Chapter 4

Optimal control problems described by PDEs

by Czesław Bajer[1], Andrzej Myśliński[2], Antoni Żochowski[2] and Bartłomiej Dyniewicz[1]

4.1 Introduction

The mathematical theory of optimal control of processes governed by ordinary or partial differential equations has rapidly emerged into a separate field of the applied mathematics. The optimal control of ordinary differential equations is of interest in the field of aviation and space technology, robotics, chemical processes and power plants to name just a few of the various applications. However in many cases the processes to be optimized cannot be adequately modeled by the ordinary differential equations. Therefore for description of such processes partial differential equations (PDEs), involving functions of several variables, have to be formulated. The list of applications of the PDEs includes, among others, sound, electrostatics, electrodynamics, heat conduction, diffusion, advection, electromagnetic waves, elasticity or fluid problems, freezing processes and many other physical, biological or financial phenomena. These seemingly distinct different phenomena can be formalized identically in terms of PDEs, which shows that they are governed by the same underlying dynamic. Just as ordinary differential equations often model one-dimensional dynamical systems, partial differential equations often model multidimensional systems. There are many types of partial differential equations [111, 317]. Some linear, second-order partial differential equations can be classified as parabolic, hyperbolic or elliptic. Others such as the Euler-Tricomi equation have different types in different regions. The classification provides a guide to appropriate initial and boundary conditions, and to smoothness of the solutions. Here we confine to the two-dimensional elliptic ones as the model equations.
The mathematical analysis of optimal control problems for systems described by the partial differential equations consists in the investigation of the:

1. existence, uniqueness and regularity of the solutions to the partial differential equations,

[1] Institute of Fundamental Research in Technology, Polish Academy of Sciences, 02-106 Warsaw
[2] Systems Research Institute, Polish Academy of Sciences, 01-447 Warsaw

2. existence and regularity of the optimal control,
3. necessary optimality conditions and the adjoint equations,
4. second-order sufficient optimality conditions,
5. numerical methods for solving the optimal control problem and the partial differential equation.

Since our main interest is in numerical methods for solving PDEs constrained optimal control problems we confine to provide the necessary optimality conditions for these problems. The discussion on other topics the reader can find in the literature (see [16, 17, 29, 53, 56, 81, 86, 90, 92, 102, 106, 111, 154, 155, 160, 166, 172, 181, 186, 194, 248]).

In this chapter first we recall the notion of Sobolev spaces and Sobolev imbeddings theorems used as the main tool in the analysis of PDEs. The existence results for linear and nonlinear abstract elliptic problems are also recalled. As a special case of these abstract problems the model second order linear elliptic boundary value problems in the operator and variational forms are formulated. Next the optimal control problems for the second order elliptic PDEs are considered. The optimal control existence results as well as the first order necessary optimality condition for this class of optimal control problems where control to state operator is used are recalled. Finally the first order necessary optimality condition for optimal control problems governed by the second order linear elliptic equations is formulated using Lagrange multiplier techniques.

4.2 Functional spaces

Consider domain $\Omega \subset \mathscr{R}^N$, $N = 2,3$. Assume this domain is bounded and is enough regular, i.e., it has at least Lipschitz continuous boundary Γ. This boundary consists from two disjoint parts Γ_1 and Γ_2 such that $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\bar{\Gamma}_1 \cap \bar{\Gamma}_2 = \emptyset$. Following [248] let us introduce the required functional spaces.

Let $\mathbf{N}$ denote a set of integers and $k \in \mathbf{N} \cup \{0\}$. Then $\mathscr{C}^k(\Omega)$ denotes the set of all continuous real functions defined in Ω, whose derivatives up to the order k are continuous in Ω. $\mathscr{C}^\infty(\Omega) = \bigcap_{k=0}^{\infty} \mathscr{C}^k(\Omega)$. $\mathscr{C}_0^k(\Omega)$, $k \geq 0$, k is an integer, stands for the subset of $\mathscr{C}^k(\Omega)$ containing all functions vanishing in a neighbourhood of the boundary Γ. By $\mathscr{C}(\bar{\Omega})$ we denote the Banach space of all functions which are continuous on the closure $\bar{\Omega}$ and are endowed with the norm

$$\|f\|_{\mathscr{C}(\bar{\Omega})} = \sup_{x \in \Omega} |\, f \,| \,. \tag{4.1}$$

$\mathscr{C}^k(\bar{\Omega})$, $k \geq 1$, where k is an integer, denotes the space of all functions which have the first k derivatives continuous in $\bar{\Omega}$, equipped with the norm

$$\|f\|_{\mathscr{C}^k(\bar{\Omega})} = \sum_{|\alpha|=0}^{k} \sum_{\alpha} \|D^\alpha f\|_{\mathscr{C}(\bar{\Omega})}. \tag{4.2}$$

Here

$$D^{\alpha} f = \frac{\partial^{\alpha_1} ... \partial^{\alpha_N}}{\partial x_1^{\alpha_1} ... \partial x_N^{\alpha_N}} f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_N^{\alpha_N}} f, \tag{4.3}$$

where $\alpha = (\alpha_1, ..., \alpha_N)$, $\alpha_i \geq 1$, $1 \leq i \leq N$, are integers and $\mid \alpha \mid = \alpha_1 + \alpha_2 + ... + \alpha_N$. $\mathscr{C}_0^k(\bar{\Omega})$ denotes the subset of $\mathscr{C}^k(\bar{\Omega})$ containing all functions vanishing in a neighbourhood of Γ. $\mathscr{C}^{\infty}(\bar{\Omega}) = \bigcap_{k=0}^{\infty} \mathscr{C}^k(\bar{\Omega})$. $\mathscr{D}(\Omega)$ is the linear space of infinitely many times differentiable functions with compact support in Ω. The sequence ϕ_k converges to ϕ in $\mathscr{D}(\Omega)$ if and only if there exists a compact set $\bar{\mathscr{O}} \subset \Omega$ such that for all $k = 1, 2, ...$, support $\phi_k \subset \bar{\mathscr{O}}$ and for all α derivatives

$$D^{\alpha} \phi_k \to D^{\alpha} \phi \quad \text{uniformly on} \quad \bar{\mathscr{O}} \quad \text{for} \quad k \to \infty. \tag{4.4}$$

4.2.1 $\mathscr{L}^p$ spaces

By $\mathscr{L}^p(\Omega)$, $1 \leq p < \infty$, we denote the Banach space of all real measurable functions whose $p-th$ power is integrable in the sens of Lebesgue, with the norm

$$\|f\|_{\mathscr{L}^p(\Omega)} = (\int_{\Omega} \mid f \mid^p dx)^{\frac{1}{p}}. \tag{4.5}$$

For $p = 2$ the resulting space is a Hilbert space with the inner - product

$$(u, v) = \int_{\Omega} uv dx \quad \forall u, v \in \mathscr{L}^2(\Omega). \tag{4.6}$$

For $p = \infty$ the norm is defined as,

$$\|f\|_{\mathscr{L}^{\infty}(\Omega)} = \text{esssup}_{x \in \Omega} \mid f \mid. \tag{4.7}$$

Recall from [2] **Hölder inequality**. For any functions $f \in \mathscr{L}^p(\Omega)$, $g \in \mathscr{L}^q(\Omega)$ hold

$$\|fg\|_{\mathscr{L}^1(\Omega)} \leq \|f\|_{\mathscr{L}^p(\Omega)} \|g\|_{\mathscr{L}^q(\Omega)}, \quad \frac{1}{p} + \frac{1}{q} = 1, \ p \geq 1. \tag{4.8}$$

4.2.2 Sobolev spaces

Sobolev spaces are the main tool in modern analysis of boundary value problems [2]. They appear naturally in solving boundary value problems and in calculus of variation. For detailed description of Sobolev spaces see [2, 16, 154, 166, 248].

Let $s \geq 0$, be an integer and let p satisfying $1 \leq p < \infty$ be given. The Sobolev space $\mathscr{W}^{s,p}(\Omega)$ is defined as the set of functions which belong to $\mathscr{L}^p(\Omega)$ together with their derivatives up to order s, i.e.,

$$\mathscr{W}^{s,p}(\Omega) = \{\phi \in \mathscr{L}^p(\Omega) \ : \ D^\alpha \phi \in \mathscr{L}^p(\Omega), \ \forall \quad | \alpha | \leq \ s\}. \tag{4.9}$$

The Sobolev space is equipped with the norm

$$\|f\|^p_{\mathscr{W}^{s,p}(\Omega)} = \sum_{|\alpha|=0}^{s} \sum_{\alpha} \|D^\alpha f\|_{\mathscr{L}^p(\Omega)}, \tag{4.10}$$

where the second summation is taken over all $\alpha_1, ..., \alpha_N$ such that $| \ \alpha \ |= \alpha_1 + \alpha_2 + ... + \alpha_N$. Sobolev spaces are also defined for real $s \geq 0$ [2]. Assume $s = m + \mu$, $m \geq 0$, is an integer, $0 \leq \mu \leq 1$. Function u belongs to $\mathscr{W}^{s,p}(\Omega)$ if and only if $u \in \mathscr{W}^{m,p}(\Omega)$ and

$$\|u\|^p_{\mathscr{W}^{s,p}(\Omega)} = \|u\|^p_{\mathscr{W}^{m,p}(\Omega)} + \sum_{|\alpha|=m} \int\int_{\Omega\times\Omega} \frac{| \ D^\alpha u(x) - D^\alpha u(y) \ |^p}{\|x-y\|^{(N+\mu)p}_{\mathscr{R}^N}} dxdy. \tag{4.11}$$

The expressions:

$$| \ u \ |_{s,p} = (\sum_{|\alpha|=s} \int_\Omega | \ D^\alpha u \ |^p \ dx)^{\frac{1}{p}}, \quad p \in [1,\infty), \tag{4.12}$$

$$| \ u \ |_{s,\infty} = \sum_{|\alpha|=s} \|D^\alpha u\|_{\mathscr{L}^\infty(\Omega)}, \quad p = \infty, \tag{4.13}$$

define the seminorms in the corresponding Sobolev spaces. $\mathscr{W}^{s,p}(\Omega)$ for $p = 2$ is a Hilbert space. It is denoted by $\mathscr{H}^s(\Omega)$. $\mathscr{W}^{s,p}_0(\Omega)$, $s > 0$, is the closure of the space $\mathscr{D}(\Omega)$ in $\mathscr{W}^{s,p}(\Omega)$. Similarly, for $p = 2$, by $\mathscr{H}^s_0(\Omega)$ we shall denote the completion in the norm of $\mathscr{H}^s(\Omega)$ of smooth functions with the compact support. For real $s > 0$, we denote by $\mathscr{W}^{-s,q}(\Omega)$, $p \in [1,\infty)$, the dual space of $\mathscr{W}^{s,p}_0(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. The dual space of $\mathscr{W}^{s,p}_0(\Omega)$ is identified as a space of distributions in Ω.
We shall also need Sobolev spaces on manifolds, which are only boundaries of open subsets of $\mathscr{R}^N$. Let Ω be an open subset of $\mathscr{R}^N$ having $\mathscr{C}^{k,1}$ boundary Γ (for details see [154, 248]). By O we denote a neighbourhood of a point $x \in \Gamma$ such that $O = \{(y_1, ..., y_N) \ | \ -a_j < y_j < a_j, \ 1 \leq j \leq N\}$ and $\{y_1, ..., y_N\}$ denote new orthogonal coordinates. Moreover $O' = \{(y_1, ..., y_{N-1}) \ | \ -a_j < y_j < a_j, \ 1 \leq j \leq N-1\}$. Let φ denote $\mathscr{C}^{k,1}$ continuous function describing the boundary Γ. Let us define also by $\Phi(y) = \{y_1, ..., y_{N-1}, \varphi(y_1, ..., y_{N-1})\}$. Recall from [2, 154]:

Definition 4.1. Let Ω be a bounded open subset of $\mathscr{R}^N$ with a boundary Γ of class $\mathscr{C}^{k,1}$, where k is nonnegative integer. Let Γ_0 be an open subset of Γ. A distribution u on Γ_0 belongs to $\mathscr{W}^{s,p}(\Gamma_0)$ with $| \ s \ |\leq k+1$ if $u(\ \Phi) \in \mathscr{W}^{s,p}(O' \cap \Phi^{-1}(\Gamma_0 \cap O))$ for all possible O and φ.

It is well known, that $\mathscr{C}^\infty(\bar{\Omega})$ is dense in $\mathscr{W}^{s,p}(\Omega)$ and $\mathscr{C}^\infty_0(\bar{\Omega})$ in $\mathscr{W}^{s,p}_0(\Omega)$, $p \geq 1$, $s \geq 0$. In general, the space $\mathscr{D}(\Omega)$ is not dense in $\mathscr{W}^{s,p}(\Omega)$ for real $s \geq 0$. Let us recall from literature [154] two density results Lemmas for these spaces:

Lemma 4.1. *Let Ω be an open subset of $\mathscr{R}^N$ with a continuous boundary and let $\mathscr{C}_c^\infty(\bar{\Omega})$ denotes the space of $\mathscr{C}^\infty$ functions with compact support in $\mathscr{R}^N$ restricted to domain Ω. Then $\mathscr{C}_c^\infty(\bar{\Omega})$ is dense in $\mathscr{W}^{s,p}(\Omega)$ for $s \geq 0$.*

Lemma 4.2. *Let Ω be an open bounded subset of $\mathscr{R}^N$ with a Lipschitz continuous boundary. Then $\mathscr{D}(\Omega)$ is dense in $\mathscr{W}^{s,p}(\Omega)$ for $0 < s \leq \frac{1}{p}$.*

Lemma 4.2 implies that, $\mathscr{W}_0^{s,p}(\Omega)$ is the same space as $\mathscr{W}^{s,p}(\Omega)$, when $0 < s \leq \frac{1}{p}$. Let us also recall from [154, Theorem 1.4.3.1 p.25] the following continuation Lemma:

Lemma 4.3. *Let Ω be an open bounded subset of $\mathscr{R}^N$ with a Lipschitz continuous boundary. Then for every $s > 0$ there exists a continuous linear operator P_s from $\mathscr{W}^{s,p}(\Omega)$ into $\mathscr{W}^{s,p}(\mathscr{R}^N)$ such that $P_s u\,|_\Omega = u$.*

Under the assumptions of Lemma 4.3 we have $\mathscr{W}^{s,p}(\Omega) = \mathscr{W}^{s,p}(\bar{\Omega})$. Moreover P_s can be chosen independently of s. The continuation Lemma 4.3 and other continuation Lemmas in [154] are powerful tools for extending results proved in $\mathscr{R}^N$ to similar results in a bounded domain with Lipschitz continuous boundary.

The following two inequalities are useful in investigating the ellipticity of bilinear forms:

Poincare - Fridrichs inequality [2], [193, p. 2]. Let Γ be such that meas $\Gamma > 0$. For every function $f \in \mathscr{H}_0^1(\Omega)$ the following inequality holds

$$\|f\|_{\mathscr{L}^2(\Omega)} \;\leq\; \alpha\|\nabla f\|_{\mathscr{L}^2(\Omega)}, \tag{4.14}$$

with a constant $\alpha > 0$ independent on f.

First Korn inequality [2, 166, 172, 248]. Let $\Gamma_0 \subset \Gamma$ be such that meas $\Gamma_0 > 0$. For every function $u \in \{v \in \mathscr{H}^1(\Omega) \;:\; v = 0 \text{ on } \Gamma_0\}$ the inequality

$$\sum_{i,j=1}^{N} \|\varepsilon_{ij}(u)\|^2_{\mathscr{L}^2(\Omega)} \geq \alpha\|u\|^2_{\mathscr{H}^1(\Omega)}, \tag{4.15}$$

holds with a constant $\alpha > 0$ independent of u where, a strain tensor $\varepsilon_{ij}(u)$ is given by

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{4.16}$$

4.2.3 Traces

Before we proceed to the Sobolev embedding theorem, let us recall the notion of a trace of a function. Let γ be the operator, for enough smooth function u, defined by

$$\gamma u = u_{|\Gamma}. \tag{4.17}$$

The operator γ provides the restriction of u on the boundary Γ. We call it the trace mapping. The function $u_{|\Gamma}$ is called a trace of the function u on the boundary Γ. If Γ is $\mathscr{C}^{k,1}$, $k \geq 1$, Lipschitz continuous boundary of a bounded open subset of $\mathscr{R}^N$, we can define a unit outward normal $n = (n_1, n_2, ..., n_N)$ which is of class $\mathscr{C}^{k-1,1}$. The following theorem concerns the existence and properties of the trace mapping.

Theorem 4.1 ([154], Theorem 1.5.1.2, p. 37). *Let Ω be a bounded open subset of $\mathscr{R}^N$ with a $\mathscr{C}^{k,1}$ boundary Γ. Assume that $s - \frac{1}{p}$ is not an integer, $s \leq k+1$, $s - \frac{1}{p} = l + \mu$, where $0 < \mu < 1$, $l \geq 0$ an integer. Then the mapping*

$$u \to \{\gamma u, \gamma \frac{\partial u}{\partial n}, ..., \gamma \frac{\partial^l u}{\partial n^l}\}, \tag{4.18}$$

defined for $u \in \mathscr{C}^{k,1}(\bar{\Omega})$, has a continuous extension as an operator from

$$\mathscr{W}^{s,p}(\Omega) \text{ onto } \prod_{j=0}^{l} W^{s-j-\frac{1}{p},p}(\Gamma). \tag{4.19}$$

This operator has a right continuous inverse which does not depend on p.

The space of traces $W^{s-j-\frac{1}{p},p}(\Gamma)$ is defined as in Definition 4.1.

4.2.4 Sobolev Embedding Theorems

The most outstanding result concerning the Sobolev spaces is the famous embedding theorem derived by Sobolev himself. The main statement is the following:

Theorem 4.2 ([2, 3, 154, 166, 172]). *Let Ω be an open bounded subset of $\mathscr{R}^N$ with Lipschitz continuous boundary Γ. Then the following inclusion holds:*

$$\mathscr{W}^{s,p}(\Omega) \subset \mathscr{W}^{t,q}(\Omega), \tag{4.20}$$

for $t \leq s$, $q \geq p$, $s - \frac{N}{p} = t - \frac{N}{q}$. Moreover

$$\mathscr{W}^{s,p}(\Omega) \subset \mathscr{C}^{k,\alpha}(\Omega), \tag{4.21}$$

for $k < s - \frac{N}{p} < k+1$, where $\alpha = s - k - \frac{N}{p}$, and k is a nonnegative integer. $\mathscr{C}^{k,\alpha}(\Omega)$ denotes the space of functions Hölder continuous together with their derivatives up to the order k, with the Hölder constant equal α.

By Theorem 4.2 the following inclusions take place for $k \in \mathbf{N}$ and $p \in [1, \infty)$ [2, 186]:

$$\mathscr{W}^{k,p}(\Omega) \subset \mathscr{L}^{q^\star}(\Omega) \text{ for } \frac{1}{q^\star} = \frac{1}{p} - \frac{k}{N}, \quad k < \frac{N}{p},$$

$$\mathscr{W}^{k,p}(\Omega) \subset \mathscr{L}^q(\Omega) \text{ for } q \in [1+\infty), \quad k = \frac{N}{p}.$$

Moreover the inclusion

$$\mathscr{W}^{k,p}(\Omega) \subset \mathscr{L}^q(\Omega) \text{ for } q \in [1, q^\star), \quad k < \frac{N}{p},$$

is compact.

The compactness of the embedding $\mathscr{H}^1(\Omega) \subset \mathscr{L}^2(\Omega)$ is known as the Rellich theorem. The similar theorem can be formulated for the trace mapping. Note that, one of many consequences of Sobolev's embedding is the continuity of the functions belonging to $\mathscr{W}^{s,p}(\Omega)$ where $s > \frac{N}{p}$. It is even continuity up to the boundary which allows to consider the values on the boundary of such functions. Similar theorem can be formulated for the trace mapping (4.17). Recall [193, Theorem 1.1, p.2]:

Theorem 4.3. *Let $\Omega \subset \mathscr{R}^N$ be a bounded domain with a piecewise smooth boundary Γ and let Γ_r be the intersection of Ω with an r - dimensional smooth hyperplane, where $r < N$. Then for every function $f \in \mathscr{W}^{s,p}(\Omega)$, where $s \geq 1$, $p > 1$, $N > sp$, $r > N - sp$, there exists a trace $f_{|\Gamma_r}$ on Γ_r such that,*

$$f_{|\Gamma_r} \in \mathscr{L}^q(\Gamma_r), \;\; q \leq \frac{pr}{N - sp}, \tag{4.22}$$

$$\|f_{|\Gamma_r}\|_{\mathscr{L}^q(\Gamma_r)} \leq \alpha \|f\|_{\mathscr{W}^{s,p}(\Omega)}. \tag{4.23}$$

For $N = sp$, q can take any value from $1 \leq q < \infty$. If $sp > N$, then $f \in \mathscr{C}^k(\bar{\Omega})$, $k = s - 1 - [\frac{N}{p}]$, and

$$\|f\|_{\mathscr{C}^k(\bar{\Omega})} \leq \alpha \|f\|_{\mathscr{W}^{s,p}(\Omega)}, \tag{4.24}$$

where α is a positive constant independent of f. $[\frac{N}{p}]$ denotes the integer part of $\frac{N}{p}$. For $N > sp$, $q < \frac{pr}{N-sp}$, the embedding of $\mathscr{W}^{s,p}(\Omega)$ in $\mathscr{L}^q(\Gamma_r)$ is compact. Moreover, if $sp > N$, the embedding of $\mathscr{W}^{s,p}(\Omega)$ in $\mathscr{C}^k(\bar{\Omega})$ is compact.

The following inclusions result from Theorem 4.3 for $p \in [1, \infty)$ [166, Theorem 1.13, p. 10]:

$$\mathscr{W}^{1,p}(\Omega) \subset \mathscr{L}^{q^\star}(\Gamma), \text{ where } q^\star = \frac{Np - p}{N - p}, \quad 1 \leq p < N,$$

$$\mathscr{W}^{1,p}(\Omega) \subset \mathscr{L}^q(\Gamma), \text{ for any } q \in [1, \infty), \quad p \geq N.$$

The latter embedding is compact. For proofs of Theorems 4.2, 4.3 see, e.g., [2]. The embedding theorems are used, among others, in analysis of boundary value problems. Assume we are able to build a solution to some given problem, which belongs to $\mathscr{W}^{s,p}(\Omega)$ with s large enough. Then we know this solution is differentiable in the usual sense up to order (strictly) less than $s - \frac{N}{p}$.

4.2.5 Green formula

Green's formulas are an important tool in analysis and, particularly, in the theory of boundary value problems for ordinary and partial differential operators. They are used to transform differential equations in operator form into equivalent variational form or vice-versa. They connect the values of the N-fold integral over a domain Ω in an N-dimensional Euclidean space $\mathscr{R}^N$ with an $(N-1)$-fold integral along the piecewise smooth boundary $\partial\Omega$ of this domain. These formulas are obtained by integration by parts of integrals of the divergence of a vector field that is continuous in $\bar{\Omega} = \Omega \cup \partial\Omega$ and that is continuously differentiable in Ω. Green formula for the second order elliptic operators is based on

Theorem 4.4. *Let Ω be a bounded open subset of $\mathscr{R}^N$ with a Lipschitz continuous boundary Γ. Then for every $u \in \mathscr{W}^{1,p}(\Omega)$ and $v \in \mathscr{W}^{1,q}(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$\int_\Omega D_i uv dx + \int_\Omega u D_i v dx = \int_\Gamma \gamma u \gamma v n_i d\sigma, \tag{4.25}$$

where $D_i u = \frac{\partial u}{\partial x_i}$ is the differential operator with respect to x_i, $1 \leq i \leq N$, n_i denotes the $i-th$ component of the normal vector n to the boundary Γ.

From (4.25) it follows that for $u \in \mathscr{H}^2(\Omega)$ and $v \in \mathscr{H}^1(\Omega)$

$$\int_\Omega \text{grad}\, u \text{ grad}\, v\, dx = -\int_\Omega \triangle uv dx + \int_\Gamma \frac{\partial u}{\partial n} v ds, \tag{4.26}$$

where the symbol grad $= (\partial/\partial x_1, ..., \partial/\partial x_N)$ denotes the gradient, $\triangle = \partial^2/\partial x_1^2 + ... + \partial^2/\partial x_N^2$ is the Laplacian and $\partial/\partial n$ stands for the normal derivative operator along Γ.

4.3 Linear elliptic boundary value problems

Let V be a real Hilbert space and $V^\star$ it's dual. The norm and the scalar product in V will be denoted by $\|.\|$, and $(.,.)$, respectively. The value of $f \in V^\star$ at $v \in V$ will be denoted by $\langle f, v \rangle$. The norm of $f \in V^\star$ is defined in a standard way:

$$\|f\|_{V^\star} = \sup_{v \in V,\, v \neq 0} \frac{\langle f, v \rangle}{\|v\|}. \tag{4.27}$$

By $a : V \times V \to \mathscr{R}$ we denote a bilinear form such that:

$$\exists M > 0 \;:\; |\, a(u,v)\,| \leq M\|u\|\|v\| \;\; \forall u, v \in V \quad \text{(boundedness)}, \tag{4.28}$$

$$\exists \alpha > 0 \;:\; |\, a(v,v)\,| \geq \alpha\|v\|^2 \;\; \forall v \in V \quad \text{(V - ellipticity)}. \tag{4.29}$$

By an abstract linear elliptic equation we call a triplet $\{V, a, f\}$ where $f \in V^\star$. Any element $u \in V$ satisfying

$$a(u,v) = \langle f, v\rangle \quad \forall v \in V, \tag{4.30}$$

is called a (weak) solution of the linear equation. The existence and uniqueness of solutions to the equation (4.30) follows from Lax - Milgram Lemma [86],

Lemma 4.4. *Let the bilinear form $a\ :\ V \times V \to \mathscr{R}$ be bounded and $V - elliptic$. Then for any $f \in V^\star$ there exists a unique solution $u \in V$ to the equation (4.30) and*

$$\|u\| \ \leq\ \frac{1}{\alpha}\|f\|_{V^\star}. \tag{4.31}$$

The bilinear form a defines a linear mapping $A\ :\ V \to V^\star$:

$$a(u,v) = (Au, v)\ \ \forall u, v \in V. \tag{4.32}$$

The abstract linear equation (4.30) is equivalent to the operator equation:

$$Au = f \quad \text{in } V^\star. \tag{4.33}$$

From (4.28), (4.29) it follows, that $A \in \mathscr{L}(V, V^\star)$ and $A^{-1} \in \mathscr{L}(V^\star, V)$. The element u satisfying (4.33) is called a strong solution to this operator equation.
Before we introduce an abstract variational inequality, let us recall notions of a closed subset and a convex subset of V. Let K be nonempty subset of V. Recall

Definition 4.2. A set $K \subset V$ is said to be convex if $\lambda u_1 + (1-\lambda)u_2 \in K$ for every $u_1, u_2 \in K$, $\lambda \in (0,1)$.

Definition 4.3. A set $K \subset V$ is called closed if $u_n \to u$ strongly in V, $u_n \in K$, implies that $u \in K$.

By an abstract elliptic inequality we call a triplet $\{K, a, f\}$ where $a\ :\ V \times V \to \mathscr{R}$ is a bilinear form and $f \in V^\star$. Any element $u \in K$ satisfying:

$$a(u, v-u) \geq (f, v-u) \quad \forall v \in K, \tag{4.34}$$

is called a (weak) solution to the abstract elliptic inequality $\{K, a, f\}$. The following analogy of Lemma 4.4 can be proved [2]

Lemma 4.5. *Let K be nonempty closed convex subset of V and let $a\ :\ V \times V \to \mathscr{R}$ satisfy (4.28), (4.29). Then for any $f \in V^\star$, (4.34) has a unique solution $u \in K$. Moreover if u_i are solutions to (4.34) for f_i, $i = 1, 2$, then*

$$\|u_1 - u_2\| \ \leq\ \frac{1}{\alpha}\|f_1 - f_2\|_{V^\star}. \tag{4.35}$$

In many problems the bilinear form a is symmetric in V, i.e.,

$$a(u,v) = a(v,u) \quad \forall u, v \in V. \tag{4.36}$$

In this case problems (4.30) and (4.34) are equivalent to an abstract minimization problem of the following quadratic functional $J : V \to \mathscr{R}$ defined by:

$$J(v) = \frac{1}{2}(Av, v) - (f, v), \tag{4.37}$$

on V or K, respectively. The equivalence between the problems (4.30) and (4.34) and the minimization of the functional (4.37) is given in []

Lemma 4.6. *Let a bilinear form a satisfy (4.28), (4.29) and (4.36). Then*
(i) u solves (4.30) if and only if

$$J(u) = \min_{v \in V} J(v). \tag{4.38}$$

(ii) u solves (4.34) if and only if

$$J(u) = \min_{v \in K} J(v). \tag{4.39}$$

4.4 Nonlinear elliptic problems

In this section some basic results concerning the existence of solutions to nonlinear elliptic equations and inequalities are recalled. These results generalize Lemmas 4.4, 4.5, 4.6 concerning the existence of solutions to the linear elliptic equations or inequalities.
Let V be a reflexive Banach space and $V^\star$ its dual space. Let $T : V \to V^\star$ be a mapping, in general, nonlinear. We shall look for the solutions of the following equation

$$T(u) = f \text{ in } V^\star \;\Leftrightarrow\; \langle T(u), v\rangle \;=\; \langle f, v\rangle \;\; \forall v \in V, \tag{4.40}$$

or the inequality

$$u \in K \; : \; \langle T(u), v\rangle \;\geq\; \langle f, v\rangle \;\; \forall v \in K, \tag{4.41}$$

where $f \in V^\star$ is a given element and K is a nonempty, closed and convex subset of V. Recall [166, p. 37]:

Definition 4.4. The operator $T : V \to V^\star$ is monotone if

$$\langle T(u) - T(u_1), u - u_1\rangle \;\geq\; 0 \;\; \forall u, u_1 \in V. \tag{4.42}$$

Definition 4.5. T is strongly monotone in V if there exists a strictly increasing function $\alpha : [0, \infty) \to \mathscr{R}$ such that $\alpha(0) = 0$, $\lim_{t\to\infty} \alpha(t) = \infty$ and for any $u, v \in V$ the following condition holds:

$$\langle T(u) - T(v), u - v\rangle \;\geq\; \alpha(\|u - v\|)\|u - v\|. \tag{4.43}$$

Definition 4.6. T is locally Lipschitz continuous in V if there exists a positive constant $M(r)$ such that for any $u, v \in B_r = \{v \in V \;|\; \|v\| \leq r\}$ the following condition holds

$$\|T(u) - T(v)\|_{V^\star} \le M(r)\|u - v\|. \tag{4.44}$$

The existence and uniqueness of solutions to (4.40) and (4.41) follows from:

Theorem 4.5 ([166], Theorem 1.31, p. 37). *Let $T : V \to V^\star$ be strongly monotone and locally Lipschitz in V. Then the equation (4.40) and the inequality (4.41) have unique solutions $u \in V$ and $u \in K$, respectively, for any right hand side $f \in V^\star$.*

We shall need a notion of Gateaux derivative of a functional.

Definition 4.7. Let V be a normed space and let $J : V \to \mathscr{R}$ be any functional. The functional J is called directionally differentiable at a point $u \in V$ in the direction $v \in V$ if for any $v \in V$ there exists a unique element $DJ_u(v)$ such that

$$DJ_u(v) = \lim_{t \to 0^+} \frac{J(u + tv) - J(u)}{t}, \tag{4.45}$$

called directional derivative. The limit (4.45) is called the Gateaux differential if it holds for any $t \to 0$. DJ_u is called Gateaux derivative of the functional J at a point $u \in V$. The functional J is called differentiable if it has the derivative DJ_u at every point $u \in V$.

$T : V \to V^\star$ is called the potential operator if there exists a functional $\Psi : V \to \mathscr{R}$, called the potential of T, which is Gateaux differentiable at any point $u \in V$ and such that its Gateaux derivative $D\Psi(u) = T(u)$ for any $u \in V$. In this case the equation (4.40) and the inequality (4.41) has the following form

$$u \in V \ : \ \langle D\Psi(u), v\rangle \ = \ \langle f, v\rangle \quad \forall v \in V, \tag{4.46}$$

$$u \in K \ : \ \langle D\Psi(u), v - u\rangle \ \ge \ \langle f, v\rangle \quad \forall v \in K. \tag{4.47}$$

These problems are related to the minimization of the functional Ψ in V and K respectively. The proof of the following theorem can be found in literature [166]:

Theorem 4.6. *Let $\Psi : V \to \mathscr{R}$ be the potential of an operator T such that*

for any $u, v, z \in V$ fixed, the function

$$t \to \langle D\Psi(u + tv), z\rangle \text{ is continuous in } \mathscr{R}, \tag{4.48}$$

$$\langle D\Psi(u + v), v\rangle - \langle D\Psi(u), v\rangle \ \ge \ \alpha(\|v\|)\|v\|, \tag{4.49}$$

with the function α having the same properties as in Definition 4.5. Then there exists unique solutions to the following minimization problems:

$$u \in V \ : \ \Psi(u) = \min_{v \in V} \Psi(v), \tag{4.50}$$

$$u \in K \ : \ \Psi(u) = \min_{v \in K} \Psi(v). \tag{4.51}$$

Moreover, (4.50) is equivalent to (4.40) and (4.51) to (4.41).

4.5 Second order elliptic equations

We shall deal mainly with the second order elliptic or parabolic equations. Recall from [193, 317] the general form of the second order elliptic operator $\mathscr{A}$ in the domain Ω:

$$\mathscr{A}y = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial y}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial y}{\partial x_i} + c(x)y, \tag{4.52}$$

where $x \in \Omega$, $y = y(x)$. Moreover $a_{ij}(x), b_i(x)$, $i, j = 1, ..., n$ and $c(x)$ are given measurable functions on $\mathscr{R}^N$ satisfying for some constants $\kappa > 0$ and $K \in (0, +\infty)$ for all values of the arguments and $\zeta \in \mathscr{R}^N$ conditions, i.e.,

$$\begin{aligned} \mid b \mid + \mid c \mid \leq K, \;\; c \leq 0, \\ a_{ij}(x) \in \mathscr{L}^{\infty}(\Omega), \;\; a_{ij} = a_{ji}, \\ \kappa^2 \mid \zeta \mid^2 \leq \;\; a_{ij}(x)\zeta_i\zeta_j \leq \kappa^{-1} \mid \zeta \mid^2, \end{aligned} \tag{4.53}$$

where $b = (b^1, ..., b^n)$. Moreover it is assumed that there is a function $\omega(\varepsilon), \varepsilon \geq 0$, such that $\omega(0+) = 0$ and for all $i, j = 1, 2, ..., n$, $x, y \in \mathscr{R}^n$, we have

$$\mid a_{ij}(x) - a_{ij}(y) \mid \leq \omega(\mid x - y \mid).$$

Consider the second order elliptic boundary value problem in domain Ω: find a function y such that

$$\mathscr{A}y - \lambda y = f \text{ in } \Omega, \tag{4.54}$$

where $f = f(x)$ is a given function in Ω. Recall from [193] the following existence result:

Theorem 4.7. *There exists constants $\lambda_0 \geq 1$ and N depending only on $p \in (1, +\infty)$, K, κ, ω, n such that the estimate*

$$\lambda \|y\|_{\mathscr{L}^p(\Omega)} + \lambda^{1/2} \|\frac{\partial y}{\partial x}\|_{\mathscr{L}^p(\Omega)} + \|\frac{\partial^2 y}{\partial x^2}\|_{\mathscr{L}^p(\Omega)} \leq N \|\mathscr{A}y - \lambda y\|_{\mathscr{L}^p(\Omega)},$$

holds true for any $y \in \mathscr{W}^{2,p}(\Omega)$ and $\lambda \geq \lambda_0$. Furthermore for any $\lambda \geq \lambda_0$ and $f \in \mathscr{L}^p(\Omega)$ there exists a unique $y \in \mathscr{W}^{2,p}(\Omega)$ satisfying (4.54).

The following boundary condition is imposed on function y on the boundary $\partial\Omega$ of domain Ω:

$$\mathscr{W}y = g(x), \tag{4.55}$$

where g is a given function and the boundary operator $\mathscr{W}$ is defined as

$$\mathscr{W}y = \alpha(x)\frac{\partial y}{\partial n} + \beta(x)y, \tag{4.56}$$

where $\frac{\partial y}{\partial n}$ denotes the normal derivative of function y on $\partial\Omega$. Functions $\alpha(x)$ and $\beta(x)$ are given functions on $\partial\Omega$. Remark that for $\alpha(x) = 0$ condition (4.55) becomes Dirichlet condition and for $\beta(x) = 0$ becomes Neumann condition. We can also split the boundary $\partial\Omega$ into two disjoint parts

$$\partial\Omega = \partial_D\Omega \cup \partial_N\Omega, \tag{4.57}$$

and impose Dirichlet or Neumann boundary conditions on $\partial_D\Omega$ or $\partial_N\Omega$ respectively, i.e.,

$$y = g_D \quad \text{on} \quad \partial_D\Omega, \quad \frac{\partial y}{\partial n} = g_N \quad \text{on} \quad \partial_N\Omega, \tag{4.58}$$

where g_D and g_N are a given functions.

4.5.1 Model elliptic equation

Let us apply the results of Section 4.5 to the model problem. As a model elliptic equation we shall consider the Poisson equation

$$-\triangle y + y = f \;\text{ in }\; \Omega, \tag{4.59}$$

with boundary conditions

$$y = 0 \;\text{ on }\; \partial_D\Omega \;\text{ and }\; \frac{\partial y}{\partial n} = u \;\text{ on }\; \partial_N\Omega, \tag{4.60}$$

where $f \in \mathscr{L}^2(\Omega)$ is a given function, y denotes the unknown (state) function. Function $u \in \mathscr{L}^2(\partial_N\Omega)$ we shall call the control function defined on $\partial\Omega$. Boundary $\partial\Omega$ is assumed to be smooth enough with outward pointing unit normal n.

Since in system (4.59)-(4.60) the control function u acts on the boundary of domain Ω this system is called the boundary control problem. In the PDEs control theory are also considered distributed control systems where the control function u acts in the whole domain Ω or its subdomain. In this case $f = u$ or $f = u + F$, where F is a given function.

The existence of the unique solution to (4.59)-(4.60) follows from Theorem 4.7.

For the sake of finite element approximation we shall consider the boundary value problem (4.59)-(4.60) in the variational form [166, 172, 186, 193, 248]. Using Green formula (4.25) we transform the boundary value problem (4.59)-(4.60) into the equivalent one: for a given function u find $y \in V$ satisfying:

$$a(y,\varphi) = l(u;\varphi) \quad \forall\varphi \in V, \tag{4.61}$$

where the bilinear form $a(\cdot,\cdot) : V \times V \to \mathscr{R}$ and the linear form $l(u;\cdot) : V \to \mathscr{R}$ for given $u \in U$ is defined by

$$a(y,\varphi) = \int_\Omega (\nabla y \nabla \varphi + y\varphi)dx, \tag{4.62}$$

$$l(u;\varphi) = \int_\Omega f\varphi dx + \int_{\partial_N \Omega} u\varphi ds, \tag{4.63}$$

and V is a subspace of Sobolev space $\mathscr{H}^1(\Omega)$ defined as follows

$$V = \{v \in \mathscr{H}^1(\Omega) : v = 0 \text{ on } \partial\Omega\}.$$

For a given $u \in \mathscr{L}^2(\partial_N \Omega)$ the existence of a unique solution $y \in V$ to the state system (4.61) follows from Lax-Milgram Lemma 4.4.

Recall, the application of the Lax-Milgram Lemma 4.4 to the homogeneous Dirichlet boundary value problems requires the application of the Friedrich's inequality (4.14). The application of this Lemma to the boundary value problem with the mixed boundary condition (4.56) requires additional assumption of the form:

$$\int_\Omega \lambda^2 dx + \int_{\partial\Omega} \frac{\alpha(x)}{\beta(x)}^2 ds > 0,$$

providing the functions defined as in (4.54) and (4.56) satisfy $\lambda(x) \in \mathscr{L}^\infty(\Omega)$, $\alpha(x), \beta(x) \in \mathscr{L}^\infty(\partial\Omega)$ and $\beta(x) \neq 0$.

4.6 Necessary optimality conditions for ODE constrained optimal control problems

Let us formulate the optimal control problem for system governed by PDEs. We have to define the set of admissible functions as well as the cost functionals.
We denote by U a space of control functions and by $U_{ad} \subset U$ a bounded, closed and convex set of admissible controls. Taking into account boundary value systems (4.54)-(4.56) or (4.61) we can consider either distributed control where the control function depends on $x \in \Omega$ or the boundary control where the control function depends on $x \in \partial\Omega$. Therefore a natural choice for the control space U is Hilbert space $\mathscr{L}^2(\Omega)$ or $\mathscr{L}^2(\partial\Omega)$. One can consider also parameter control problem where the coefficients of the elliptic operator are control functions.
Usually the control function $u \in U$ is assumed bounded, i.e., for almost every x in Ω or $\partial\Omega$ it holds

$$u_{min}(x) \leq u(x) \leq u_{max}(x), \tag{4.64}$$

where $u_{min}(x)$ and $u_{max}(x)$ are given bounded functions. The set of admissible controls we define as follows

$$U_{ad} = \{u \in U; u \text{ satisfies (4.64)}\}. \tag{4.65}$$

We shall consider the cost functional $J(y,u) = J(u) : U_{ad} \to \mathscr{R}$ defined as follows:

$$J(y,u) = \alpha_1 J_1(y,u) + \alpha_2 J_2(y,u) + \frac{\alpha_3}{2} R(u), \tag{4.66}$$

where $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ and $\alpha_3 > 0$ are given real constants such that $\alpha_1^2 + \alpha_2^2 \neq 0$. The cost functionals J_1 and J_2 deal with the distributed control and boundary control problem, respectively. The term $R(u)$ denotes a regularization functional depending on the control function u only. This term is added to ensure the radial unboundedness of the cost functional as well as the existence of the optimal solutions to the optimal control problem. The regularization term in the form of the bound imposed on the norm of the control function may be also added to the admissible set U_{ad} ensuring its compactness in the control space U [217]. In the majority of practical applications the cost functionals are quadratic therefore we confine to consider the quadratic cost functionals only having the following form:

$$J_1(y,u) = \int_\omega (Cy(u) - y_d)^2 dx, \tag{4.67}$$

$$J_2(y,u) = \int_{\partial\Omega} (Cy(u) - y_d)^2 ds, \tag{4.68}$$

$$R(u) = \int_\Omega (Nu,u)dx \quad \text{or} \quad R(u) = \int_{\partial\Omega} (Nu,u)ds, \tag{4.69}$$

where C and N are given real numbers, y_d is a given function, a set ω is a given subset of Ω.
The functionals (4.67) and (4.68) are called in literature tracking type functionals. It means that the aim of the optimal control problem is to find such control function u that the distance between the solution y to the state problem and the prescribed function y_d measured by $\mathscr{L}^2$ norm is minimal. In many industrial based optimal control applications also other types of the cost functionals are considered including, among others, the total energy of the system, the compliance or the stress of the construction [217], the fundamental frequency of the free vibrations of the body or the drag of the body moving in a fluid.

The dependence of the cost functional (4.66) on the control function u is indirect: the cost functional depends on the solution y to the boundary value problem, i.e., (4.59) -(4.60) or (4.61), and on the other hand this solution y depends on the control function u appearing on the right hand side of the state equation. In order to deal directly with the control function u only we can introduce control-to-state operator S [317, p. 50] defined as the mapping $S : U \ni u \to y(u) \in V \subset \mathscr{L}^2(\Omega)$. Therefore $y(u) = Su$ and the cost functionals (4.67) and (4.68) can be written, respectively, as

$$f_1(u) = \int_\omega (CSu - y_d)^2 dx, \tag{4.70}$$

$$f_2(u) = \int_{\partial\Omega} (CSu - y_d)^2 ds. \tag{4.71}$$

The use of the operator S has the advantage that the adjoint operator $S^\star$ also acts in the space $\mathscr{L}^2(\Omega)$. Using (4.70) - (4.71) we can replace the cost functional (4.66) by the equivalent reduced functional

$$f(u) = \alpha_1 f_1(u) + \alpha_2 f_2(u) + \frac{\alpha_3}{2} R(u), \tag{4.72}$$

in terms of the control function u only.

Using (4.64), (4.65) and (4.72) an optimal control problem takes the form: find control function $u^\star \in U_{ad}$ such that,

$$f(u^\star) = \min_{u \in U_{ad}} f(u). \tag{4.73}$$

Remark this optimal control problem is formulated in terms of the control function u only. Due to operator S the solution y to the PDEs problem can be calculated, i.e., $y^\star = y(u^\star) = Su^\star$.

4.6.1 Basic results for the reduced optimal control problems

Let us recall from [317, Therem 2.14, p. 50] the following existence result for problem (4.73):

Theorem 4.8. *Let $\{U, \|\cdot\|_U\}$ and $\{H, \|\cdot\|_H\}$ denote the Hilbert spaces and let a nonempty, closed, bounded, and convex set $U_{ad} \subset U$ as well as some $y_d \in H$ and constant $N \geq 0$ be given. Then the quadratic Hilbert space optimization problem*

$$\min_{u \in U_{ad}} f(u) \overset{def}{=} \frac{1}{2} \|Su - y_d\|_H^2 + \frac{N}{2} \|u\|_U^2 \tag{4.74}$$

admits an optimal solution $u^\star$. If $N > 0$ or S is injective, then the solution is uniquely determined.

Let us formulate the first order necessary optimality condition for the optimal control problem (4.73). Recall first from [317, Lemma 2.21, p. 63] the fundamental result:

Lemma 4.7. *Let C denote a nonempty and convex subset of a real Banach space U and let the real-valued mapping f be Gateaux differentiable in an open subset of U containing C. If $u^\star \in C$ is a solution to the problem*

$$\min_{u \in C} f(u), \tag{4.75}$$

then it solves the variational inequality

$$f'(u^\star)(u - u^\star) \geq 0 \quad \forall u \in C. \tag{4.76}$$

$f'(\cdot)$ denotes Gateaux derivative of $f(\cdot)$ defined by (4.45). Conversely, if $u^\star \in C$ solves the variational inequality (4.76) and f is convex then $u^\star$ is a solution to the minimization problem (4.75).

This Lemma states a necessary and in the case of convexity also sufficient first-order necessary optimality condition. Applying this result to the problem (4.74) we get [317]:

Lemma 4.8. *Let real Hilbert spaces U and H, a nonempty and convex set $U_{ad} \subset U$, some $y_d \in H$, and a constant $N \geq 0$ be given. Moreover let $S : U \to H$ denote a continuous linear operator. Then $u^\star \in U_{ad}$ is a solution to the optimization problem (4.74) if and only if $u^\star$ solves the variational inequality:*

$$(S^\star(Su^\star - y_d) + Nu^\star, u - u^\star)_U \geq 0 \;\; \forall u \in U_{ad}, \tag{4.77}$$

where $S^\star$ denotes the adjoint operator of the operator S.

Remark the inequality (4.77) is equivalent to

$$(Su^\star - y_d, Su - Su^\star)_H + N(u^\star, u - u^\star)_U \geq 0 \;\; \forall u \in U_{ad},$$

which avoids the adjoint operator $S^\star$.

4.6.2 The Lagrange method

Lagrange multiplier technique is usually used to find solution of the constrained optimal control problems. Let us describe this technique in general form and next use it to formulate the necessary optimality condition for PDEs constrained optimal control problems.

4.6.3 Saddle point problem formulation

Consider the constrained minimization problem of the form: find $u^\star \in K \subset V$ minimizing the cost functional $J(\cdot) : V \to \mathscr{R}$ over a cone K in the space V:

$$\min_{v \in K} J(v) \tag{4.78}$$

For the sake of simplicity, assume the closed convex set $K \subset V$ is given as

$$K = \{v \in V : f(v) \leq 0, \;\; g(v) = 0\}, \tag{4.79}$$

where $f : V \to \Lambda_1$ is convex Gateaux differentiable function, $g : V \to \Lambda_2$ is linear Gateaux differentiable function, Λ_1, Λ_2 are reflexive Banach spaces. Function g may be interpreted as a PDE constraint and function f as the unilateral variational inequality. Using Lagrange multipliers approach we replace the constrained minimization problem (4.78) by the unconstrained minimization problem for the associated Lagrangian.

Denote by $\lambda_1^{\sim} \in \Lambda_1^{set}$ and $\lambda_2^{\sim} \in \Lambda_2^{set}$ the Lagrange multipliers associated with the constraints in (4.79) where

$$\Lambda_1^{set} = \{\lambda_1^{\sim} \in \Lambda_1^{\star} : \lambda_1^{\sim} \geq 0\}, \;\; \Lambda_2^{set} = \Lambda_2^{\star}.$$

$\Lambda_1^{\star}$ (resp. $\Lambda_2^{\star}$) denotes a dual space of Λ_1 (resp. Λ_2). Let us introduce the Lagrangian $\mathscr{L}(.,.,.) : V \times \Lambda_1^{set} \times \Lambda_2^{set} \to \mathscr{R}$ associated with the problem (4.78):

$$\mathscr{L}(v, \lambda_1^{\sim}, \lambda_2^{\sim}) = J(v) + \langle \lambda_1^{\sim}, f(v) \rangle_{\Lambda_1^{\star} \times \Lambda_1} + \langle \lambda_2^{\sim}, g(v) \rangle_{\Lambda_2^{\star} \times \Lambda_2}. \tag{4.80}$$

Problem (4.78) is equivalent [317] to the problem

$$\min_{v \in V} \max_{\lambda_1^{\sim} \in \Lambda_1,\, \lambda_2^{\sim} \in \Lambda_2} \mathscr{L}(v, \lambda_1^{\sim}, \lambda_2^{\sim}). \tag{4.81}$$

Applying [81, Theorem 4.18, p. 226, Theorem 5.3, p. 252] and [109, Theorems 1.5, 1.6, 2.2 p. 174 - 179] to problems (4.78) or (4.81) we can formulate the following necessary optimality condition:

Lemma 4.9. *Let V, Λ_1, Λ_2 be reflexive Banach spaces, and $K \subset V$ be a set given by (4.79). Assume the functional $J : V \to \mathscr{R}$ is lower semicontinuous, function $f : V \to \Lambda_1$ is convex and lower semicontinuous, function $g : V \to \Lambda_2$ is linear, sets K, Λ_1^{set}, Λ_2^{set} have a nonempty interior. Then there exist a saddle point $(u^{\star}, \lambda_1^{\star}, \lambda_2^{\star}) \in K \times \Lambda_1^{set} \times \Lambda_2^{set}$ of the Lagrangian $\mathscr{L}$ defined as follows: for all $v \in K$ and $\lambda_1^{\sim} \in \Lambda_1^{set}$, $\lambda_2^{\sim} \in \Lambda_2^{set}$ holds,*

$$\mathscr{L}(u^{\star}, \lambda_1^{\sim}, \lambda_2^{\sim}) \leq \mathscr{L}(u^{\star}, \lambda_1^{\star}, \lambda_2^{\star}) \leq \mathscr{L}(v, \lambda_1^{\star}, \lambda_2^{\star}). \tag{4.82}$$

Assume moreover that functional $J : V \to \mathscr{R}$ and functions $f : V \to \Lambda_1$, $g : V \to \Lambda_2$ are Gateaux differentiable. Then a saddle point $(u^{\star}, \lambda_1^{\star}, \lambda_2^{\star}) \in K \times \Lambda_1^{set} \times \Lambda_2^{set}$ satisfies,

$$(DJ_{u^{\star}} + \lambda_1^{\star} Df_{u^{\star}} + \lambda_2^{\star} Dg_{u^{\star}})(v - u^{\star}) \geq 0 \;\; \forall v \in K, \tag{4.83}$$

$$f(u^{\star})(\lambda_1^{\sim} - \lambda_1^{\star}) \leq 0 \;\; \forall \lambda_1^{\sim} \in \Lambda_1^{set}, \lambda_1^{\star} \geq 0, \;\; f(u^{\star}) \leq 0, \tag{4.84}$$

$$g(u^{\star})\lambda_2^{\star} = 0, \;\; g(u^{\star}) = 0. \tag{4.85}$$

$DJ_{u^{\star}}$ denotes Gateaux derivative of the functional J with respect to u at a point $u^{\star}$ defined by (4.45).

4.6.4 Adjoint state approach

We apply Lemma 4.9 to formulate the necessary optimality condition for the second order elliptic PDEs constrained problems. Let U_{ad} and $J(y,u)$ be the set of the admissible controls and the cost functional given by (4.66) and (4.65), respectively. For the sake of simplicity we set $\alpha_1 = 1$ and $\alpha_2 = 0$ in (4.66), i.e., we confine to

consider the distributed control only. We also assume there are no constraints on the control function u, i.e., $U = U_{ad} = L^2(\Omega)$.
By V we denote a subspace of the Sobolev space $\mathscr{H}^1(\Omega)$

$$V = \{v \in \mathscr{H}^1(\Omega) : v = g_D \ \text{ on } \ \partial_D\Omega\}.$$

For a given $u \in U_{ad}$ we denote by $y \in V$ the solution to the second order elliptic equation:

$$a(y,\varphi) = l(u;\varphi) \quad \forall \varphi \in V, \tag{4.86}$$

where the following bilinear form $a(,\cdot,\cdot) : V \times V \to \mathscr{R}$ and the linear form $l(u;\cdot) : V \to \mathscr{R}$ are defined by

$$a(y,\varphi) = \int_\Omega [\sum_{i,j=1}^n a_{ij}(x)\frac{\partial y}{\partial x_i}\frac{\partial \varphi}{\partial x_j} + \sum_{i=1}^n b_i(x)\frac{\partial y}{\partial x_i}\varphi + c(x)y\varphi]dx, \tag{4.87}$$

$$l(u;\varphi) = \int_\Omega (u+f)\varphi dx + \int_{\partial_N\Omega} g_N \varphi ds. \tag{4.88}$$

Consider an optimal control problem: find $u^\star \in U_{ad}$ such that

$$J(y^\star,u^\star) = \min_{u \in U_{ad}} J(y,u). \tag{4.89}$$

Due to (4.86) problem (4.89) is the constrained minimization problem. In order to formulate the necessary optimality condition for this problem let us first apply directly (4.76). Under our assumptions the derivatives of the cost functional (4.66) with respect to y or u are equal to:

$$\frac{\partial J(y,u)}{\partial y}h = \frac{\partial J_1(y,u)}{\partial y}h \quad \forall h \in V, \tag{4.90}$$

$$\frac{\partial J(y,u)}{\partial u}h = \frac{\partial J_1}{\partial u}h + \frac{\partial J_1(y,u)}{\partial y}\frac{dy}{du}h + \frac{\alpha_3}{2}\frac{dR(u)}{du}h \quad \forall h \in U, \tag{4.91}$$

where the rules for total differentials where applied. Remark in order to calculate (4.91) one has to calculate the derivative $\frac{dy}{du}$ of y with respect to u. In order to do it one has to differentiate the state equation (4.86) with respect to u and to use the inverse of the elliptic operator. Usually the calculation of the inverse elliptic operator is complicated and too costly for numerical calculations.
In order to simplify the formulation of the necessary optimality condition as well as to avoid the use of the inverse operator let us introduce the adjoint state and apply the Lagrange method.

Using Lagrange multiplier $p \in V$ as well as the Lagrangian function $\mathscr{L}(\cdot,\cdot,\cdot) : V \times U \times V \to \mathscr{R}$ for the optimization problem (4.89):

$$\mathscr{L} = \mathscr{L}(y,u,p) = J_1(y,u) + \frac{\alpha_3}{2}R(u) + a(y,p) - l(u;p), \tag{4.92}$$

we can transform it into the unconstrained minimization problem: find $(y^\star, u^\star, p^\star) \in V \times U \times V$ satisfying:

$$\mathscr{L}(y^\star, u^\star, p^\star) = \max_{p \in V} \min_{u \in U_{ad}} \mathscr{L}(y, u, p). \tag{4.93}$$

The Lagrange multiplier p associated with the boundary value problem constraint (4.86) is called also the adjoint state and besides mathematical has also many physical or economic interpretations.
In order to formulate first order necessary optimality condition let us calculate the derivatives of (4.93) with respect to p, y, u:

$$\int_\omega \frac{\partial \mathscr{L}(y,u,p)}{\partial p} \varphi dx = a(y,\varphi) - l(u,\varphi) \ \ \forall \varphi \in V, \tag{4.94}$$

$$\begin{aligned} \int_\omega \frac{\partial \mathscr{L}(y,u,p)}{\partial y} \varphi dx = \\ \int_\Omega [\sum_{i,j=1}^n a_{ij}(x) \frac{\partial p}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial \varphi}{\partial x_i} p + c(x) p \varphi] dx \\ + \int_\omega \frac{\partial J_1(y,u)}{\partial y} \varphi dx \ \ \forall \varphi \in V, \end{aligned} \tag{4.95}$$

$$\int_\omega \frac{\partial \mathscr{L}(y,u,p)}{\partial u} \varphi dx = \int_\omega (\frac{\partial J_1(y,u)}{\partial y} \frac{dy}{du} + \frac{\alpha_3}{2} \frac{dR(u)}{du} - p) \varphi dx \ \forall \varphi \in U. \tag{4.96}$$

The adjoint state $p \in V$ is defined as satisfying the condition:

$$\int_\omega \frac{\partial \mathscr{L}(y,u,p)}{\partial y} \varphi dx = 0 \ \ \forall \varphi \in V, \tag{4.97}$$

what implies the adjoint equation

$$a(p,\varphi) = -\int_\omega \frac{\partial J_1(y,u)}{\partial y} \varphi dx \ \ \forall \varphi \in V. \tag{4.98}$$

The derivative (4.96) contains the derivative $\frac{dy}{du}$. Regard that setting $\varphi = \frac{dy}{du} h$, $h \in U$, in (4.98) we obtain

$$a(p, \frac{dy}{du} h) = -\int_\omega \frac{\partial J_1(y,u)}{\partial y} \frac{dy}{du} h dx \ \ \forall h \in U, \tag{4.99}$$

Differentiating (4.86) with respect to u and setting $\varphi = p$ we have

$$a(p, \frac{dy}{du} h) = \int_\omega p h dx \ \ \forall h \in U, \tag{4.100}$$

Using (4.99) and (4.100) in (4.96) we obtain

$$\int_\omega \frac{\partial \mathscr{L}(y,u,p)}{\partial u}\varphi dx = -\int_\omega (\frac{\alpha_3}{2}\frac{dR(u)}{du} - p)\varphi dx \ \ \forall \varphi \in U. \tag{4.101}$$

The introduction of the adjoint state has two advantages: the first order necessary optimality conditions simplify and the use of the derivative of the state function with respect to the control function and the inverse elliptic operator is avoided. Moreover the form o the cost functional gradient or directional derivative also simplifies.
Remark that at optimal point $u^\star$, from (4.86) and (4.89) it follows that the derivative (4.101) is equal to:

$$\int_\omega \frac{\partial \mathscr{L}(y^\star,u^\star,p^\star)}{\partial u}\varphi dx = \int_\omega \frac{\partial J(y^\star,u^\star)}{\partial u}\varphi dx = \int_\omega (p^\star - \frac{\alpha_3}{2}\frac{\partial R(u^\star)}{\partial u})\varphi dx. \tag{4.102}$$

The gradient determined by (4.102) is called the reduced gradient.

Based on Lemma 4.9 the first order necessary optimality condition for problem (4.89) takes the form: if $u^\star \in U$ is an optimal solution to the problem (4.89) then there exists Lagrange multiplier $p^\star \in V$ such that the following conditions hold:

$$a(y^\star,\varphi) = l(u^\star;\varphi) \ \ \forall \varphi \in V, \tag{4.103}$$

$$a(p^\star,\varphi) = -\int_\omega \frac{\partial J(y^\star,u^\star)}{\partial y}\varphi dx \ \ \forall \varphi \in V. \tag{4.104}$$

$$\int_\omega \frac{\partial J(y^\star,u^\star)}{\partial u}\varphi dx = \int_\omega (p^\star - \frac{\alpha_3}{2}\frac{\partial R(u^\star)}{\partial u})\varphi dx \ \ \forall \varphi \in V. \tag{4.105}$$

4.6.5 Model optimal control problem

Let us apply the Lagrange multiplier technique from the previous subsection to formulate the necessary optimality conditions for the model boundary optimal control problem.
We denote by $U_{ad} \subset U = \mathscr{L}^2(\Omega)$ the set of admissible controls given by (4.64). The cost functional $J(\cdot)$ transforms $J(\cdot) : U \to \mathscr{R}$ and is given by

$$J(y(u)) \stackrel{def}{=} \frac{1}{2}\int_\omega (y(u) - y_d)^2 dx + \frac{1}{2}\alpha \int_{\partial_N \Omega} u^2 ds. \tag{4.106}$$

The optimal control problem is formulated as follows: find $u^\star \in U_{ad}$ such that

$$J(u^\star) = \min_{u \in U_{ad}} J(y(u)), \tag{4.107}$$

where $y(u) \in V$ denotes the solution to the state equation (4.61) depending on $u \in U_{ad}$, $y_d \in \mathscr{L}^2(\Omega)$ denotes a given function and $\alpha > 0$ is a given constant. The second term in the goal functional (4.106) is added to ensure the existence of optimal solution to the problem (4.107).

The Lagrangian function $\mathscr{L}(.,.,.,.,.) : V \times U \times V \times U \times U \to \mathscr{R}$ for the problem (4.107) has the form:

$$\mathscr{L}(y,u,p,\lambda_1,\lambda_2) = \frac{1}{2}\int_\omega (y(u)-y_d)^2 dx + \frac{1}{2}\alpha \int_{\partial_N \Omega} u^2 ds + \\ a(y,p) - l(u;p) + \int_{\partial_N \Omega} (\lambda_1(u_{min}-u) + \lambda_2(u-u_{max}))ds \tag{4.108}$$

The derivatives of the Lagrangian (4.108) have the form

$$\int_\omega \frac{\mathscr{L}(y,u,p,\lambda_1,\lambda_2)}{\partial y}\varphi = \int_\omega (y(u)-y_d)\varphi dx + a(p,\varphi) \quad \forall \varphi \in V, \tag{4.109}$$

$$\int_\omega \frac{\mathscr{L}(y,u,p,\lambda_1,\lambda_2)}{\partial u}\varphi = \\ \int_\omega (y(u)-y_d)\frac{dy(u)}{du} h dx + \alpha \int_{\partial_N \Omega} uhds - l(h;p) + \\ \int_{\partial_N \Omega} (-\lambda_1 h + \lambda_2 h) ds \quad \forall h \in U, \tag{4.110}$$

$$\int_\omega \frac{\mathscr{L}(y,u,p,\lambda_1,\lambda_2)}{\partial p}\varphi = a(y,\varphi) - l(u;\varphi) \quad \forall \varphi \in V, \tag{4.111}$$

$$\int_\omega \frac{\mathscr{L}(y,u,p,\lambda_1,\lambda_2)}{\partial \lambda_1}\varphi = \int_{\partial_N \Omega} (u_{min}-u) h ds \quad \forall h \in U, \tag{4.112}$$

$$\int_\omega \frac{\mathscr{L}(y,u,p,\lambda_1,\lambda_2)}{\partial \lambda_2}\varphi = \int_{\partial_N \Omega} (u-u_{max}) h ds \quad \forall h \in V. \tag{4.113}$$

The adjoint function $p \in V$ is defined as an element satisfying:

$$\int_\Omega (\nabla p \nabla \varphi + p\varphi) dx = \int_\Omega (y-y_d)\varphi dx \;\; \forall \varphi \in V. \tag{4.114}$$

Therefore the necessary optimality condition takes form: if $u^\star \in U_{ad}$ is an optimal solution to (4.61) there exists Lagrange multipliers $(p^\star, \lambda_1^\star, \lambda_2^\star) \in V \times U \times U$ such that the following conditions hold:

$$\int_\Omega (\nabla y^\star \nabla \varphi + y^\star \varphi) dx = \int_\Omega f\varphi dx + \int_{\partial_N \Omega} u^\star \varphi ds \quad \forall \varphi \in V, \tag{4.115}$$

$$\int_\Omega (\nabla p^\star \nabla \varphi + p^\star \varphi) dx = \int_\Omega (y^\star - y_d)\varphi dx \quad \forall \varphi \in V, \tag{4.116}$$

$$\int_{\partial_N \Omega} (p^\star - \alpha u^\star)\varphi ds = 0 \quad \forall \varphi \in V, \tag{4.117}$$

$$\int_{\partial_N \Omega} (\tilde{\lambda}_1 - \lambda_1^\star)(u_{min} - u^\star) ds \leq 0, \;\; \forall \tilde{\lambda}_1 \in U, \;\; \lambda_1^\star \geq 0, \tag{4.118}$$

$$\int_{\partial_N \Omega} (\tilde{\lambda}_2 - \lambda_2^\star)(u^\star - u_{max})ds \leq 0, \ \forall \tilde{\lambda}_2 \in U, \ \lambda_2^\star \leq 0, \tag{4.119}$$

where $y^\star = y(u^\star)$.