Reference :

Cz. Bajer, Time integration methods – still questions. In: Theoretical Foundations of Civil Engineering. W. Szcześniak, ed., v. 1. pp 45-54, Warsaw 2002.

Time integration methods – still questions

Czesław Bajer

Institute of Fundamental Technological Research, Polish Academy of Sciences, Świętokrzyska 21, 00-049 Warsaw, Poland

Streszczenie. W pracy omówione zostały niestosowane w praktyce metody całkowania równań różniczkowych ruchu. Są one niezwykle skuteczne numerycznie oraz mają korzystniejsze własności od powszechnie stosowanej metody Newmarka czy metody różnic centralnych. Niestety, poza miejscem w publikacjach samych twórców nie weszły do szerszych zastosowań. Metody Bossaka, Parka-Housnera, Hilbera-Hughesa-Taylora, elementów czasoprzestrzennych, przedstawione w niniejszej pracy, mogą być bezpośrednio zastosowane w komercyjnych pakietach obliczeniowych bez istotnych modyfikacji.

1. Introduction

Time integration methods for several of years were intensively investigated. In hundreds of publications both new methods and their properties were broadly described. Unfortunately nowadays in the engineering practice only few of them are in regular use. Usually the selection is done taking into account the accessibility of procedures rather than numerical quality. Newmark method and the central difference method are employed in almost totality of structural dynamic analysis problems. The simplicity is a great advantage of these methods. However, alternative computational schemes are not more complicated. They enable the user a wide range of useful properties instead.

The decision: implicit or explicit methods, depends on the problem to be solved. Refined spatial mesh decreases the approximation error and strongly increases the computational time, because of both the total number of degrees of freedom increase and the time step decrease involved by the stability criterion. However, experiences in the practical use of time integration methods are low. Some properties of the methods are described in the academic literature. The Wilson method is too dissipative in lower modes. It requires a time step smaller than needed for required accuracy. The Houbolt method is even more dissipative than the Wilson method. It has no parameter to control this property. The damping is controlled in practice by the time step value.

In the opinion of the author, the best time integration method should has the following features:

- should be unconditionally stable,
- should have the numerical dissipation controlled by a parameter (in a particular case should have no dissipation),
- the numerical dissipation should affect higher modes; lover modes should not be affected,
- numerical effort should be low enough, comparing with explicit methods,
- should permit computations of non-inertia structures with the motion forced kinematically.

The last point of the above list concerns for example crashworthiness problems.

In the paper we recall some efficient schemes, rarely used, with interesting non-classical features. The algorithms which perform computations in practice are as simple as in commonly applied methods and can be alternatively introduced to computational codes.

The following features are important in practical use:

- computational cost,
- accuracy (phase error),
- stability,
- damping of high and low frequencies,
- scheme of the propagation of information (important in wave problems),
- type of inertia matrices, fundamental for finite displacements and rotations.

Below we discuss the following methods:

- implicit methods: Newmark, Bossak, Hilber-Hughes-Taylor, space-time element method,
- semi-implicit methods: Park-Housner, Trujillo.

The scheme of numerical methods applied to structural dynamics is depicted in Fig. 1.

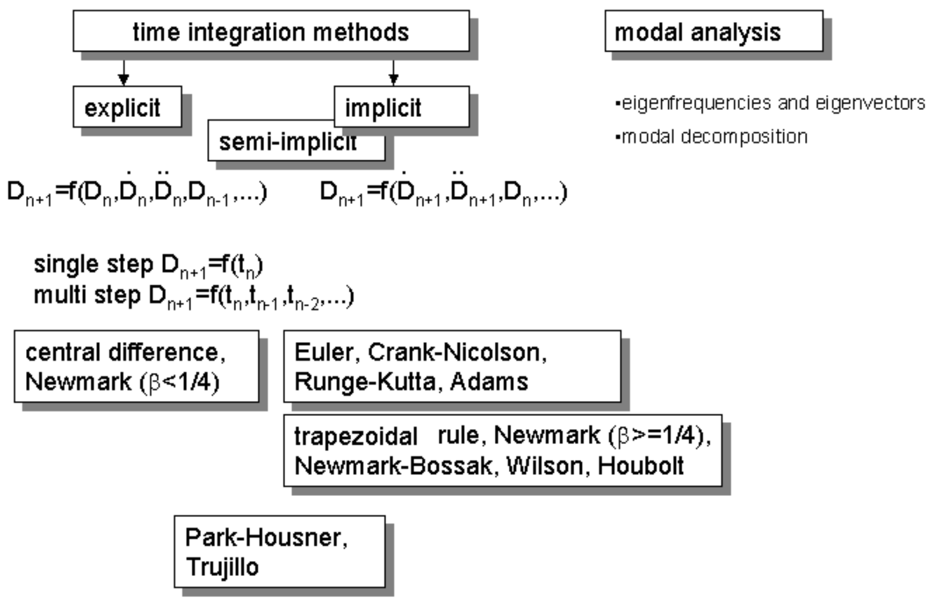


Fig. 1. Numerical methods employed in structural dynamics.

Although discrete methods are broadly applied to structural dynamics, the qualitative progress has not been made for several recent years. The following questions are still open.

- Parabolic type of the solution of the hyperbolic differential equation; numerical velocity of the information flow is higher than the physical wave speed.
- Inertia matrix does not result in accurate period of vibration, especially if applied to finite rotations.
- Spurious oscillations in fine meshes are hardly eliminated.

That is why so many methods have been elaborated, with the hope to improve at least one of the mentioned features.

2. The Newmark method

The Newmark method [1], well known and commonly applied in computations, is presented here since it is a particular case of the methods described in successive paragraphs. The force equilibrium is determined in t_{i+1}. Three equations (Tab. 1) allow to determine the displacement vector by solving the system of equations. Complementary computations are carried out to determine velocity and acceleration vector.

Table 1. The Newmark scheme.

▪ $\mathbf{u}_{n+1}=\mathbf{u}_n+h\mathbf{v}_n+h^2(1/2-\beta)\mathbf{a}_n+h^2\beta\mathbf{a}_{n+1}$
▪ $\mathbf{v}_{n+1}=\mathbf{v}_n+h(1-\gamma)\mathbf{a}_n+h\gamma\mathbf{a}_{n+1}$
▪ $\mathbf{Ma}_{n+1}+\mathbf{Cv}_{n+1}+\mathbf{Ku}_{n+1}=\mathbf{F}_{n+1}$

In the case of $\gamma=1/2$ and $\beta \geq 1/4$ we have unconditionally stable scheme. The maximum numerical damping of higher frequencies is achieved for $\beta={}^1/_4(\gamma+{}^1/_2)^2$, with $\gamma>{}^1/_2$.

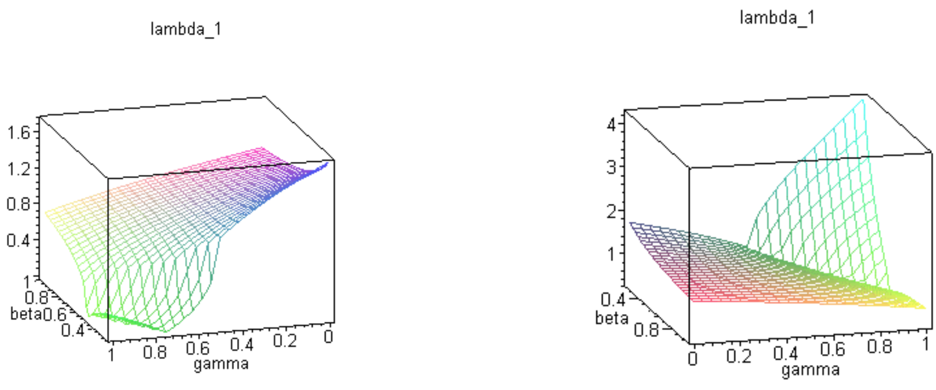


Fig. 2. Spectral radius of the transfer matrix for Newmark method.

3. The Bossak method

The Bossak method [2] is the extension of the Newmark method. The acceleration **a** is taken prior to t_{i+1}. The method can successfully replace the Newmark method in all cases.

Table 2. The Bossak scheme.

- $\mathbf{u}_{n+1}=\mathbf{u}_n+h\mathbf{v}_n+h^2(1/2-\beta)\mathbf{a}_n+h^2\beta\mathbf{a}_{n+1}$
- $\mathbf{v}_{n+1}=\mathbf{v}_n+h(1-\gamma)\mathbf{a}_n+h\gamma\mathbf{a}_{n+1}$
- $\mathbf{M}(1-\alpha_B)\,\mathbf{a}_{n+1}+\mathbf{M}\alpha_B\mathbf{a}_n+\mathbf{C}\mathbf{v}_{n+1}+\mathbf{K}\mathbf{u}_{n+1}=\mathbf{F}_{n+1}$

In the case of $\alpha_B=0$ we have the Newmark method.

Stability conditions are fulfil for

$$\alpha_B \leq 1/2 \qquad \beta_B \geq \gamma_B/2 \geq 1/4 \qquad \alpha_B + \gamma_B \geq 1/4$$

The spectral radius, which determine the numerical dissipation, is depicted in Fig. 3. The following values of parameters are assumed:
B1: $\alpha_B= -0.1$, $\beta_B=0.3025$, $\gamma_B=0.6$,
B2: $\alpha_B= -0.1$, $\beta_B=0.5000$, $\gamma_B=0.6$,
B3: $\alpha_B= +0.1$, $\beta_B=0.3025$, $\gamma_B=0.6$.

The set of Hilber-Hughes-Taylor parameters are as follows:
HHT1: $\alpha_H= -0.1$, $\beta=0.3025$, $\gamma=0.6$,
HHT2: $\alpha_H= -0.3$, $\beta=0.3025$, $\gamma=0.6$.

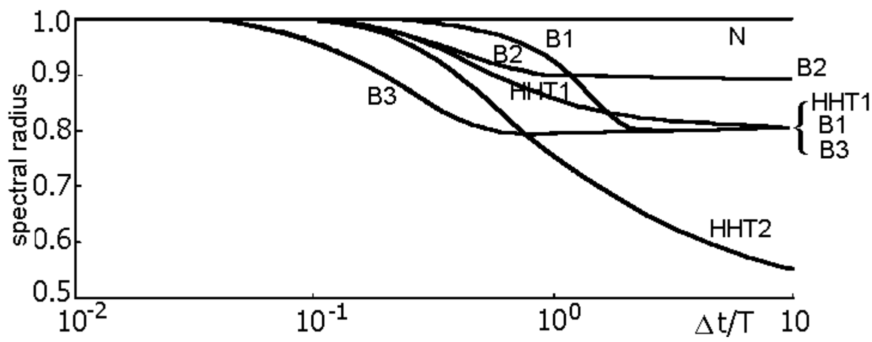


Fig. 3. Spectral radius for selected time integration methods [2].

The Bossak method is characteristic of a good damping in high frequencies range and less sensible to the wrong choice than Hilber-Hughes-Taylor method.

4. Hilber-Hughes-Taylor method

The elastic forces are taken here between t_{n+1} and t_{n+1} (α_H is negative in the original publication [3]), i.e. in $t_{n+1}+\alpha_H h$. In the case of $\alpha_H=0$ we have the Newmark method. The effect of the artificial damping for this method is depicted in Fig. 3. The authors of this method do not give the range of application, mutual relation between parameters α_H, β and γ and their influence on the stability condition. Numerical tests performed by the author of the present paper proved that the change of the parameters should be done with attention.

The method can be considered as the alternative to the Bossak method. However, since it contributes potential forces not clearly definite, applications to nonlinear problems should be investigated.

Table 4. The Hilber-Hughes-Taylor scheme.

- $\mathbf{u}_{n+1}=\mathbf{u}_n+h\mathrm{v}_n+h^2(1/2-\beta)\mathbf{a}_n+h^2\beta\mathbf{a}_{n+1}$
- $\mathbf{v}_{n+1}=\mathbf{v}_n+h(1-\gamma)\mathbf{a}_n+h\gamma\mathbf{a}_{n+1}$
- $\mathbf{Ma}_{n+1}+(1+\alpha_H)\mathbf{Ku}_{n+1}-\alpha_H\mathbf{K}\mathbf{u}_n=\mathbf{F}_{n+1}$

5. The Park-Housner method

The Park-Housner method [4] is the example of semi-implicit methods. It employs the best features of both the implicit and explicit methods: low numerical cost and memory storage requirements with unconditional stability. The mass matrix **M** is diagonal. The stiffness matrix **K** is split into a sum of triangular matrices. Two systems of equations with triangular matrices are to be solved. Finally resulting displacement and velocity vectors are computed (Tab. 5).

Table 5. Park-Housner algorithm.

- Form **K** and diagonal **M**
- Split **K** into $\mathbf{K}_L$ and $\mathbf{K}_U$ ($\mathbf{K}=\mathbf{K}_L+\mathbf{K}_U$, $\mathbf{K}_L=\mathbf{K}_U^T$)
- Build the matrices of systems of equations:
  $\mathbf{L}=\mathbf{M}(\mathbf{I}+\alpha\beta h^2\mathbf{M}^{-1}\mathbf{K}_L)$, $\mathbf{U}=\mathbf{I}+\alpha\beta h^2\mathbf{M}^{-1}\mathbf{K}_\mathbf{U}$,
  $\mathbf{g}_{n+1}=\alpha\beta h^2[\beta\mathbf{f}_{n+1}+(1-\beta)\mathbf{f}_n]+\mathbf{M}(\mathbf{u}_n+\beta h\mathbf{v}_n)$
- Solve the systems of equations (triangular matrices):
  $\mathbf{Ly}_{n+1}=\mathbf{g}_{n+1}$, $\mathbf{Uu}^*_{n+1}=\mathbf{y}_{n+1}$
- Solve:
  $\mathbf{u}_{n+1}=1/\beta[\mathbf{u}^*_{n+1}-(1-\beta)\mathbf{u}_n]$, $\mathbf{v}_{n+1}=1/(\alpha h)(\mathbf{u}_{n+1}-\mathbf{u}_n)-(1-\alpha)/\alpha\,\mathbf{v}_n$

The stability analysis exhibits for which values α and β the stability is ensured (Fig. 4).

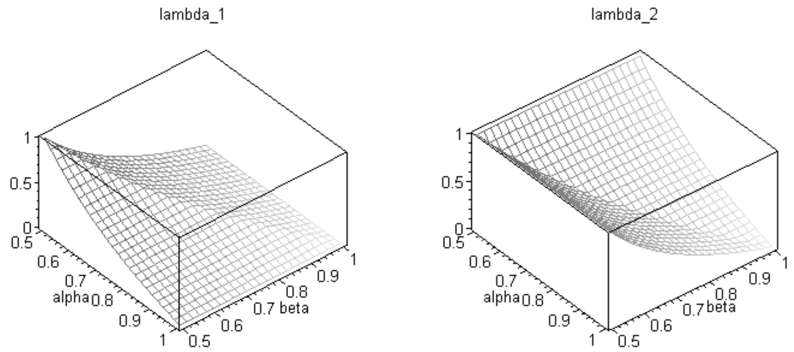


Fig. 4. Spectral radius of the transfer matrix for Park-Housner method.

In only one point ($\alpha=1/2$, $\beta=1/2$) we have the unconditional stability of the method without artificial amplitude decrease. The damping for $\alpha,\beta>0.5$ is significant and affects low frequencies considerably.

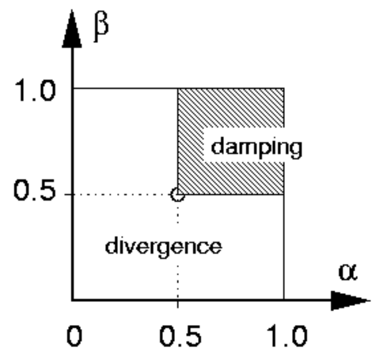


Fig. 4. Stability domain for Park-Housner method.

The incoincidence with the theoretical solution (Fig. 7) concerns the oscillation decay. In means that the system response is trapezoidal rather than rectangular one.

6. Trujillo method

In the Trujillo semi-implicit method the inertia matrix **M** is diagonal. Matrices **K** and **C** are symmetric and positive definite. The restriction of a diagonal matrix **M** may not be severe. **K** and **C** are split into lower and upper triangular matrices: $\mathbf{K}_L+\mathbf{K}_U=\mathbf{K}$, $\mathbf{C}_L+\mathbf{C}_U=\mathbf{C}$. The symmetric splitting was investigated in the original paper [5].

Table 4. Trujillo scheme.

$$\left(\mathbf{M}+\mathbf{C}_L\frac{h}{2}+\mathbf{K}_L\frac{h^2}{8}\right)\mathbf{v}_{j+1/2}=\left(\mathbf{M}-\mathbf{C}_U\frac{h}{2}-\mathbf{K}_L\frac{h^2}{8}\right)\mathbf{v}_j-\mathbf{K}\frac{h}{2}\mathbf{u}_j+\left(\mathbf{f}_j+\mathbf{f}_{j+1}\right)\frac{h}{4}$$

$$\mathbf{u}_{j+1/2}=\mathbf{u}_j+\frac{h}{4}\left(\mathbf{v}_j+\mathbf{v}_{j+1}\right)$$

$$\left(\mathbf{M}+\mathbf{C}_U\frac{h}{2}+\mathbf{K}_U\frac{h^2}{8}\right)\mathbf{v}_{j+1}=\left(\mathbf{M}-\mathbf{C}_L\frac{h}{2}-\mathbf{K}_U\frac{h^2}{8}\right)\mathbf{v}_{j+1/2}-\mathbf{K}\frac{h}{2}\mathbf{u}_{j+1/2}+\left(\mathbf{f}_j+\mathbf{f}_{j+1/2}\right)\frac{h}{4}$$

$$\mathbf{u}_{j+1}=\mathbf{u}_{j+1/2}+\frac{h}{4}\left(\mathbf{v}_{j+1/2}+\mathbf{v}_{j+1}\right)$$

$$\mathbf{K}=\mathbf{K}_L+\mathbf{K}_U,\quad \mathbf{K}=\mathbf{K}_L+\mathbf{K}_U,\quad h=2\Delta t,\quad t=jh=2j\Delta t$$

The test example depicted in Fig. 7 shows that results better coincide with theoretical response, then those obtained by Park-Housner method.

7. Space-time element method

The space-time finite element method has two formulations. The early one is expressed in terms of displacements [6,7]. The time space could be split into rectangular, triangular or even less regular meshes. The displacement formulation results in the three-level scheme

$$\mathbf{A}\mathbf{q}_{n-1}+\mathbf{B}\mathbf{q}_n+\mathbf{C}\mathbf{q}_{n+1}=\mathbf{F}_i.$$

The second type of formulation is expressed in terms of velocities [8,9]. The step-by-step scheme is the following

$$\mathbf{A}\mathbf{v}_n+\mathbf{B}\mathbf{v}_{n+1}+\mathbf{s}_n=\mathbf{F}_n.$$

A, **B**, **C** are the square matrices, **q** and **v** are the displacement and velocity vectors, respectively, **F** is the vector of external forces and **s** is the vector of nodal potential forces, computed at the end of the previous time step.

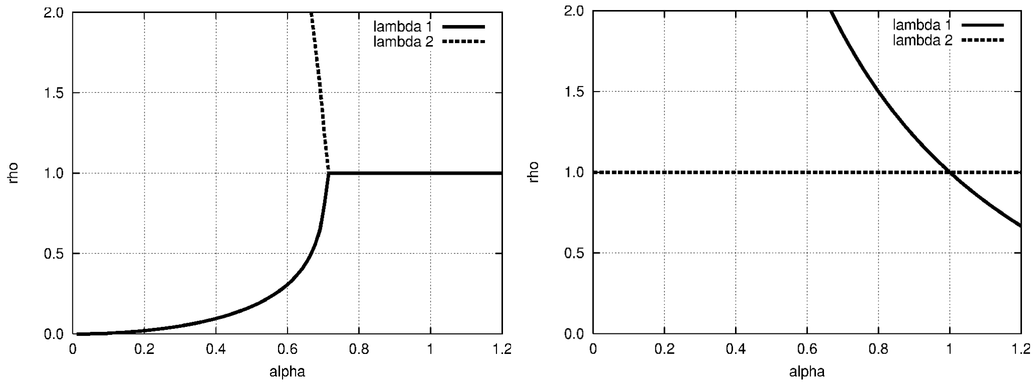
Table 6. The algorithm of the space-time element method (velocity formulation).

▪ Form the matrices **K** and **M**	
▪ Right-hand-side vector:	$\mathbf{F}=\mathbf{f}_{n+1}-\left[\mathbf{K}(1-\frac{\alpha}{2})\alpha h-\frac{1}{h}\mathbf{M}\right]\mathbf{v}_n$
▪ Solve the equation:	$\left[\mathbf{K}\frac{\alpha^2}{2}h+\frac{1}{h}\mathbf{M}\right]\mathbf{v}_{n+1}=\mathbf{F}$
▪ Displacements:	$\mathbf{x}_{n+1}=\mathbf{x}_n+h\left[\alpha\mathbf{v}_n+(1-\alpha)\mathbf{v}_{n+1}\right]$
▪ Nodal forces:	$\mathbf{s}_{n+1}=\mathbf{K}\,\mathbf{x}_{n+1}$

The numerical dissipation is performed by modifying the formula for displacements

$$\mathbf{x}_{n+1}=\mathbf{x}_n+h\left[(1-\beta)\mathbf{v}_n+\beta\mathbf{v}_{n+1}\right], \quad \beta=1-\alpha/(1+\gamma)\ .$$

The important advantage of this method is that it can be directly employed both to dynamic and quasistatic analysis. For $\alpha=1.0$ the same procedure can be used even if the mass density is equal to zero. In such a case the kinematic boundary conditions should force the motion. In the case of positive mass density the unconditional stability is obtained for $\alpha\geq\sqrt{2}/2$.



M>0 M=0

Fig. 5. Spectral radiuses with non zero inertia and without inertia.

The time integration methods (many of them) can be described in the following form:

$$\mathbf{q}_{i+1} = \mathbf{A}\ \mathbf{q}_i - \mathbf{q}_{i-1}$$

The comparison given in Tab. 7 presents the group of methods, that can be considered as a particular case of the space-time finite element formulation.

Table 7. Operators for time integration methods.

method	operator **A**
central difference method Newmark $\beta=0$, $\gamma=1/2$ space-time elem. $\alpha=0$	$2-\kappa$
Newmark $\beta=1/6$, $\gamma=1/2$	$4(3-\kappa)/(6+\kappa)$
trapezoidal rule Crank-Nicolson Newmark $\beta=1/4$, $\gamma=1/2$ space-time elem. $\alpha=0.707$	$2(4-\kappa)/(4+\kappa)$
space-time elem. $\alpha=1$	$4/(2+\kappa)$

8. Information flow.

The information flow between nodes is important for every wave propagation problem. Especially strongly non-linear problems are sensitive. We can say that all the discrete methods exhibit parabolic-type propagation of disturbances. Implicit methods give infinite speed of the information flow while in explicit methods the speed is limited to the diagonal of the mesh. Practically in both types the wave propagation exceeds the physical wave speed.

The wave front in the same time is not sufficiently sharp. In certain problems it can be essential. Fig. 6 shows the flow of information between joints in one time layer and between successive time layers. The arrows show how the external impulses flow from joint to joint and how they perturb the mesh.

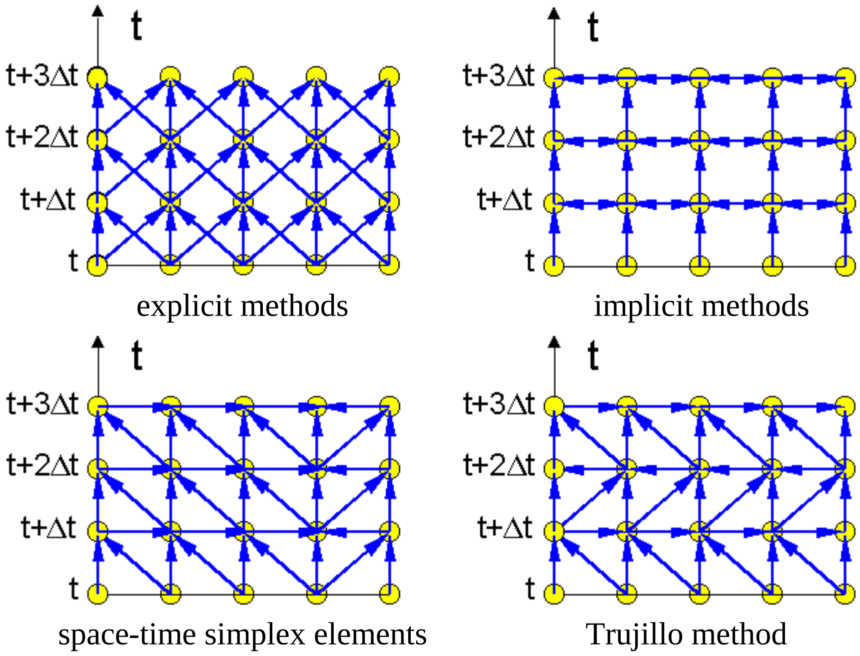


Fig. 6. The information flow in selected time integration schemes.

9. Inertia matrix.

Dynamic response strongly depends on the form of inertia matrix. Several forms of inertia matrices are described in the literature. The diagonal mass matrix is the simplest one. It is efficient in numerical calculations carried on by central difference method or semi-implicit methods. The consistent matrix, derived directly from the shape functions results in more accurate simulation of wave propagation ore vibrations with both the transverse and rotatory degrees of freedom. Another way [10] of the lumping scheme also does not take into account complete rotational degrees of freedom.

We can notice that for most of purposes all the methods give sufficient accuracy. However, the best approach to the theoretical line is obtained with the consistent mass matrix. Park-Housner and Trujillo methods have the feature observed for lumped mass matrix (since they use such a matrix). The split of stiffness matrix in the case of semi-implicit methods does not worsen results. However, the Trujillo method exhibits better quality.

We should emphasize that even if the artificial damping is applied, the spurious vibrations locally dominate. The decay of oscillations, when consistent and lumped mass matrices are applied, are similar. However, in the first case the divergence is according to time, in the second case it is opposed.

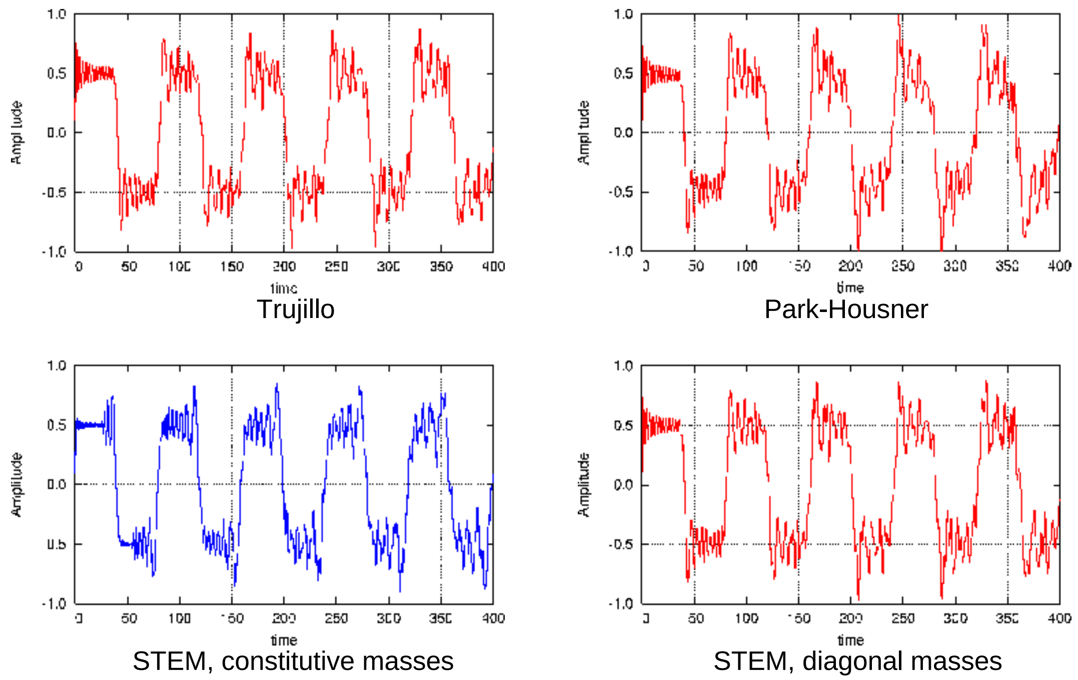


Fig. 7. Vibrating bar solved by different methods (no damping).

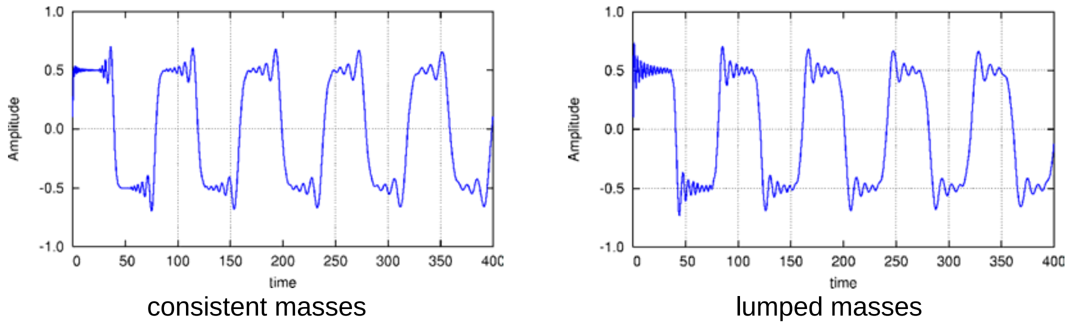


Fig. 8. Artificial damping effect with constitutive and lumped mass matrices.

11. Conclusions.

The comparison of time integration method given in this paper exhibits non classical methods, elaborated and described in the literature. Although the question of the efficient numerical tool was discussed in many papers (for example [11-15]), supplementary tests proved essential advantages of the methods presented in the present paper:

- semi-implicit methods are efficient and give sufficiently good results both in dynamic and wave analysis,
- the Bossak and Hilber-Hughes-Taylor methods are the alternative to the Newmark method,
- the space-time element method enables both dynamic and quasi-static analysis.

References

1. N.M. Newmark. A method of computation for structural dynamics. *Proc. A.S.C.E.*, **8**:67-94, 1959.
2. W.L. Wood, M. Bossak and O.C. Zienkiewicz. An alpha modification of Newmark's method. *Int. J. Num. Meth. Eng.*, **15**:1562-1566, 1981.
3. H.M. Hilber, T.J.R. Hughes and R.L. Taylor. Improved numerical dissipation for time integration algorithms in structural dynamics. *Earthquake Eng. And Struc. Dyn.*, **5**:283-292, 1977.
4. K. Park and J.M. Housner. Semi-implicit transient analysis procedures for structural dynamics analysis. *Int. J. Num. Meth. Eng.*, **18**:609-622, 1982.
5. D.M. Truhillo. An unconditionally stable explicit algorithm for structural dynamics. *Int. J. Num. Meth. Eng.*, **11**:1579-1592, 1977.
6. Z. Kączkowski. On the application of non-rectangular space-time elements. *Mech. Teoret. i Stosow.*, **21**(4):531-542, 1883.
7. C. Bajer. Trangular and tetrahedral space-time finite elements in vibration analysis. *Int. J. Numer. Meth. Engng.*, **23**:2031-2048, 1986.
8. C. Bajer. Space-time finite element formulation for the dynamical evolutionary process. *Appl. Math. and Comp. Sci.*, **3**(2)251-268, 1993.
9. C. Bajer and C. Bohatier. The soft way method and the velocy formulation. Comput. and Struct., **55**(6):1015-1025, 1995.
10. E. Hinton, T. Rock and O.C. Zienkiewicz. A note on mass lumping and related processes in the finite element method. *Earthquake Engng. Struct. Dyn.*, **4**(3):245-249, 1976.
11. R. Mullen and T. Belytschko. An analysis of an unconditionally stable explicit method. *Comp. and Struct.*, **16**:691-696, 1983.
12. T.J.R. Hughes and T. Belytschko. A précis of developments in computational methods for transient analysis. *J. Appl. Mech.*, **50**:1033-1041 (1983).
13. J. Cushman. Difference schemes or element schemes? *Int. J. Num. Meth Engng.*, **14**:1643-1651, 1979.
14. R. Mullen. An analysis of an unconditionally stable explicit method. *Comp. and Struct.*, **16**(6):691-696, 1983.
15. K.C. Park. Practical aspects of numerical time integration. *Comp. and Struct.*, **7**:343-353, 1977.

Supported by KBN: 8 T12C 046 21