution is characteristic of all features of the critical, sub-critical and over-critical motion. Results exhibit discontinuity of the mass trajectory at the end support point. The closed solution in the case of massless string is analysed and the discontinuity is mathematically proved. Numerical results obtained for inertial string demonstrate similar features. Small vibrations are analysed and that is why the effect discussed in the paper is of pure mathematical interest. However, the phenomenon can increase the complexity in discrete solutions.

1. Introduction

Inertial moving loads are frequently treated in engineering problems. Some problems as train-track interaction, vehicle-bridge interaction, pantograph collectors in railways, magnetic rails, guideways in robotic solutions, etc. are of real practical importance. The problem has been widely treated in literature (e.g. Szcześniak, 1990; Frýba, 1972). Recent papers contribute the analysis of complex problems of structures subjected to moving inertial load (Jia-Jang Wu, 2005) or oscillator (Metrikine and Verichev, 2001; Pesterev et al., 2003; Biondi and Muscolino, 2005). Variable speed was analysed in (Andrianov and Awrejcewicz, 2006; Michaltsos, 2002). Unfortunately, the beam is subjected by massless forces. Equivalent mass influence is analysed by Gavrilov, 2006). However, detailed investigation in the field does not result in numerical procedures implemented to commercial codes. One can perform a simulation of extremely complex problems except problems with moving loads.

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We must emphasise here that string or bar vibrations represent purely hyperbolic differential equation, with all attributes of wave front propagation. Beams, both Bernoulli-Euler and Timoshenko type, exhibit additionally parabolic properties of the solution. In this case final results and diagrams have strong contribution of bending and wave phenomena are diluted by low gradient bending. This is the reason why the beam bending problem always results in acceptable trajectories. Differences between theoretically correct solution and those obtained by a certain approach in the case of low range speed moving mass are not well visible. Most of results are accepted; even they do not give correct solutions. The crucial point in the formulation of the term contributing the moving mass into the differential equation is the difference between $\delta(\upsilon t,t)\partial^2 u(x,t)/\partial t^2$ and $\delta(\upsilon t,t)\partial^2 u(\upsilon t,t)/\partial t^2$. Further implementation to the first case term of the so-called Renaudot formula, which in fact is the chain rule derivative, does not result in the identical final mathematical form as in the case of the second term.

In the paper we present two different approaches to the problem of moving mass (Fig. 1). The first one is derived from the purely mathematical analysis performed by the Fourier method. The second one is based on the Lagrange equation of the second type.

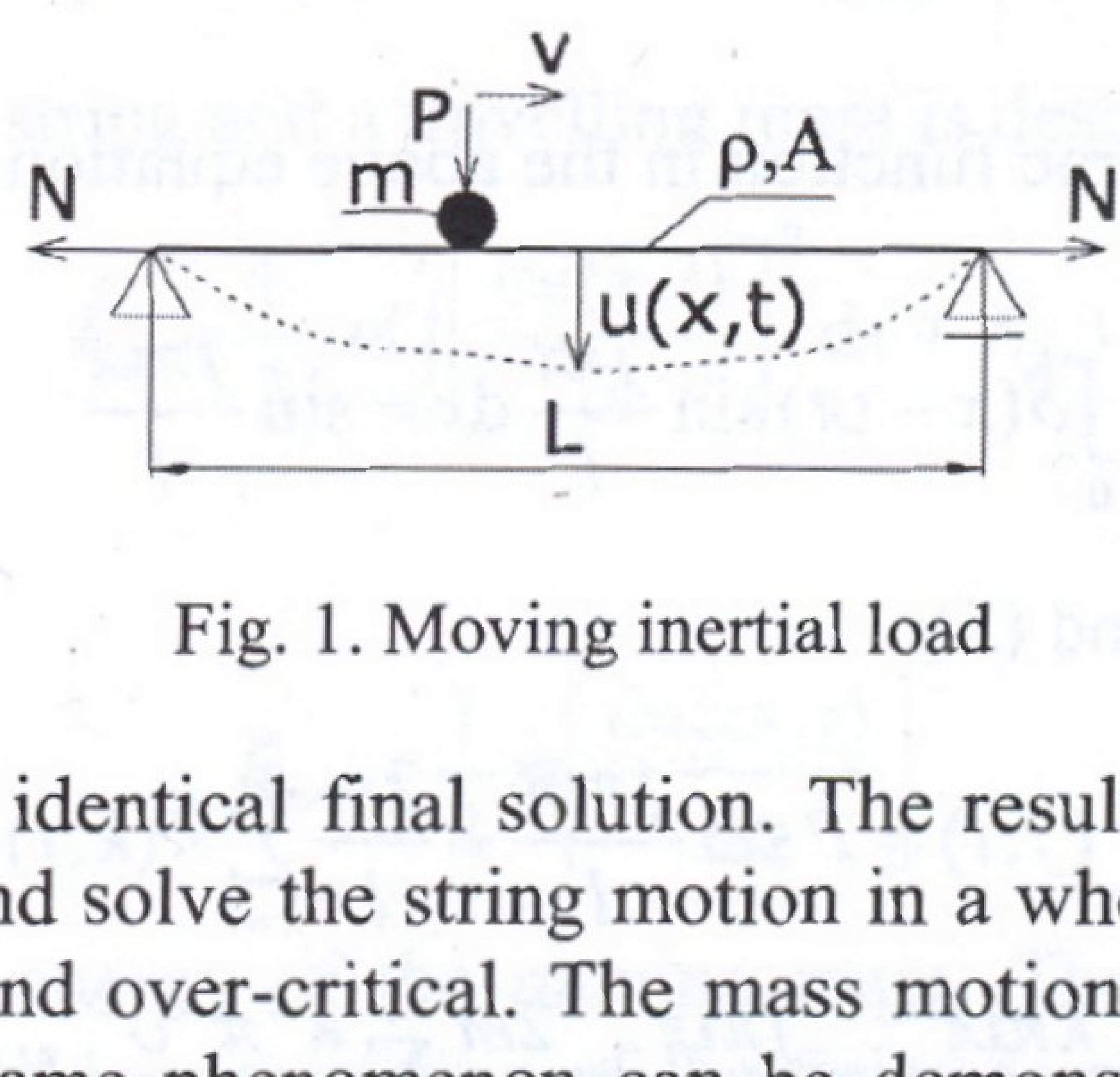

Fig. 1. Moving inertial load

Both ways result in identical final solution. The resulting differential equation allows us to describe and solve the string motion in a whole range of the velocity: under-critical, critical and over-critical. The mass motion exhibits discontinuity at the end support. The same phenomenon can be demonstrated in the case of the Timoshenko beam. The Bernoulli-Euler beam model is free of this feature.

2. Mathematical analysis

2.1. The Fourier solution

The solution can be obtained by direct mathematical solving of the original equation (9), see Dyniewicz and Bajer (2007). In order to reduce partial differential equation to ordinary differential equation, we apply Fourier sine integral transformation in a finite range (*i.e.* finite length of the string), (1), (2):

$$V(j,t)=\int_0^l u(x,t)\sin\frac{j\pi x}{l}\mathrm{d}x \tag{1}$$

$$u(x,t)=\frac{2}{l}\sum_{j=1}^{\infty}V(j,t)\sin\frac{j\pi x}{l} \tag{2}$$

We can present each of the functions as a infinite sum of sine functions (2) with respective coefficients (1). Then the expansion of the moving mass acceleration in a series has a form

$$\frac{\partial^2 u(\upsilon t,t)}{\partial t^2}=\frac{2}{l}\sum_{k=1}^{\infty}\left[\ddot{V}(k,t)\sin\frac{k\pi\upsilon t}{l}+\frac{2k\pi\upsilon}{l}\dot{V}(k,t)\cos\frac{k\pi\upsilon t}{l}-\frac{k^2\pi^2\upsilon^2}{l^2}V(k,t)\sin\frac{k\pi\upsilon t}{l}\right] \tag{3}$$

The integral transformation (1) of the equation (9) with consideration of (3) can be performed

$$N\frac{j^2\pi^2}{l^2}V(j,t)+\rho A\ddot{V}(j,t)=P\sin\frac{j\pi ct}{l}-m\frac{\partial^2 u(\upsilon t,t)}{\partial t^2}\int_0^l\delta(x-\upsilon t)\sin\frac{j\pi x}{l}\mathrm{d}x \tag{4}$$

The integral with delta Dirac function in the above equation is as follows

$$\int_0^l\delta(x-\upsilon t)\sin\frac{j\pi x}{l}\mathrm{d}x=\sin\frac{j\pi\upsilon t}{l} \tag{5}$$

Let us consider now (3) and (5):

$$N\frac{j^2\pi^2}{l^2}V(j,t)+\rho A\ddot{V}(j,t)=P\sin\frac{j\pi\upsilon t}{l}-\frac{2m}{l}\sum_{k=1}^{\infty}\ddot{V}(k,t)\sin\frac{k\pi\upsilon t}{l}\sin\frac{j\pi\upsilon t}{l}$$
$$-\frac{2m}{l}\sum_{k=1}^{\infty}\frac{2k\pi\upsilon}{l}\dot{V}(k,t)\cos\frac{k\pi\upsilon t}{l}\sin\frac{j\pi\upsilon t}{l}+\frac{2m}{l}\sum_{k=1}^{\infty}\frac{k^2\pi^2\upsilon^2}{l^2}V(k,t)\sin\frac{k\pi\upsilon t}{l}\sin\frac{j\pi\upsilon t}{l} \tag{6}$$

Finally, the motion equation after Fourier transformation can be written

$$\rho A\ddot{V}(j,t)+\alpha\sum_{k=1}^{\infty}\ddot{V}(k,t)\sin\omega_k t\sin\omega_j t+2\alpha\sum_{k=1}^{\infty}\omega_k\dot{V}(k,t)\cos\omega_k t\sin\omega_j t+$$
$$+\Omega^2V(j,t)-\alpha\sum_{k=1}^{\infty}\omega_k^2V(k,t)\sin\omega_k t\sin\omega_j t=P\sin\omega_j t \tag{7}$$

where

$$\omega_k = \frac{k\pi\upsilon}{l}, \quad \omega_j = \frac{j\pi\upsilon}{l}, \quad \Omega^2 = N\frac{j^2\pi^2}{l^2}, \quad \alpha = \frac{2m}{l} \tag{8}$$

2.2. The Lagrange equation

Let us consider a string of the length l, cross-sectional area A, mass density ρ, tensile force N, subjected to a mass m accompanied by a force P (Fig. 1), moving with a constant speed υ. The motion equation of the string under moving inertial load with a constant speed υ has a form

$$-N\frac{\partial^2 u(x,t)}{\partial x^2} + \rho A\frac{\partial^2 u(x,t)}{\partial t^2} = \delta(x-\upsilon t)P - \delta(x-\upsilon t)m\frac{\partial^2 u(\upsilon t,t)}{\partial t^2} \tag{9}$$

We impose boundary conditions

$$u(0,t)=0, \quad u(l,t)=0 \tag{10}$$

and initial conditions

$$u(x,0)=0, \quad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0 \tag{11}$$

The kinetic energy of a string and a travelling mass is described by the equation

$$E_k = \frac{1}{2}\rho A\int_0^l \left[\frac{\partial u(x,t)}{\partial t}\right]^2 \mathrm{d}x + E_{km} \tag{12}$$

where

$$E_{km} = \frac{1}{2}m\left[\frac{\partial u(\upsilon t,t)}{\partial t}\right]^2 \tag{13}$$

contributes the kinetic energy of the moving mass. The potential energy of the string can be determined by computing of the δx to δs change of its infinitesimal segment. The work $N(\delta x - \delta s)$ integrated in space allow us to compute the potential energy of the string

$$E_p = \int_0^l N(\delta s - \delta x) = N\int_0^l \left\{\sqrt{1+\left[\frac{\partial u(x,t)}{\partial x}\right]^2} - 1\right\}\mathrm{d}x \tag{14}$$

We apply the expansion of (14) into the Maclaurin series and we consider only the first term of it

$$E_p = N\int_0^l \left\{ \sqrt{1+\left[\frac{\partial u(x,t)}{\partial x}\right]^2} - 1 \right\} \mathrm{d}x \approx \frac{1}{2} N \int_0^l \left[\frac{\partial u(x,t)}{\partial x}\right]^2 \mathrm{d}x \tag{15}$$

If we neglect next terms of the series, we assume higher powers of $\partial u(x,t)/\partial x$ to be nearly equal to zero. The equation (15) can be applied to the problem of small displacements of the string only. Finally the potential energy of the system, *i.e.* the string and the moving constant force P gains a form

$$E_p = \frac{1}{2} N \int_0^l \left[\frac{\partial u(x,t)}{\partial x}\right]^2 \mathrm{d}x - Pu(\upsilon t, t) \tag{16}$$

The examined string has a finite length. It is convenient to use standing waves for description of its displacements. We assume the solution in the following form:

$$u(x,t) = \sum_{i=1}^{\infty} U_i(x)\xi_i(t) \tag{17}$$

$\xi_i(t)$ are the generalized coordinate functions. In order to compute both the kinetic and the potential energy and to determine its derivatives required, we express them by generalized coordinates. We derive first the equation (17) with respect to t

$$\frac{\partial u(x,t)}{\partial t} = \sum_{i=1}^{\infty} U_i(x)\dot{\xi}_i(t) \tag{18}$$

and with respect to spatial variable x

$$\frac{\partial u(x,t)}{\partial x} = \sum_{i=1}^{\infty} U_i'(x)\xi_i(t) \tag{19}$$

The displacement of the string in the contact point with a travelling mass is given by the equation.

$$u(\upsilon t,t) = \sum_{i=1}^{\infty} U_i(\upsilon t)\xi_i(t) \tag{20}$$

The transverse velocity of the moving mass is expressed by a composite derivative. It expresses the load travelling along the string

$$\frac{\partial u(\upsilon t,t)}{\partial t} = \upsilon \sum_{i=1}^{\infty} U_i'(x)\xi_i(t)\Bigg|_{x=\upsilon t} + \sum_{i=1}^{\infty} U_i(x)\dot{\xi}_i(t)\Bigg|_{x=\upsilon t} \tag{21}$$

According to the above equation the velocity $\partial u(vt,t)/\partial t$ is expressed as a function of both generalized coordinates and the derivative of generalized coordinates with respect to time

$$\frac{\partial u(\upsilon t,t)}{\partial t} = f(\xi_i, \dot{\xi}_i) \tag{22}$$

After rearrangement of the equation (12), with respect to (18), the total energy is given the by the following form

$$E_k = \frac{1}{2}\rho A \sum_{i,j=1}^{\infty} \dot{\xi}_i(t)\dot{\xi}_j(t) \int_0^l U_i(x)U_j(x)\mathrm{d}x + \frac{1}{2}m\left[\frac{\partial u(\upsilon t,t)}{\partial t}\right]^2 \tag{23}$$

We assume orthogonal functions (24) which fulfil boundary conditions (10)

$$U_i(x) = \sin\frac{i\pi x}{l} \tag{24}$$

The orthogonality of functions $U_i(x)$ allows us to write

$$\int_0^l U_i(x)U_j(x)\mathrm{d}x = \begin{cases} \frac{1}{2}l & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{25}$$

The kinetic energy of the system (23) according to (24) and (25) is described by the relation

$$E_k = \frac{1}{4}\rho A l \sum_{i=1}^{\infty} \dot{\xi}_i^2(t) + \frac{1}{2}m\left[\frac{\partial u(\upsilon t,t)}{\partial t}\right]^2 \tag{26}$$

In the case of the potential energy (19), with respect to the (16) integrated by parts, has the following form

$$E_p = \frac{1}{2}N\sum_{i,j=1}^{\infty} \xi_i(t)\xi_j(t)\int_0^l U_i'(x)U_j'(x)\mathrm{d}x - Pu(\upsilon t,t) =$$

$$= -\frac{1}{2}N\sum_{i,j=1}^{\infty} \xi_i(t)\xi_j(t)\int_0^l U_i''(x)U_j(x)\mathrm{d}x - P\sum_{i=1}^{\infty} U_i(\upsilon t)\xi_i(t) \tag{27}$$

We can derive the function (24)

$$U_i''(x) = -\frac{i^2\pi^2}{l^2}U_i(x) \tag{28}$$

The equation (27) with respect to (28) can be written in the form

$$E_p = \frac{1}{2}N\sum_{i,j=1}^{\infty} \frac{i^2\pi^2}{l^2}\xi_i(t)\xi_j(t)\int_0^l U_i(x)U_j(x)\mathrm{d}x - P\sum_{i=1}^{\infty} U_i(\upsilon t)\xi_i(t) \tag{29}$$

Finally the potential energy of the string, with respect to (25) can be described by the equation

$$E_p = \frac{1}{4} Nl \sum_{i=1}^{\infty} \frac{i^2 \pi^2}{l^2} \xi_i^2(t) - P \sum_{i=1}^{\infty} \xi_i(t) \sin \frac{i\pi \upsilon t}{l} \tag{30}$$

Now, when we have kinetic and potential energy described in generalized coordinates and the derivative of generalized coordinates with respect to time, we can formulate the Lagrange equation, which general form is given by the equation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E_k}{\partial \dot{\xi}_i}\right) - \frac{\partial E_k}{\partial \xi_i} + \frac{\partial E_p}{\partial \xi_i} = 0 \tag{31}$$

In order to obtain the Lagrange equation describing our problem, we must compute respective required terms. From (13) and (22) we have derivative of kinetic energy of travelling mass E_{km} with respect to ξ_i and $\dot{\xi}_i$.

$$\frac{\partial E_{km}}{\partial \xi_i} = m \frac{\partial u(\upsilon t, t)}{\partial t} \frac{\mathrm{d}}{\mathrm{d}\xi_i}\left(\frac{\partial u(\upsilon t, t)}{\partial t}\right) \tag{32}$$

$$\frac{\partial E_{km}}{\partial \dot{\xi}_i} = m \frac{\partial u(\upsilon t, t)}{\partial t} \frac{\mathrm{d}}{\mathrm{d}\dot{\xi}_i}\left(\frac{\partial u(\upsilon t, t)}{\partial t}\right) \tag{33}$$

We compute the derivative of the kinetic energy for hole system (26) with respect to ξ_i and $\dot{\xi}_i$, taking into account (32) and (33).

$$\frac{\partial E_k}{\partial \xi_i} = \frac{\partial E_{km}}{\partial \xi_i} = m\left[\upsilon^2 \sum_{i,j=1}^{\infty} \frac{ij\pi^2}{l^2} \cos\frac{i\pi\upsilon t}{l} \cos\frac{j\pi\upsilon t}{l} \xi_j(t) + \upsilon \sum_{i,j=1}^{\infty} \frac{i\pi}{l} \cos\frac{i\pi\upsilon t}{l} \sin\frac{i\pi\upsilon t}{l} \dot{\xi}_j(t)\right] \tag{34}$$

$$\frac{\partial E_k}{\partial \dot{\xi}_i} = \frac{1}{2}\rho Al \sum_{i=1}^{\infty} \dot{\xi}_i(t) + \frac{\partial E_{km}}{\partial \dot{\xi}_i} = \frac{1}{2}\rho Al \sum_{i=1}^{\infty} \dot{\xi}_i(t) + m\left[\upsilon \sum_{i,j=1}^{\infty} \frac{j\pi}{l} \sin\frac{i\pi\upsilon t}{l} \cos\frac{i\pi\upsilon t}{l} \xi_j(t) + \right.$$
$$\left. + \sum_{i,j=1}^{\infty} \frac{j\pi}{l} \sin\frac{i\pi\upsilon t}{l} \sin\frac{i\pi\upsilon t}{l} \dot{\xi}_j(t)\right] \tag{35}$$

The derivative of the potential energy (30) with respect to generalized coordinates ξ_i.

$$\frac{\partial E_p}{\partial \xi_i} = \frac{1}{2} Nl \sum_{i=1}^{\infty} \frac{i^2\pi^2}{l^2} \xi_i(t) - P \sum_{i=1}^{\infty} \sin\frac{i\pi\upsilon t}{l} \tag{36}$$

The derivative of (35) with respect to t is as follows.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E_k}{\partial \dot{\xi}_i}\right)=\frac{1}{2}\rho Al\sum_{i=1}^{\infty}\ddot{\xi}_i(t)+m\left\{\sum_{i,j=1}^{\infty}\frac{j\pi\upsilon}{l}\frac{\mathrm{d}}{\mathrm{d}t}\left[\sin\frac{i\pi\upsilon t}{l}\cos\frac{j\pi\upsilon t}{l}\xi_j(t)\right]+\right.$$

$$\left.+\sum_{i,j=1}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}\left[\sin\frac{i\pi\upsilon t}{l}\sin\frac{j\pi\upsilon t}{l}\dot{\xi}_j(t)\right]\right\} \tag{37}$$

Finally the Lagrange equation (31) in the case of our problem of the inertial string subjected to moving inertial force has a following form

$$\frac{1}{2}\rho Al\sum_{i=1}^{\infty}\ddot{\xi}_i(t)+m\sum_{i,j=1}^{\infty}\frac{j\pi\upsilon}{l}\frac{\mathrm{d}}{\mathrm{d}t}\left[\sin\frac{i\pi\upsilon t}{l}\cos\frac{j\pi\upsilon t}{l}\right]\xi_j(t)+m\sum_{i,j=1}^{\infty}\frac{j\pi\upsilon}{l}\sin\frac{i\pi\upsilon t}{l}\cos\frac{j\pi\upsilon t}{l}\dot{\xi}_j(t)+$$

$$+m\sum_{i,j=1}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}\left[\sin\frac{i\pi\upsilon t}{l}\sin\frac{j\pi\upsilon t}{l}\right]\dot{\xi}_j(t)+m\sum_{i,j=1}^{\infty}\sin\frac{i\pi\upsilon t}{l}\sin\frac{j\pi\upsilon t}{l}\ddot{\xi}_j(t)+\frac{1}{2}Nl\sum_{i=1}^{\infty}\frac{i^2\pi^2}{l^2}\xi_i(t)-$$

$$-m\left[\upsilon^2\sum_{i,j=1}^{\infty}\frac{ij\pi^2}{l^2}\cos\frac{i\pi\upsilon t}{l}\cos\frac{j\pi\upsilon t}{l}\xi_j(t)+\upsilon\sum_{i,j=1}^{\infty}\frac{i\pi}{l}\cos\frac{i\pi\upsilon t}{l}\sin\frac{j\pi\upsilon t}{l}\dot{\xi}_j(t)\right]=P\sum_{i=1}^{\infty}\sin\frac{i\pi\upsilon t}{l} \tag{38}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sin\frac{i\pi\upsilon t}{l}\cos\frac{j\pi\upsilon t}{l}\right]=\frac{i\pi\upsilon}{l}\cos\frac{i\pi\upsilon t}{l}\cos\frac{j\pi\upsilon t}{l}-\frac{j\pi\upsilon}{l}\sin\frac{i\pi\upsilon t}{l}\sin\frac{j\pi\upsilon t}{l} \tag{39}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sin\frac{i\pi\upsilon t}{l}\sin\frac{j\pi\upsilon t}{l}\right]=\frac{i\pi\upsilon}{l}\cos\frac{i\pi\upsilon t}{l}\sin\frac{j\pi\upsilon t}{l}+\frac{j\pi\upsilon}{l}\sin\frac{i\pi\upsilon t}{l}\cos\frac{j\pi\upsilon t}{l} \tag{40}$$

We have the differential equation of variable coefficients. Finally (38) can be written in a following form

$$\ddot{\xi}_i(t)+\frac{2m}{\rho Al}\sum_{j=1}^{\infty}\ddot{\xi}_j(t)\sin\frac{i\pi\upsilon t}{l}\sin\frac{j\pi\upsilon t}{l}+\frac{4m}{\rho Al}\sum_{j=1}^{\infty}\frac{i\pi\upsilon}{l}\dot{\xi}_j(t)\sin\frac{i\pi\upsilon t}{l}\cos\frac{j\pi\upsilon t}{l}+$$

$$\frac{N}{\rho A}\frac{i^2\pi^2}{l^2}\xi_i(t)-\frac{2m}{\rho Al}\sum_{j=1}^{\infty}\frac{j^2\pi^2\upsilon^2}{l^2}\xi_j(t)\sin\frac{i\pi\upsilon t}{l}\sin\frac{j\pi\upsilon t}{l}=\frac{2P}{\rho Al}\sin\frac{i\pi\upsilon t}{l} \tag{41}$$

These two methods lead us to the identical differential equation of variable coefficients (41).

2.3. The massless string

We can consider the particular case of our problem: the massless string. The solution is given by a sum and can be derived from our solution (38). This particular solution first was published by Stokes in the case of the beam (Stokes, 1883), then by Smith (1964) and Frýba (1972) for massless string:

$$y(\tau) = \frac{4\alpha}{\alpha - 1}\tau(\tau - 1)\sum_{k=1}^{\infty}\prod_{i=1}^{k}\frac{(a+i-1)(b+i-1)}{c+i-1}\frac{\tau^k}{k!} \tag{42}$$

where $\tau = \upsilon t / l > 0$ is time parameter, $\alpha = Nl/(2m\upsilon^2) > 0$ determines the dimensionless parameter. Parameters a, b and c are given below:

$$a_{1,2} = \frac{3 \pm \sqrt{1+8\alpha}}{2}, \quad b_{1,2} = \frac{3 \mp \sqrt{1+8\alpha}}{2}, \quad c = 2 \tag{43}$$

In the case of $\alpha = 1$ the initial problem has a closed solution. Here we consider the case of $\alpha \neq 1$.

Proof

In (42) we will include the term $\tau(\tau - t)$ into the sum. Thus the equation can be reduced to the following form:

$$(1-\tau)\sum_{k=1}^{\infty}\frac{(a_k)(b_k)}{(c_k)}\frac{\tau^k}{k!} = \frac{ab\tau}{c} + \sum_{k=2}^{\infty}\frac{(a_{k-1})(b_{k-1})}{(c_{k-1})}\left(\frac{(a+k-1)(b+k-1)}{k(c+k-1)} - 1\right)\frac{\tau^k}{(k-1)!} \tag{44}$$

In the above relation $(a_k) = a(a+1)...(a+k-1)$, $(b_k) = b(b+1)...(b+k-1)$ and $(c_k) = c(c+1)...(c+k-1)$. By using Rabbe criterion one can show that for $a+b<c+2$ the limit

$$\lim_{\tau\to 1}\left[\frac{4\alpha}{\alpha-1}\tau(1-\tau)\sum_{k=1}^{\infty}\prod_{i=1}^{k}\frac{(a+i-1)(b+i-1)}{c+i-1}\frac{\tau^k}{k!}\right] \text{ is finite.}$$

Now we can estimate the value of the sum (44). The sum of the first two–three terms, depending on parameters, including $ab\tau/c$, is positive. Next terms are all positive (remember, that P is negative in our numerical example). This proves that the sum (44) is finite and is greater than 0.

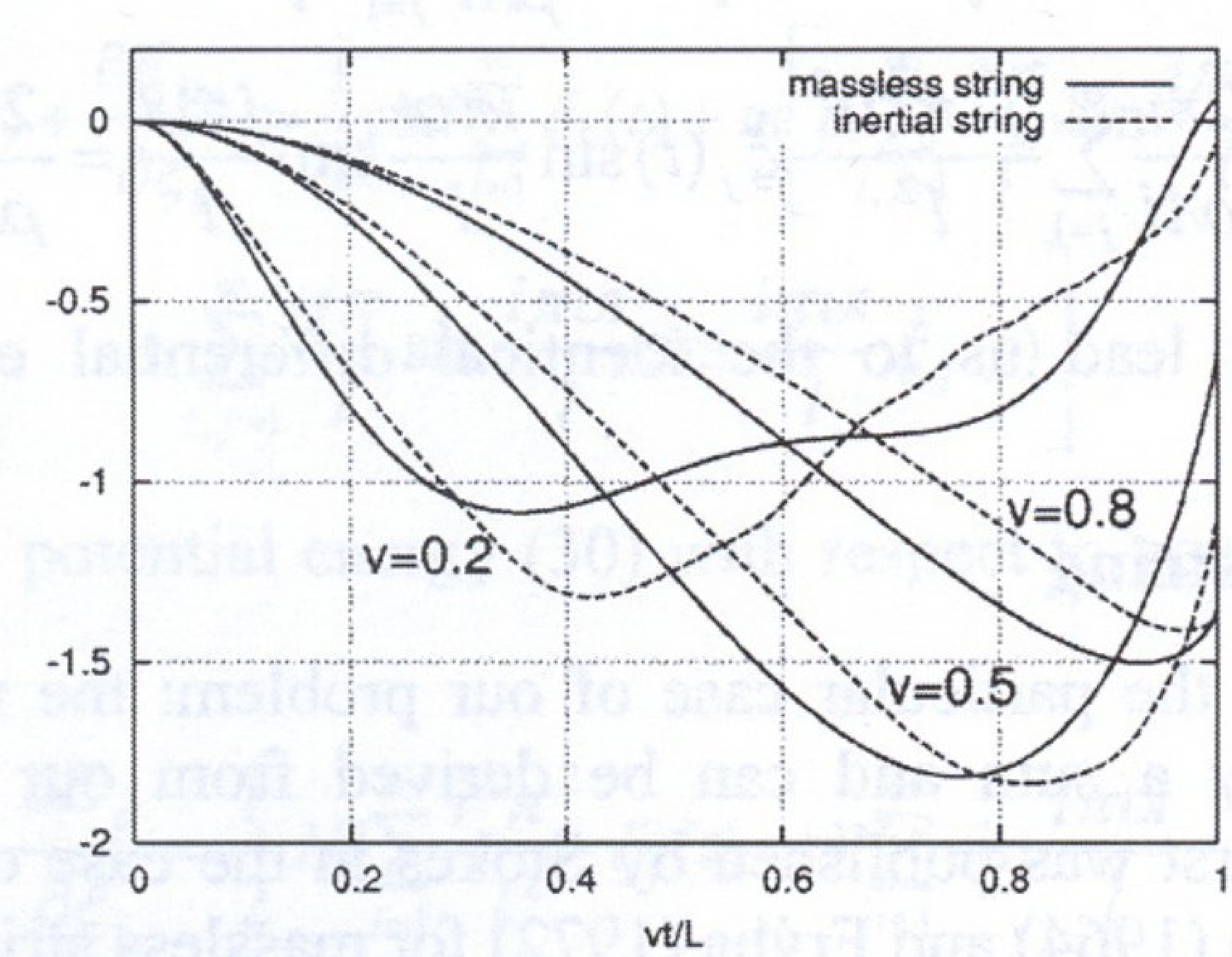

Fig. 2. Comparison of particle's trajectory moving on massless and inertial string

We can observe the same properties of the solution in the case of inertial string. Comparative plot is presented in Fig. 2. We can emphasise that in the case of lower $m/\rho Al$ ratio the coincidence of each pair of curves is higher. However, analytical proof of discontinuity in the case of inertial string is impossible to be obtained, because of the numerical integration stage.

2.4. The beam under the moving mass

The motion of the Bernoulli-Euler beam under the moving mass is described by the equation

$$EI\frac{\partial^4 u(x,t)}{\partial x^4}+\rho A\frac{\partial^2 u(x,t)}{\partial t^2}=\delta(x-\upsilon t)P-\delta(x-\upsilon t)m\frac{\partial^2 u(\upsilon t,t)}{\partial t^2} \tag{45}$$

with boundary conditions

$$u(0,t)=0, \qquad u(l,t)=0, \qquad \left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{x=0}=0, \qquad \left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{x=l}=0 \tag{46}$$

and initial conditions

$$u(x,0)=0, \qquad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}=0 \tag{47}$$

The Fourier transform method carried on in a way as in the case of the string results in he following equation

$$\ddot{V}(j,t)+\alpha\sum_{k=1}^{\infty}\ddot{V}(k,t)\sin\omega_k t\sin\omega_j t+2\alpha\sum_{k=1}^{\infty}\omega_k\dot{V}(k,t)\cos\omega_k t\sin\omega_j t+$$
$$+\Omega^2 V(j,t)-\sum_{k=1}^{\infty}\omega_k^2 V(k,t)\sin\omega_k t\sin\omega_j t=\frac{P}{\rho A}\sin\omega_j t \tag{48}$$

where

$$\omega_k=\frac{k\pi\upsilon}{l}, \qquad \omega_j=\frac{j\pi\upsilon}{l}, \qquad \Omega^2=\frac{EI}{\rho A}\frac{j^4\pi^4}{l^4}, \qquad \alpha=\frac{2m}{\rho Al} \tag{49}$$

The equation (48) can not be easily solved and we must integrate it in a numerical way. We use the matrix notation here

$$\mathbf{M}\begin{bmatrix}\ddot{V}(1,t)\\ \ddot{V}(2,t)\\ \vdots\\ \ddot{V}(n,t)\end{bmatrix}+\mathbf{C}\begin{bmatrix}\dot{V}(1,t)\\ \dot{V}(2,t)\\ \vdots\\ \dot{V}(n,t)\end{bmatrix}+\mathbf{K}\begin{bmatrix}V(1,t)\\ V(2,t)\\ \vdots\\ V(n,t)\end{bmatrix}=\mathbf{P} \tag{50}$$

or in a short form

$$\mathbf{M}\ddot{\mathbf{V}}+\mathbf{C}\dot{\mathbf{V}}+\mathbf{K}\mathbf{V}=\mathbf{P} \tag{51}$$

In the case of the Timoshenko we follow the similar procedure as in the case of the Bernoulli-Euler beam. The resulting equation has the following short form

$$\boldsymbol{\Gamma}\ddddot{\mathbf{V}}+\mathbf{U}\dddot{\mathbf{V}}+\mathbf{M}\ddot{\mathbf{V}}+\mathbf{C}\dot{\mathbf{V}}+\mathbf{K}\mathbf{V}=\mathbf{P}$$

3. Results

Results of the semi-analytical solution are depicted in Fig. 3. The diagram can be compared with displacements of the string under moving oscillator. The analysis of the spring-mass system motion was performed for a relatively rigid spring. However, for significantly high spring rigidity the convergence of the solution was poor of completely lost.

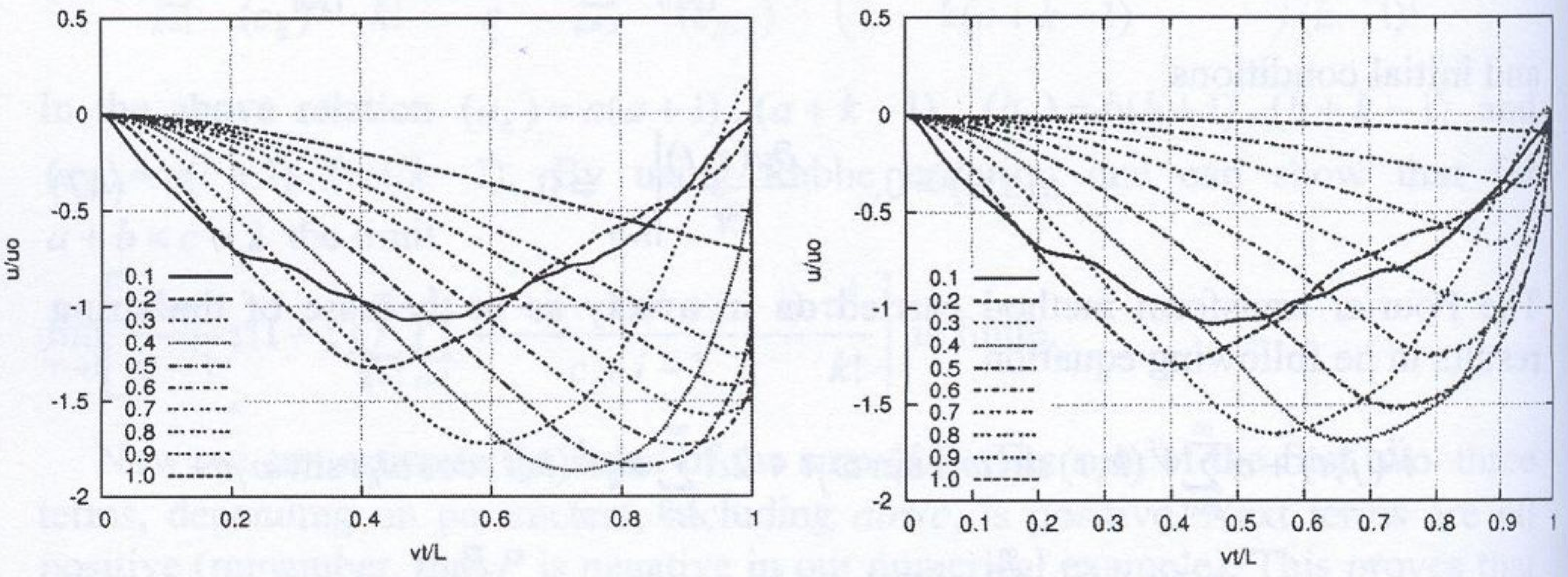

Fig. 3. Semi-analytical solution (left) and displacements of the string under the oscillator (right)

More detailed presentation of the string motion is given in Fig. 4.
We can notice the sharp edge of the wave and reflection from both supports. Moreover, the wave reflection from the travelling mass is clearly visible, especially for the case $v = 1.2c$. Both the mass trajectory and waves are depicted.

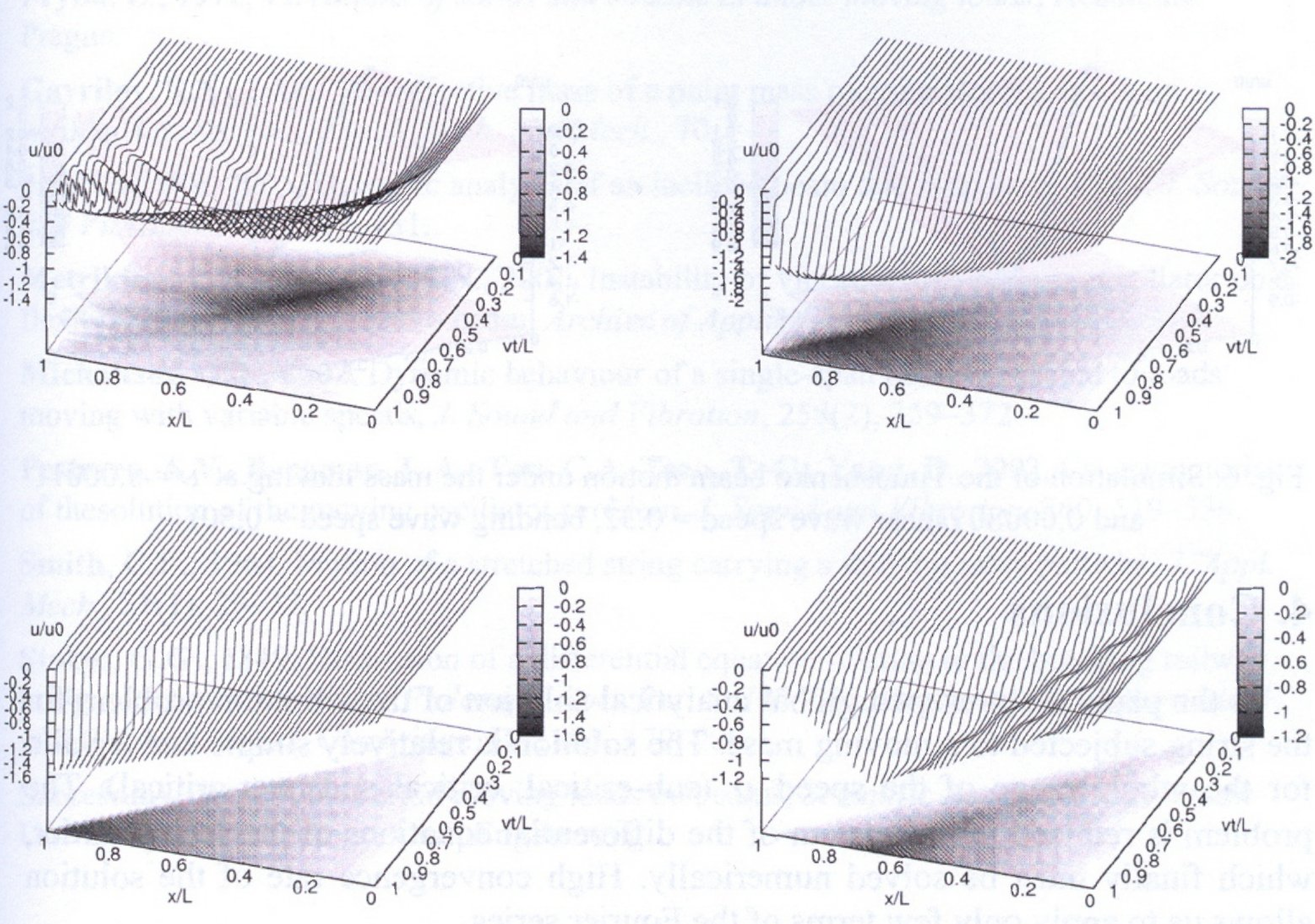

Fig. 4. Simulation of the string motion under the mass moving at υ : 0.2c, 0.5c, 1.0c, 1.5c

Respective diagrams for a beam (Figs. 5, 6).

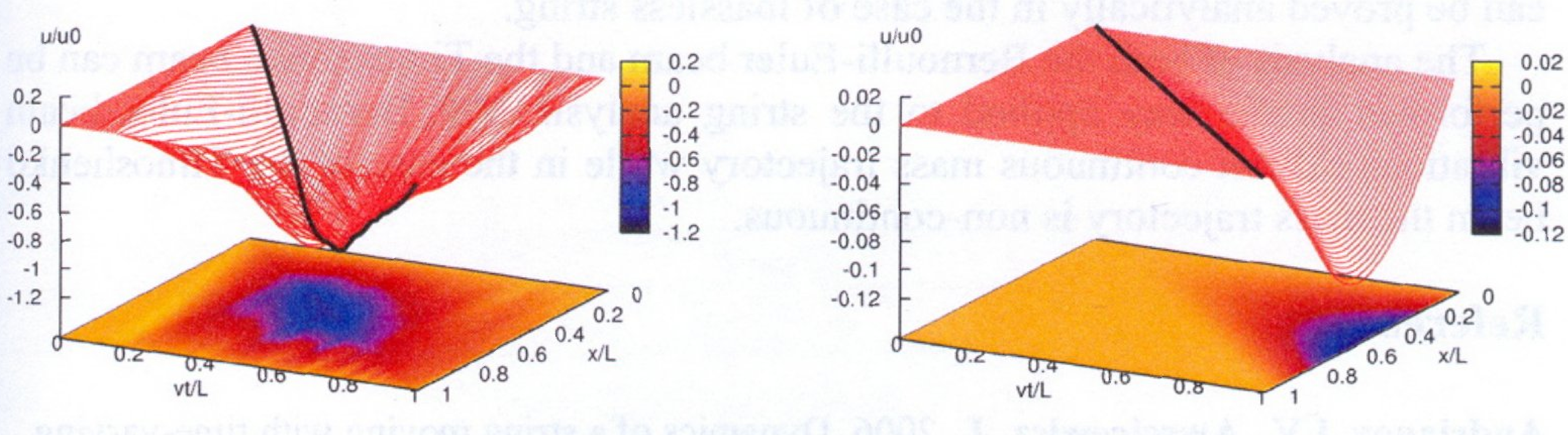

Fig. 5. Simulation of the Timoshenko beam motion under the mass moving at υ : 0.1, 1.0 (shear wave speed = 0.63, bending wave speed = 1.00)

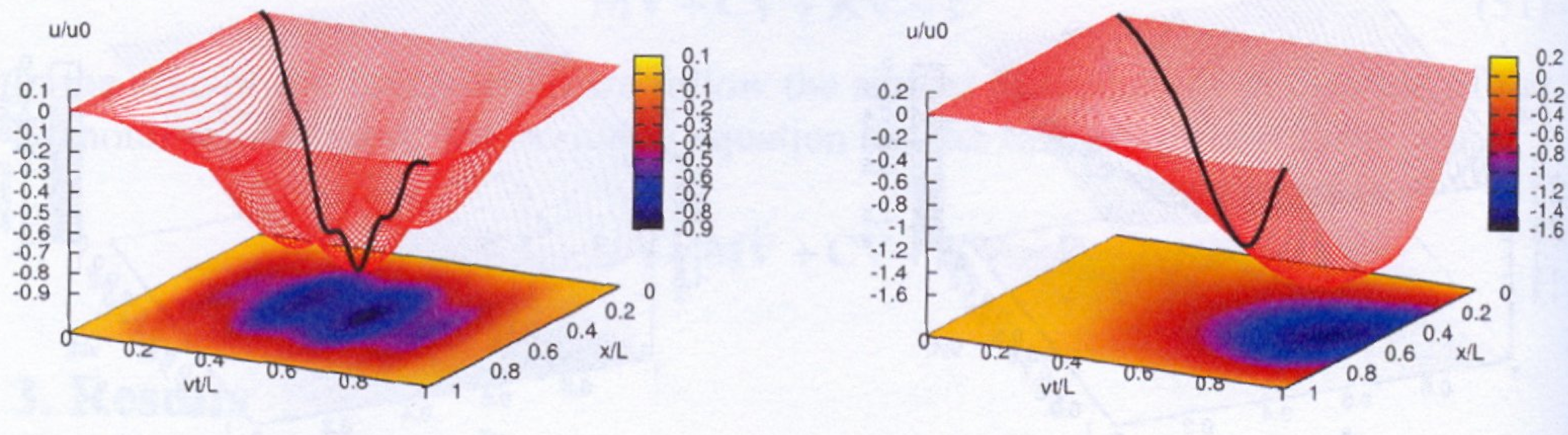

Fig. 6. Simulation of the Timoshenko beam motion under the mass moving at $\upsilon = 0.00011$ and 0.00080 (shear wave speed = 0.32, bending wave speed = 0.50)

4. Conclusions

In the paper we present a global analytical solution of the vibration problem for the string subjected to a moving mass. The solution is relatively simple and is valid for the whole range of the speed υ (sub-critical, critical and over-critical). The problem is reduced to the system of the differential equations of the second order, which finally must be solved numerically. High convergence rate of the solution allows us to apply only few terms of the Fourier series.

The analysis of results exhibits a jump of the mass in the neighbourhood of the end support. The force acting on the mass is, however, limited to the tensile force N. The mass can not be accelerated to the appropriate vertical velocity to arrive directly at the support in a smooth way. Discontinuity of the solution at $x=L$ exists in the case of $\nu>0$ (for each non-zero moving speed, *ie.* $0<\upsilon<c$, $\upsilon=c$ and $\upsilon>c$). It can be proved analytically in the case of massless string.

The analysis of both the Bernoulli-Euler beam and the Timoshenko beam can be performed in the way applied to the string analysis. The Bernoulli-Euler beam vibrations exhibit continuous mass trajectory while in the case of the Timoshenko beam the mass trajectory is non-continuous.

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