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The influence of inertia in moving load problems

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1 Introduction

Several ways of modelling are considered in problems of structures vibrating under a travelling load. Two main kinds of a load, a moving massless force or a moving inertial force (Figure 1) result in different mathematical and numerical treatment. Displacements in time also significantly differ. What is more, solutions differ qualitatively.

The analysis of the moving massless force is relatively simple and has been treated in numerous publications [1, 2]. We include in this group all the papers devoted to the travelling oscillator, i.e. a mass particle joined to the base with a spring. Although the authors call this type of a load an inertial one, we consider it as a massless force generated only by the particle's inertia. The inertial force moving over the structure is rarely reported in the literature [3, 4, 5, 6, 7]. In the present paper we focus our attention on the true inertial travelling load and on solutions of the problem in a finite domain. The existing solutions which deal with the problem are not satisfactory. Examples of practical problems are presented in Figure 2.

The solution in a form of series was presented in [5]. However, the discontinuity of the mass trajectory, ie. the fundamental feature of the solution was exhibited in [8]. The broad discussion of the discontinuity was also proved there. This phenomenon has not been previously reported in literature. In this paper we present the comparison of displacements of a structure under the massless and inertial moving load in the case of a string, and Bernoulli-Euler and Timoshenko beams. A similar analysis can be performed in the case of a plate, however it will not be included in this text.

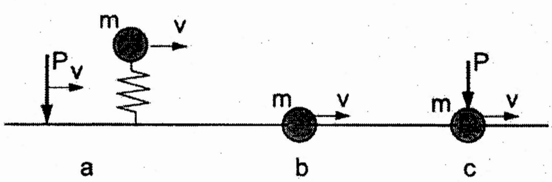

Figure 1: Moving loads: massless load (a), inertial load (b), and inertial load with a massless force (c).

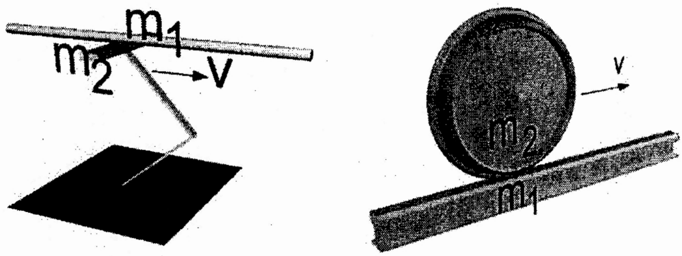

Figure 2: Examples of practical applications: a power receiver and a wheel-rail interaction.

In the paper [5] the author deals with the problem of a moving mass by the integro-differential equation. In [8] the same solution is obtained with the Fourier method applied directly to the motion differential equation. In both approaches the mathematical treatment of the distribution function results in serious questions on the continuity of the solution. The detailed discussion of the problem was given in [9]. Finally, in the paper [10] the Lagrange equation of the second kind was applied. This approach allows us to avoid troubles with treatment of the Dirac delta in the consequent mathematical solution applied to the differential equation of the motion. Especially we do not transform the distribution function.

Finally we must mention here about numerical procedures derived for the same class of problems. It must be emphasised here that the correct numerical solution with the use of the classical time integration method (for example the Newmark method) can not be obtained with the approaches given in literature. The intuitive approach to discrete analysis with the ad hoc lumping of forces and masses to neighbouring nodes with the use of the Renaudot formula always fails. Sometimes, especially in the case of beams, numerical solutions are limited, but significantly inaccurate. In the case of a string such a strategy results in divergent solutions. We emphasise here that the travelling mass problem is not trivial, even if at first sight that seems to be the case. Up to date numerical solution can be obtained with the regular space-time analysis of the finite element carrying the mass particle. Such an element, called the space-time finite element, exhibits almost absolute precision, comparing with the semi-analytical solution [11, 12, 13]. The problem starts to be important in the case of a multipoint load and mutual interaction induced by the vehicle frame. This problem should be investigated separately.

2 Mathematical formulation

Let us consider a string, a simply supported Bernoulli-Euler beam and a Timoshenko beam of the length l, cross-sectional area A, mass density ρ, subjected to a mass m moving at a constant speed v. We assume a permanent contact of the mass with the structure.

There are two components of the moving load. As the first term we consider the massless gravity force mg proportional to the gravity acceleration g, constant at each time step. The second term, the inertial force $m\ddot{u}$ varies together with variation of vertical displacements. In the first case the analytical solutions exist, e.g. [2]. The analysis of the inertial moving mass effect is much more complex. Since the closed solution can not be determined in an analytical way we must integrate the intermediate form of the solution numerically. In order to reduce the partial differential equation to the system of ordinary differential equations, we apply the Fourier sine integral transformation in a finite range $< 0; l >$.

$$u(x,t) = \frac{2}{l}\sum_{j=1}^{\infty} \xi_j(t)\,\sin\frac{j\pi x}{l}\,, \tag{1}$$

where

$$\xi_j(t) = \int_0^l u(x,t)\,\sin\frac{j\pi x}{l}\,\mathrm{d}x\,. \tag{2}$$

The Fourier series presented above fulfil boundary conditions. We restrict the solution (1) to the finite number of terms ($j = 150$).

String under a moving inertial force can be written in the following form

$$-N\frac{\partial^2 u(x,t)}{\partial x^2} + \rho A\frac{\partial^2 u(x,t)}{\partial t^2} = \delta(x-vt)m\left(g - \frac{\mathrm{d}^2 u(vt,t)}{\mathrm{d}t^2}\right)\,, \tag{3}$$

where N is the tensile force in the string.
With respect to (1) the moving mass acceleration is expanded into a trigonometric series fixed at a point $x = vt$

$$\frac{\mathrm{d}^2 u(vt,t)}{\mathrm{d}t^2} = \frac{2}{l}\sum_{k=1}^{\infty}\left[\ddot{\xi}_k(t)\sin\frac{k\pi vt}{l} + \frac{2k\pi v}{l}\dot{\xi}_k(t)\cos\frac{k\pi vt}{l} - \frac{k^2\pi^2v^2}{l^2}\xi_k(t)\sin\frac{k\pi vt}{l}\right]\,; \tag{4}$$

Equation (4) corresponds to the Renaudot equation which, when multiplied by the mass m, describes the transverse inertia force, the Coriolis force and the centrifugal force, respectively.

The motion of the Bernoulli-Euler beam under moving inertial load is given by the equation

$$EI\frac{\partial^4 u(x,t)}{\partial x^4} + \rho A\frac{\partial^2 u(x,t)}{\partial t^2} = \delta(x-vt)m\left(g - \frac{\mathrm{d}^2 u(vt,t)}{\mathrm{d}t^2}\right)\,, \tag{5}$$

where E is Young modulus and I is moment of inertia.

The motion equation of the Timoshenko beam under moving inertial load with a constant speed v has the following form

$$\begin{aligned} EI\frac{\partial^4 u(x,t)}{\partial x^4} - \left(\rho I + \rho k\frac{EI}{G}\right)\frac{\partial^4 u(x,t)}{\partial x^2\partial t^2} + \rho^2 k\frac{I}{G}\frac{\partial^4 u(x,t)}{\partial t^4} + \rho A\frac{\partial^2 u(x,t)}{\partial t^2} &= \\ = q(x,t) - k\frac{EI}{GA}\frac{\partial^2 q(x,t)}{\partial x^2} + \rho k\frac{I}{GA}\frac{\partial^2 q(x,t)}{\partial t^2}\,, \end{aligned} \tag{6}$$

where

$$q(x,t) = \delta(x-vt)m\left(g - \frac{\mathrm{d}^2 u(vt,t)}{\mathrm{d}t^2}\right)\,. \tag{7}$$

G is the Kirchhoff modulus and k is the coefficient of the cross-section. Delta Dirac function δ induces that the solution of the differential equations (3), (5) and (6) must satisfy the theory of distribution [14].

Below we present a brief solution with the method of Lagrange equation of the second kind. In this way we reduce the solution to the classical differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\xi}_j}\right) - \frac{\partial T}{\partial \xi_j} + \frac{\partial U}{\partial \xi_j} = 0\,. \tag{8}$$

T is the kinetic energy and U is the potential energy. The contribution of the moving inertial load is described by the equation

$$T_m = \frac{1}{2} m \left[\frac{\mathrm{d}u(vt,t)}{\mathrm{d}t}\right]^2 . \tag{9}$$

The moving gravitation force is expressed by the term

$$U_m = mgu(vt,t)\,. \tag{10}$$

Both the Fourier and the Lagrange methods lead us to the identical systems of differential equations with variable coefficients.

3 Examples

Below we present results obtained for selected structures subjected to a moving load. Two types of diagrams are characteristic for the analysis of the problem: the displacements in time of the contact point travelling together with the load and displacements of the midpoint of the span.

String

In the Figure 3 displacements in time are depicted. The left hand side column shows displacements under a travelling inertial particle and the right hand side column depicts displacements of the midpoint. The case of a massless force is plotted with the continuous line while the inertial load is plotted with a dashed line. The comparison of both curves in the case of a string exhibits the smooth form in the case of travelling mass. More, at the final stage the mass trajectory exhibits the discontinuity. It is well seen in the case of a speed v=0.5c, where c is the wave speed. However, it appears in a whole range of the speed $0 < v < c$.

Bernoulli-Euler beam

The simplest model of a beam results in similar diagrams as in a more complex Timoshenko model, presented in the next point. Differences in displacements can reach 50% and, although the mass trajectory is smooth, the acceleration values differ(Figure 4). Accelerations at the final stage of the process are significantly higher in the case of the moving inertial particle (left column in Figure 4).

Timoshenko beam

The Timoshenko beam exhibits similar properties as in the case of a string. The discontinuous trajectory can be noticed especially at a speed v=0.3 and v=0.5. We use dimensionless data. Vertical acceleration of the travelling mass is high and noticeable jumps must be taken into account in engineering calculations. Although in real structures described by nonlinear equations we observe a smooth response, we should expect

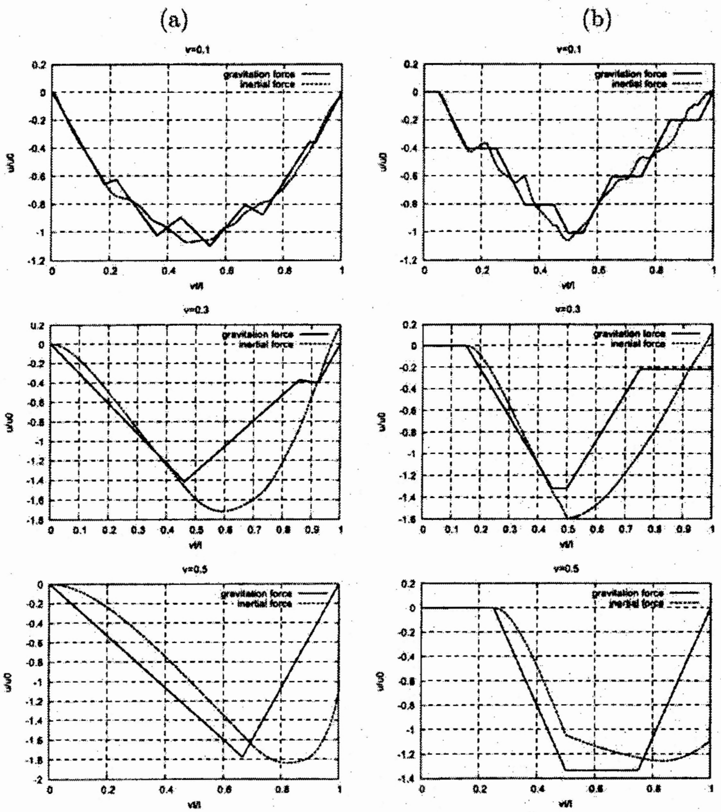

Figure 3: Moving inertial load travelling along a string: a – under the travelling mass, b – at the midpoint

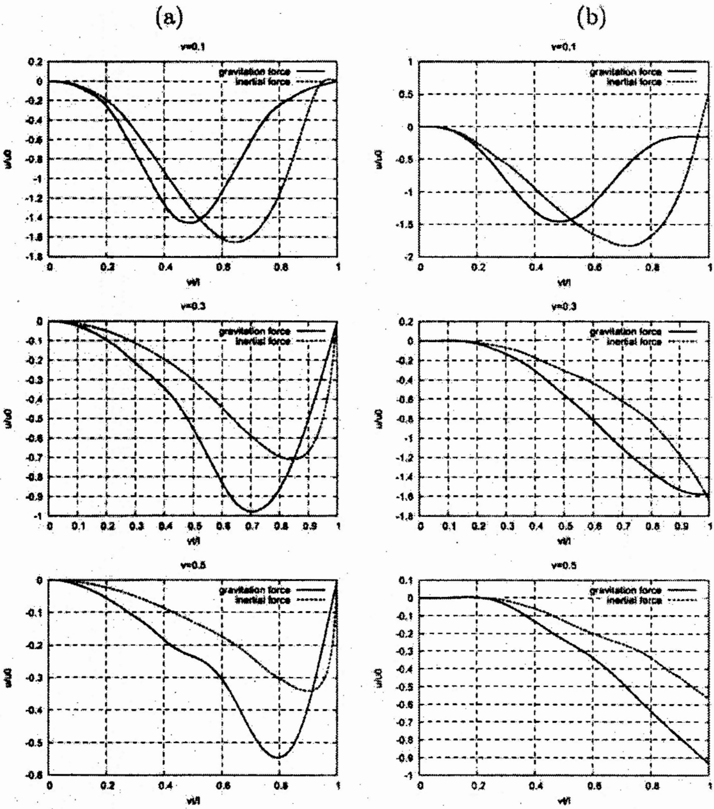

Figure 4: Moving inertial load travelling along the Bernoulli-Euler beam: a – under the travelling mass, b – at the midpoint

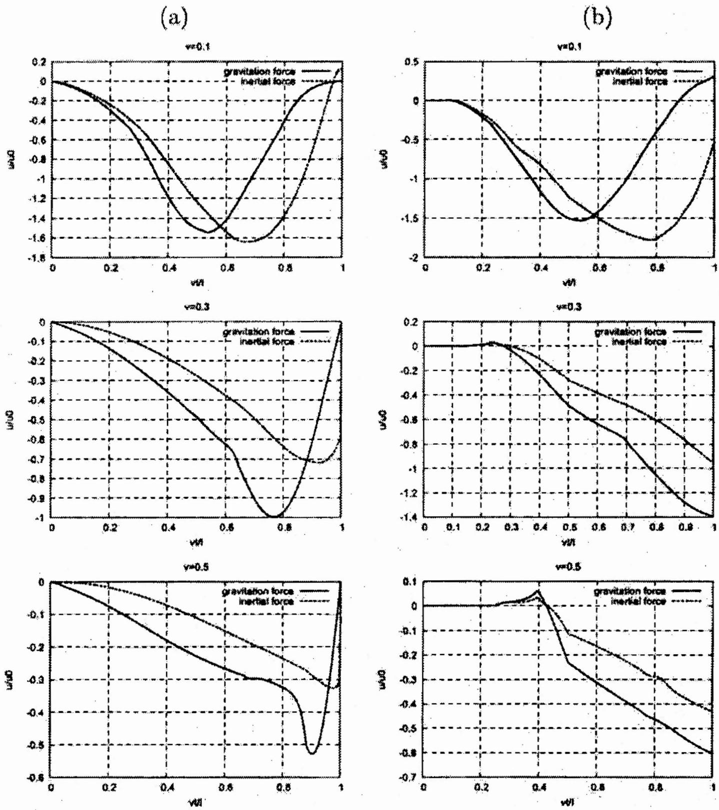

Figure 5: Moving inertial load travelling along the Timoshenko beam: a – under the travelling mass, b – at the midpoint

high vertical acceleration as a physical feature of the problem. We should emphasise e significant difference in the case of both types of a load (Figure 5). Generally we can say that in the case of inertial load displacements are lower while accelerations are higher.

4 Conclusions

We can briefly conclude that inertial loads moving along continuous structures result in higher jumps of a load near supports. In this case we can expect damages of both a structure and a vehicle. The finite element method, commonly used in engineering practice, upto now does not allow a proper analysis of inertial moving load problems. The alternate approach presented in [11, 12] will be developed for the classical Newmark time integration scheme.

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