Theoretical Foundations of Civil Engineering
Polish-Ukraïnian-Lithuanian Transactions
Warsaw, June 2008, Vol. 16, pp. 99-106

GENERAL APPROACH TO PROBLEMS WITH MOVING MASS

Bartłomiej DYNIEWICZ and Czesław BAJER
Institute of Fundamental Technological Research, Polish Academy of Sciences,
Świętokrzyska 21, 00-049 Warsaw, Poland

1. Introduction

Increasing velocity of trains results in the increase of the influence of the wave phenomenon on the deflections and stresses of structures. Dynamic effects are generated by the load of train current collectors, travelling through the power supply cable of the overhead contact line. The track or the bridge span are other places where the phenomenon occurs. Upto now the detailed analysis of the problem of the moving mass travelling on a string or a beam can not be found in the literature. The turning point in the literature devoted to moving loads was established by two historical publications [1, 2]. These analytical papers were elaborated with significant mathematical simplifications. There are some partial approaches [3]. There exist numerous review papers [4, 5, 6, 7] which discuss the problem. For a long time the main group of publications treated the problem in an analytical-numerical way [8, 9, 10, 11] or strictly numerically [12, 13, 14]. In our paper [10] we give the full semianalytical solution of the string vibrating under a moving inertial load. In the case of the massless string we proved the discontinuity of the mass trajectory. This phenomenon exists in solutions of strings and Timoshenko beams in the small displacement range and in such a case can be considered as a paradoxical solution. In a real case the mass trajectory is continuous, although stresses in the final stage of the motion can be significant.

In the paper we consider the solution derived from the Lagrange equation of the second kind. In a particular case the formulation is identical to the solution obtained by a direct analysis of the differential equation with the Dirac delta component.

In this paper, we consider a cable as the string-beam model, since it has a certain flexible stiffness. The Bernoulli-Euler beam with additional tensile effect comprises this phenomenon (Fig. 1). In the paper the differential equation of the motion of a string-beam is derived from the Lagrange equation of the 2nd kind.

2. The Fourier solution

The string motion under the moving load, both massless and inertial, is described by the following equation

$$-N\,\frac{\partial^2 u(x,t)}{\partial x^2} + \rho A\,\frac{\partial^2 u(x,t)}{\partial t^2} = \delta(x-vt)\,P - \delta(x-vt)\,m\,\frac{\partial^2 u(vt,t)}{\partial t^2}\,. \tag{1}$$

We assume boundary conditions

$$u(0,t) = 0 \qquad u(l,t) = 0 \tag{2}$$

and initial conditions

$$u(x,0) = 0 \qquad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0\,. \tag{3}$$

The partial differential equation of the motion 1) is reduced to the ordinary differential equation by using of the Fourier transform in the finite interval $< 0, l >$

$$V(j,t) = \int_0^l u(x,t)\,\sin\frac{j\pi x}{l}\,dx\,, \tag{4}$$

where the string displacement is expressed by an infinite series

$$u(x,t) = \frac{2}{l}\sum_{j=1}^{\infty} V(j,t)\,\sin\frac{j\pi x}{l}\,. \tag{5}$$

In order to (5) the acceleration under the moving load is described as follows

$$\frac{\partial^2 u(vt,t)}{\partial t^2} = \frac{2}{l}\sum_{k=1}^{\infty}\left[\ddot V(k,t)\sin\frac{k\pi vt}{l} + \frac{2k\pi v}{l}\dot V(k,t)\,\cos\frac{k\pi vt}{l} - \frac{k^2\pi^2 v^2}{l^2}V(k,t)\sin\frac{k\pi vt}{l}\right]. \tag{6}$$

The transformed motion equation (4) has a form

$$N\,\frac{j^2\pi^2}{l^2}\,V(j,t) + \rho A\,\ddot V(j,t) = P\sin\frac{j\pi vt}{l} - m\,\frac{\partial^2 u(vt,t)}{\partial t^2}\,\sin\frac{j\pi vt}{l}\,. \tag{7}$$

Finally the substitution of (6) and rearrangement of terms in (7) results in the ordinary differential equation with variable coefficients

$$\begin{aligned}\rho A\,\ddot V(j,t) &+ \alpha\sum_{k=1}^{\infty}\ddot V(k,t)\,\sin\omega_k t\,\sin\omega_j t + 2\alpha\sum_{k=1}^{\infty}\omega_k\,\dot V(k,t)\,\cos\omega_k t\,\sin\omega_j t + \\ &+ \Omega^2\,V(j,t) - \alpha\sum_{k=1}\omega_k^2\,V(k,t)\,\sin\omega_k t\,\sin\omega_j t = P\,\sin\omega_j t\,,\end{aligned} \tag{8}$$

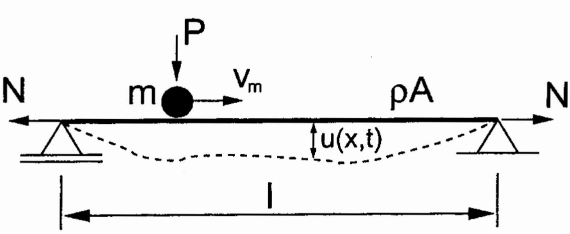

Figure 1: A string under the moving mass.

where

$$\omega_k = \frac{k\pi v}{l} , \qquad \omega_j = \frac{j\pi v}{l} , \qquad \Omega^2 = N\frac{j^2\pi^2}{l^2} , \qquad \alpha = \frac{2m}{l} . \tag{9}$$

3. The Lagrange equation

The kinetic energy of the whole system (ie. both the string and the moving mass) is given below

$$E_k = \frac{1}{2}\rho A\int_0^l \left[\frac{\partial u(x,t)}{\partial t}\right]^2 \mathrm{d}x + E_{km} , \tag{10}$$

where

$$E_{km} = \frac{1}{2}m\left[\frac{\partial u(vt,t)}{\partial t}\right]^2 . \tag{11}$$

Now we consider the infinitesimal segment of the string and we compute the potential energy of the entire string

$$E_p = \int_0^l N(\delta s - \delta x) = N\int_0^l \left\{\sqrt{1+\left[\frac{\partial u(x,t)}{\partial x}\right]^2} - 1\right\}\mathrm{d}x . \tag{12}$$

The form of the term in parenthesis is complex for further analysis so we expand it into the Maclaurin series

$$\sqrt{1+\left[\frac{\partial u(x,t)}{\partial x}\right]^2} - 1 = \frac{1}{2}\left[\frac{\partial u(x,t)}{\partial x}\right]^2 - \frac{1}{8}\left[\frac{\partial u(x,t)}{\partial x}\right]^4 + \frac{1}{16}\left[\frac{\partial u(x,t)}{\partial x}\right]^6 - \ldots . \tag{13}$$

Here we can decide what accuracy will be assumed in our analysis. We assume small displacements and in this case we can consider only the first term of the infinite expansion (13) as an accurate approach

$$E_p = N\int_0^l \left\{\sqrt{1+\left[\frac{\partial u(x,t)}{\partial x}\right]^2} - 1\right\}\mathrm{d}x \approx \frac{1}{2}N\int_0^l \left[\frac{\partial u(x,t)}{\partial x}\right]^2 \mathrm{d}x . \tag{14}$$

Finally the potential energy of the string with the moving force has the following form

$$E_p = \frac{1}{2}N\int_0^l \left[\frac{\partial u(x,t)}{\partial x}\right]^2 \mathrm{d}x - P\,u(vt,t) . \tag{15}$$

In generalized coordinates the displacements of a string are defined by an infinite series

$$u(x,t) = \sum_{i=1}^{\infty} U_i(x)\xi_i(t) . \tag{16}$$

$$\frac{\partial u(x,t)}{\partial t} = \sum_{i=1}^{\infty} U_i(x)\dot{\xi}_i(t) , \tag{17}$$

We can write the solution under the follower mass point

$$u(vt,t) = \sum_{i=1}^{\infty} U_i(vt)\xi_i(t) . \tag{18}$$

Now we compute the velocity of displacements by using of the chain rule differentiation with respect to t

$$\frac{\partial u(vt,t)}{\partial t} = v\sum_{i=1}^{\infty} U_i'(x)\xi_i(t)\bigg|_{x=vt} + \sum_{i=1}^{\infty} U_i(x)\dot{\xi}_i(t)\bigg|_{x=vt} , \tag{19}$$

and (19) is a function of both ξ_i and $\dot{\xi}_i$

$$\frac{\partial u(vt,t)}{\partial t} = f\left(\xi_i, \dot{\xi}_i\right) . \tag{20}$$

With respect to (16) the kinetic energy (10) can be computed

$$E_k = \frac{1}{2}\rho A \sum_{i,j=1}^{\infty} \dot{\xi}_i(t)\dot{\xi}_j(t) \int_0^l U_i(x)U_j(x)\mathrm{d}x + \frac{1}{2}m\left[\frac{\partial u(vt,t)}{\partial t}\right]^2 . \tag{21}$$

We assume the function $U_i(x)$, which assumes our boundary conditions (2) in a natural way

$$U_i(x) = \sin\frac{i\pi x}{l} . \tag{22}$$

This function is orthogonal with following properties

$$\int_0^l U_i(x)U_j(x)\mathrm{d}x = \begin{cases} \frac{1}{2}l & \text{if } i=j, \\ 0 & \text{if } i \neq j \end{cases} . \tag{23}$$

Finally the kinetic energy has a form

$$E_k = \frac{1}{4}\rho Al \sum_{i=1}^{\infty} \dot{\xi}_i^2(t) + \frac{1}{2}m\left[\frac{\partial u(vt,t)}{\partial t}\right]^2 . \tag{24}$$

We follow the same way in the case of the potential energy

$$E_p = \frac{1}{4}Nl\sum_{i=1}^{\infty}\frac{i^2\pi^2}{l^2}\xi_i^2(t) - P\sum_{i=1}^{\infty}\xi_i(t)\sin\frac{i\pi vt}{l} . \tag{25}$$

The general form of the Lagrange equation of the second kind is written below

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E_k}{\partial \dot{\xi}_i}\right) - \frac{\partial E_k}{\partial \xi_i} + \frac{\partial E_p}{\partial \xi_i} = 0 . \tag{26}$$

We compute required derivatives of the kinetic energy

$$\frac{\partial E_{km}}{\partial \xi_i} = m\frac{\partial u(vt,t)}{\partial t}\frac{\mathrm{d}}{\mathrm{d}\xi_i}\left(\frac{\partial u(vt,t)}{\partial t}\right) \tag{27}$$

$$\frac{\partial E_{km}}{\partial \dot{\xi}_i} = m\frac{\partial u(vt,t)}{\partial t}\frac{\mathrm{d}}{\mathrm{d}\dot{\xi}_i}\left(\frac{\partial u(vt,t)}{\partial t}\right) . \tag{28}$$

$$\frac{\partial E_k}{\partial \xi_i} = \frac{\partial E_{km}}{\partial \xi_i} = m\left[v^2\sum_{i,j=1}^{\infty}\frac{ij\pi^2}{l^2}\cos\frac{i\pi vt}{l}\cos\frac{j\pi vt}{l}\xi_j(t) + v\sum_{i,j=1}^{\infty}\frac{i\pi}{l}\cos\frac{i\pi vt}{l}\sin\frac{j\pi vt}{l}\dot{\xi}_j(t)\right] \tag{29}$$

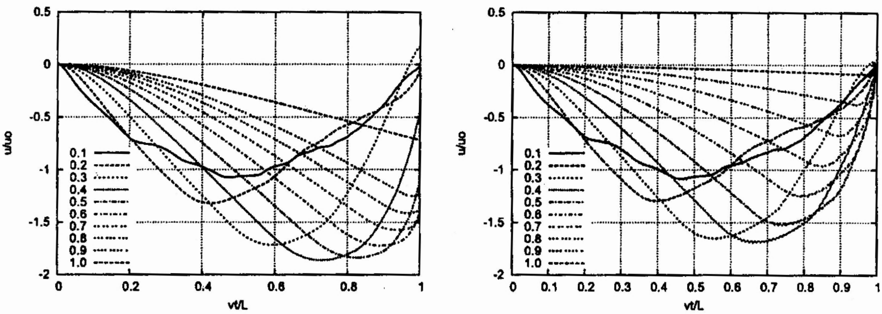

Figure 2: Semi-analytical solution (left) and displacements of the string under the oscillator (right).

$$\frac{\partial E_k}{\partial \dot{\xi}_i} = \frac{1}{2}\rho A l \sum_{i=1}^{\infty} \dot{\xi}_i(t) + \frac{\partial E_{km}}{\partial \dot{\xi}_i} = \frac{1}{2}\rho A l \sum_{i=1}^{\infty} \dot{\xi}_i(t) + m \left[v \sum_{i,j=1}^{\infty} \frac{j\pi}{l} \sin\frac{i\pi vt}{l} \cos\frac{j\pi vt}{l} \xi_j(t) + \right.$$
$$\left. + \sum_{i,j=1}^{\infty} \sin\frac{i\pi vt}{l} \sin\frac{j\pi vt}{l} \dot{\xi}_j(t) \right] \tag{30}$$

and potential energy

$$\frac{\partial E_p}{\partial \xi_i} = \frac{1}{2} N l \sum_{i=1}^{\infty} \frac{i^2\pi^2}{l^2} \xi_i(t) - P \sum_{i=1}^{\infty} \sin\frac{i\pi vt}{l} . \tag{31}$$

From rearranged Eqn. (26) we obtain the motion equation in generalized coordinates

$$\ddot{\xi}_i(t) + \frac{2m}{\rho A l} \sum_{j=1}^{\infty} \ddot{\xi}_j(t) \sin\frac{i\pi vt}{l} \sin\frac{j\pi vt}{l} + \frac{4m}{\rho A l} \sum_{j=1}^{\infty} \frac{j\pi v}{l} \dot{\xi}_j(t) \sin\frac{i\pi vt}{l} \cos\frac{j\pi vt}{l} +$$
$$+ \frac{N}{\rho A} \frac{i^2\pi^2}{l^2} \xi_i(t) - \frac{2m}{\rho A l} \sum_{j=1}^{\infty} \frac{j^2\pi^2 v^2}{l^2} \xi_j(t) \sin\frac{i\pi vt}{l} \sin\frac{j\pi vt}{l} = \frac{2P}{\rho A l} \sin\frac{i\pi vt}{l} . \tag{32}$$

We notice that the Eqn. (32) has a form similar to the Eqn. (8). Both relations differ only with the right-hand-side terms, precisely with the multiplier $2/l$. Finally displacements $u(x,t)$ are identical in both cases, since displacements (16) and (5) differ with this multiplier only.

4. Results

Results of the semi-analytical solution are depicted in Fig. 2. The diagram can be compared with displacements of the string under moving oscillator. The analysis of the spring-mass system motion was performed for a relatively rigid spring. However, for significantly high spring rigidity the convergence of the solution was poor of completely lost.

The convergence near the end point is depicted in Fig. 3. The mass trajectory is plotted for increasing number of term at the speed $v = 0.5c$. We notice that the function

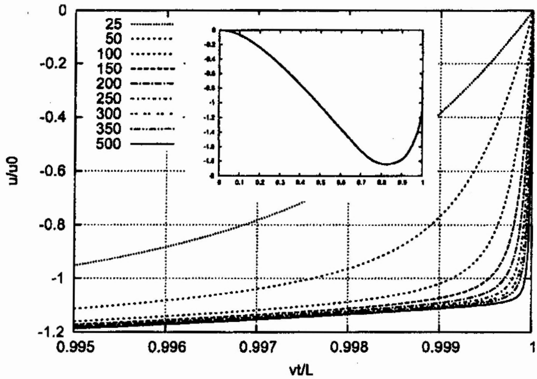

Figure 3: The convergence of the mass trajectory travelling with $v = 0.5c$ near the end point, for various number of term (25, 50,..., 500).

tends slowly to the jump at $x = l$. All characteristic lines are smooth. The convergence rate is low and especially near $x = l$ the taken number of term must be at least then 50. At high velocity range (in our case $v > 0.8$) sufficiently low time step of the integration of the resulting equation (8) must be applied (even 10^{-5}) to avoid small oscillations of the solution in the last stage.

More detailed presentation of the string motion is given in Fig. 4. We can notice the sharp edge of the wave and reflection from both supports. Moreover, the wave reflection from the travelling mass is clearly visible. Both the mass trajectory and waves are depicted. We must emphasize here that the results can not be simply compared with results of commercial codes. Moving load problems on elastic structures are not implemented in the most popular systems.

5. Conclusions

In the paper we prove the identity of the analysis performed with the Fourier transform analysis and the analysis with the Lagrange equation of the second kind. The point mass moving with the constant or variable speed involves difficulties in analysis. Especially, the first solution carried on by the Fourier method the transform of the Dirac delta function with its derivatives may require respective proofs. In the case of the Lagrange approach the analysis is clear. Moreover, it was shown that an analysis for large displacements can be performed if the second term in the expansion is taken into account. In further works more accurate analysis, ie. with two terms in the expansion of the potential energy of a string will be undertaken.

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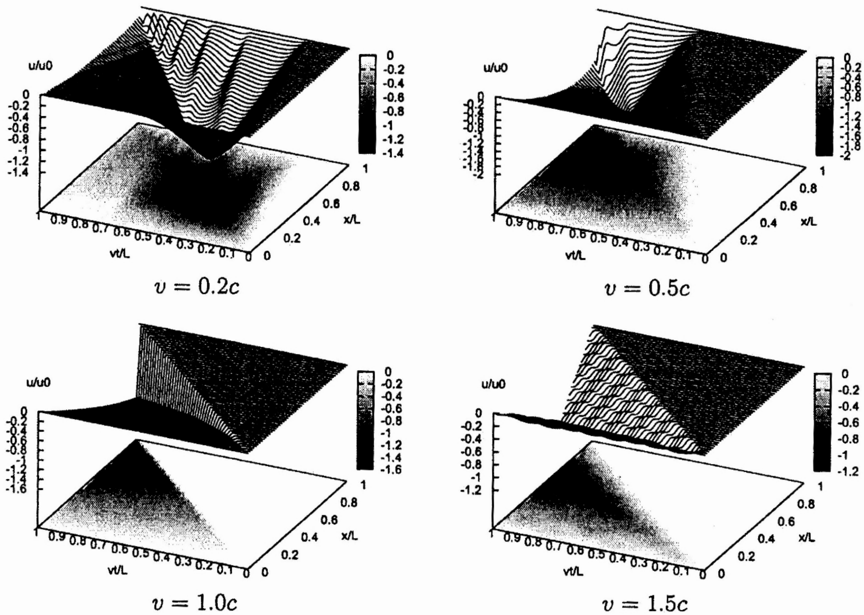

Figure 4: Simulation of the string motion under the mass moving at v=0.2c, 0.5c, 1.0c and 1.5c.

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OGÓLNA METODA ROZWIĄZYWANIA ZADAŃ Z RUCHOMĄ MASĄ

Streszczenie

W pracy przedstawiono dwa równoważne rozwiązania problemu drgań struny pod obciążeniem inercyjnym. Rozważania można stosować do również do innych konstrukcji, jak np. belki, ramy lub płyty. Pokazano, że w wyniku analizy z wykorzystaniem transformacji Fouriera uzyskuje się identyczny wynik jak w przypadku wykorzystania równania Lagrange'a drugiego rodzaju. O ile w przypadku pierwszej metody dyskusję mogą wywoływać adpowiednie przekształcenia delty Diraca, obrazującej skupioną masę, oraz jej pochodnych, to w przypadku podejścia Lagrange'a takich wątpliwości nie ma. Co więcej, pozostawiając drugi wyraz rozwinięcia w szereg enetgii potencjalnej struny możemy rozpatrywać duże przemieszczenia. Niestety, pełne uwzględnienie drugiego składnika mocno komplikuje rozważania