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Inertial load moving on a string – discontinuous solution

Bartłomiej Dyniewicz and Czesław I. Bajer

Institute of Fundamental Technological Research,
Polish Academy of Sciences,
Świętokrzyska 21, 00-049 Warsaw, Poland

Abstract

The paper deals with the analytical solution of the string motion subjected to a moving mass. Solutions published up to date do not exhibit sufficient simplicity and can not be applied to a whole range of the mass speed, also in over-critical range. We propose the analytical solution that allows us to reduce the problem to the second order matrix differential equation. In the last stage it is treated numerically. The solution is characteristic of all features of the critical, sub-critical and over-critical motion. Results are compared with the finite element method analysis performed for a rigid oscillator moving over the string.

1 Introduction

Nowadays inertial loads moving on strings and beams with the sub or super critical speed are of the special interest. Theoretical solutions are applied to many practical problems: train-track interaction, vehicle-bridge interaction, pantograph collectors in railways, magnetic rails, guideways in robotic solutions, etc. The problem was widely treated in literature. Attempts of the problem solution started in the middle of the 19th century. However, till now we do not have the complete and closed analytical solution. The term describing the concentrated mass motion is the reason of difficulties. Differential equations of variable coefficients, which, except few cases, do not have analytical solutions, are serious limits in closed solutions. These types of equations are finally solved by numerical means.

In literature numerous reviews concerning moving loads problem exist (for example Panowko [1], Jakuszew [2], Dmitrijew [3]). In most cases the moving massless constant force was considered as a moving load. This type of the problems results in closed solutions. Unfortunately, the problem of inertial loads is still open. Saller in [4] considered the moving mass for the first time. He proved, in spite of essential simplifications, significant influence of the moving mass in beam dynamics. In thirties two contributions appeared, important for the society working if the field of moving loads. The first one was written by Inglis [5]. Far simplifications were applied and the solution was expressed by only the first term of the trigonometric series. The time function fulfilled the second order differential equation of variable coefficients. This equation was derived considering the acceleration under the moving mass, expressed by the so-colled Renaudot formula. In fact it is the derivative with the constant velocity, computed with the chain rule. The final solution of the differential equation of variable coefficients was proposed by Inglis as an infinite series. It results in an approached solution.

Schallenkamp [6] proposed another approach to the problem of moving mass. However, his attempt allows to describe the motion only under the moving mass. We recall that Inglis solution is represented by the function in a whole range of spatial variable x. The method of separation of variables by the expansion of the unknown function into a sine Fourier series was applied. Boundary conditions in the beam were taken into account in a natural way. The ordinary differential equation, which describes the motion under the moving mass was expressed in generalized coordinates by using the second Lagrange equations. The generalized force was derived from the virtual work principle. Schallenkamp consideration is relatively complex and slowly converged since the final solution is expressed in terms of the triple infinite series.

Inglis and Schallenkamp works can be considered as the base for the analysis of the problem of moving mass in successive works. Bolotin [7, 8], Morgajewskij [9] and others. The excellent and important monograph in this field was written by Szcześniak [10]. One can find there hundreds of references concerning moving load on beams and strings. Most solutions, for example [11] and [12] based on semi-analytical research of the beam under moving mass. The major problem of existing solutions, besides their high complex is relatively small computational efficiency. In [11] the authors consider a simply supported beam modeled by Bernoulli-Euler theory. The equation of motion is written in the integral-differential form with a Green function terms. In order to compute this equation a dual numerical scheme has been used. A backward difference technique was applied to treat the time parameter and numerical integration was used for the spatial parameter. This way of the solution though applied to higher velocities, still requires complex mathematical operations. Each solution enables to determine displacements under the moving load only and does not give solutions in a wide range of parameters x and t. Especially we can not calculate displacements in each point of a string or a beam. Only one closed analytical solution can be found in literature. Frýba [13] proposed the solution for the inertial load, however, in the case of the massless string only. Green function terms were introduced to the formulation.

In the paper we propose the analytical solution of the string subjected to a moving inertial load. Final solution is proposed as a matrix differential equation of the second order. Numerical integration results in the solution in a full range of the velocity: under critical and over critical. Exactly the same approach can be applied to a beam with the moving mass. Numerical examples of the analytical approach are compared with numerical solutions obtained with the finite element method. The string is subjected to a moving oscillator. In the case of the rigid spring we approach to the analytical solution. However, in the case of higher speed ($v > 0.2c$) the accuracy of the FEM solution is poor. Analytical solution or at least analytical formulation of the final governing equation of the motion of the string or beam is required for mathematical investigation and optimization of the control of vibration for power receivers in railway engineering.

2 Analytical solution

Let us consider a string of the length L, cross-sectional area A, mass density ρ, tensile force N, subjected to a mass m accompanied by a force P (Fig. 1), moving with a constant speed v. The motion equation of the string under moving inertial load with a constant speed v has a form

$$-N\,\frac{\partial^2 u(x,t)}{\partial x^2} + \rho A\,\frac{\partial^2 u(x,t)}{\partial t^2} = \delta(x-vt)\,P - \delta(x-vt)\,m\,\frac{\partial^2 u(vt,t)}{\partial t^2}\,. \qquad (1)$$

We impose boundary conditions

$$u(0,t) = 0 \qquad u(l,t) = 0 \qquad (2)$$

and initial conditions

$$u(x,0) = 0 \qquad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0\,. \quad (3)$$

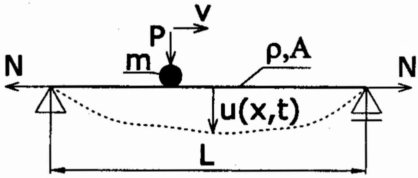


Figure 1: Moving inertial load.

In order to reduce partial differential equation to ordinary differential equation, we apply Fourier sine integral transformation in a finite range (i.e. finite length of the string) (4), (5)

$$V(j,t) = \int_0^l u(x,t)\,\sin\frac{j\pi x}{l}\,\mathrm{d}x \qquad (4)$$

$$u(x,t) = \frac{2}{l}\sum_{j=1}^{\infty} V(j,t)\,\sin\frac{j\pi x}{l}\,. \qquad (5)$$

We can present each of the function as a infinite sum of sine functions (5) with respective coefficients (4). Then the expansion of the moving mass acceleration in a series has a form

$$\frac{\partial^2 u(vt,t)}{\partial t^2} = \frac{2}{l}\sum_{k=1}^{\infty}\left[\ddot V(k,t)\,\sin\frac{k\pi vt}{l} + \frac{2k\pi v}{l}\,\dot V(k,t)\,\cos\frac{k\pi vt}{l} - \frac{k^2\pi^2 v^2}{l^2}\,V(k,t)\,\sin\frac{k\pi vt}{l}\right]. \qquad (6)$$

The integral transformation (4) of the equation (1) with consideration of (6) can be performed

$$N\,\frac{j^2\pi^2}{l^2}\,V(j,t) + \rho A\,\ddot V(j,t) = P\sin\frac{j\pi ct}{l} - m\,\frac{\partial^2 u(vt,t)}{\partial t^2}\int_0^l \delta(x-vt)\,\sin\frac{j\pi x}{l}\,\mathrm{d}x\,. \quad (7)$$

The integral with delta Dirac function in the above equation is as follows

$$\int_0^l \delta(x-vt)\,\sin\frac{j\pi x}{l}\,\mathrm{d}x = \sin\frac{j\pi vt}{l}\,, \qquad (8)$$

Let us consider now (6) and (8):

$$\begin{aligned} N\,\frac{j^2\pi^2}{l^2}\,V(j,t) + \rho A\,\ddot V(j,t) &= P\sin\frac{j\pi vt}{l} - \frac{2m}{l}\sum_{k=1}^{\infty}\ddot V(k,t)\,\sin\frac{k\pi vt}{l}\,\sin\frac{j\pi vt}{l} - \\ &- \frac{2m}{l}\sum_{k=1}^{\infty}\frac{2k\pi v}{l}\,\dot V(k,t)\,\cos\frac{k\pi vt}{l}\,\sin\frac{j\pi vt}{l} + \qquad (9)\\ &+ \frac{2m}{l}\sum_{k=1}^{\infty}\frac{k^2\pi^2 v^2}{l^2}\,V(k,t)\,\sin\frac{k\pi vt}{l}\,\sin\frac{j\pi vt}{l}\,. \end{aligned}$$

Finally, the motion equation after Fourier transformation can be written

$$\begin{aligned} \ddot{V}(j,t) \;+\; \alpha\sum_{k=1}^{\infty}\ddot{V}(k,t)\,\sin\omega_k t\,\sin\omega_j t \;+\; 2\alpha\sum_{k=1}^{\infty}\omega_k\,\dot{V}(k,t)\,\cos\omega_k t\,\sin\omega_j t \;+ \\ +\;\Omega^2\,V(j,t) \;-\; \alpha\sum_{k=1}^{\infty}\omega_k^2\,V(k,t)\,\sin\omega_k t\,\sin\omega_j t \;=\; \frac{P}{\rho A}\,\sin\omega_j t\;, \end{aligned} \tag{10}$$

where

$$\omega_k \;=\; \frac{k\pi v}{l}\,, \qquad \omega_j \;=\; \frac{j\pi v}{l}\,, \qquad \Omega^2 \;=\; \frac{N}{\rho A}\,\frac{j^2\pi^2}{l^2}\,, \qquad \alpha \;=\; \frac{2m}{\rho A l}\,. \tag{11}$$

The analytical solution for this problem does not exist. We must solve this equation numerically. The equation (10) is written in a matrix form, where matrix **M**, **C** and **K** are square matrices $(j,k = 1...n)$.

$$\mathbf{M}\begin{bmatrix}\ddot{V}(1,t)\\ \ddot{V}(2,t)\\ \vdots\\ \ddot{V}(n,t)\end{bmatrix} + \mathbf{C}\begin{bmatrix}\dot{V}(1,t)\\ \dot{V}(2,t)\\ \vdots\\ \dot{V}(n,t)\end{bmatrix} + \mathbf{K}\begin{bmatrix}V(1,t)\\ V(2,t)\\ \vdots\\ V(n,t)\end{bmatrix} = \mathbf{P} \tag{12}$$

or

$$\mathbf{M}\ddot{\mathbf{V}} + \mathbf{C}\dot{\mathbf{V}} + \mathbf{K}\mathbf{V} = \mathbf{P}\;, \tag{13}$$

where

$$\mathbf{M} = \begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix} + \alpha\begin{bmatrix}\sin\frac{1\pi vt}{l}\sin\frac{1\pi vt}{l} & \sin\frac{1\pi vt}{l}\sin\frac{2\pi vt}{l} & \cdots & \sin\frac{1\pi vt}{l}\sin\frac{n\pi vt}{l}\\ \sin\frac{2\pi vt}{l}\sin\frac{1\pi vt}{l} & \sin\frac{2\pi vt}{l}\sin\frac{2\pi vt}{l} & \cdots & \sin\frac{2\pi vt}{l}\sin\frac{n\pi vt}{l}\\ \vdots & \vdots & \ddots & \vdots\\ \sin\frac{n\pi vt}{l}\sin\frac{1\pi vt}{l} & \sin\frac{n\pi vt}{l}\sin\frac{2\pi vt}{l} & \cdots & \sin\frac{n\pi vt}{l}\sin\frac{n\pi vt}{l}\end{bmatrix}, \tag{14}$$

$$\mathbf{C} = 2\alpha\begin{bmatrix}\frac{1\pi v}{l}\sin\frac{1\pi vt}{l}\cos\frac{1\pi vt}{l} & \frac{2\pi v}{l}\sin\frac{1\pi vt}{l}\cos\frac{2\pi vt}{l} & \cdots & \frac{n\pi v}{l}\sin\frac{1\pi vt}{l}\cos\frac{n\pi vt}{l}\\ \frac{1\pi v}{l}\sin\frac{2\pi vt}{l}\cos\frac{1\pi vt}{l} & \frac{2\pi v}{l}\sin\frac{2\pi vt}{l}\cos\frac{2\pi vt}{l} & \cdots & \frac{n\pi v}{l}\sin\frac{2\pi vt}{l}\cos\frac{n\pi vt}{l}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1\pi v}{l}\sin\frac{n\pi vt}{l}\cos\frac{1\pi vt}{l} & \frac{2\pi v}{l}\sin\frac{n\pi vt}{l}\cos\frac{2\pi vt}{l} & \cdots & \frac{n\pi v}{l}\sin\frac{n\pi vt}{l}\cos\frac{n\pi vt}{l}\,,\end{bmatrix} \tag{15}$$

$$\begin{aligned} \mathbf{K} \;=\; & \begin{bmatrix}\frac{1^2\pi^2}{l^2}\frac{N}{\rho A} & 0 & \cdots & 0\\ 0 & \frac{2^2\pi^2}{l^2}\frac{N}{\rho A} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \frac{n^2\pi^2}{l^2}\frac{N}{\rho A}\end{bmatrix} - \\ & -\;\alpha\begin{bmatrix}\frac{1^2\pi^2v^2}{l^2}\sin\frac{1\pi vt}{l}\sin\frac{1\pi vt}{l} & \frac{2^2\pi^2v^2}{l^2}\sin\frac{1\pi vt}{l}\sin\frac{2\pi vt}{l} & \cdots & \frac{n^2\pi^2v^2}{l^2}\sin\frac{1\pi vt}{l}\sin\frac{n\pi vt}{l}\\ \frac{1^2\pi^2v^2}{l^2}\sin\frac{2\pi vt}{l}\sin\frac{1\pi vt}{l} & \frac{2^2\pi^2v^2}{l^2}\sin\frac{2\pi vt}{l}\sin\frac{2\pi vt}{l} & \cdots & \frac{n^2\pi^2v^2}{l^2}\sin\frac{2\pi vt}{l}\sin\frac{n\pi vt}{l}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1^2\pi^2v^2}{l^2}\sin\frac{n\pi vt}{l}\sin\frac{1\pi vt}{l} & \frac{2^2\pi^2v^2}{l^2}\sin\frac{n\pi vt}{l}\sin\frac{2\pi vt}{l} & \cdots & \frac{n^2\pi^2v^2}{l^2}\sin\frac{n\pi vt}{l}\sin\frac{n\pi vt}{l}\end{bmatrix}, \end{aligned} \tag{16}$$

$$\mathbf{P} = \frac{P}{\rho A}\begin{bmatrix}\sin\frac{1\pi vt}{l}\\ \sin\frac{2\pi vt}{l}\\ \vdots\\ \sin\frac{n\pi vt}{l}\end{bmatrix}. \tag{17}$$

When coefficients $V(j,t)$ are computed, displacements of the string (5) can be appointed as a solution of (1). It is the solution in a full range. We can calculate displacement in each point of string and and for all values of v.

3 Results

First we present the convergence rate of the series which constitutes the solution. The expected property of fast convergence of the trigonometric series was observed (Fig. 2). We denote the wave speed in the unloaded string as c ($c^2 = N/\rho A$). Furher diagrams exhibit vertical deflection of the string u related to the deflection in quasi-static mass motion in the middle of the span u_0. We can notice that the first term is already close to the exact solution. Three or five terms are sufficient for the accurate result. We must emphasize here that higher speed of the mass, for example equal to $0.9c$ or c requires even hundred term and short time step for time integration of the differential equation, since the solution exhibits small jumps. Final plot for different velocities v is given in Fig. 3.

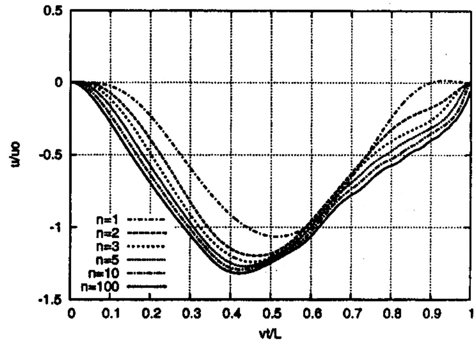


Figure 2: Trigonometric series convergence for $v = 0.2c$.

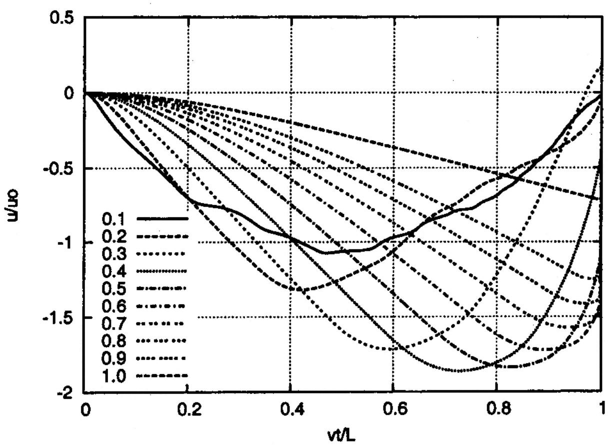


Figure 3: Displacements computed analytically.

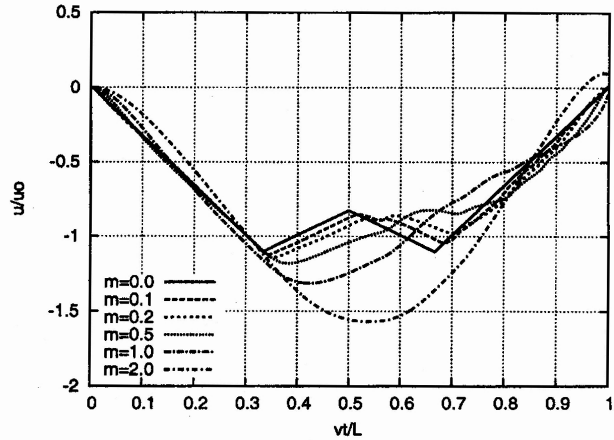


Figure 4: Displacements under the mass for different mass values at the speed $v = 0.2c$.

Let us look at the diagrams of displacements of the string in the point under the mass. Diagram for various mass related to the string mass, for the speed $v = 0.2c$ is depicted in Fig. 4. More detailed presentation of the string motion is given in Fig. 5. We can notice the sharp edge of the wave and reflection from both supports. Moreover, the wave reflection from the travelling mass is well visible, especially for the case $v = 1.2c$. Both the mass trajectory and waves are depicted. Detailed solution for the last stage of the mass motion is depicted in Fig. 6. Supersonic motion of the mass results in zero displacement. In the diagram obtained numerically this value oscillates with low amplitude. The amplitude decreases with the increase of the number of terms in a sum (Fig. 7).

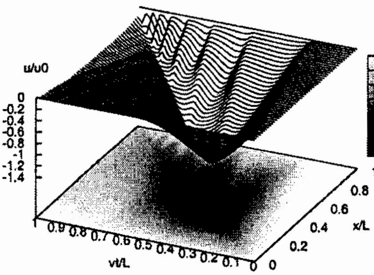
Analytical results are compared with numerical solutions obtained by the finite element method. The string was discretized by a set of 100 finite elements. It was subjected with an oscillator moving over the span. Two separate systems were considered: a string subjected to a contact force between the oscillator spring and the string, and the oscillator itself, subjected to a force P applied to a mass and displacements determined from the string motion, applied to a spring. The oscillator spring stiffness was assumed to be high enough, to simulate a rigid contact of the mass with the string. Results are depicted in Fig. 8.

Let us compare analytical solution for a massless string subjected to a moving mass given by Frýba (Fig. 9) with the analytical one (13).

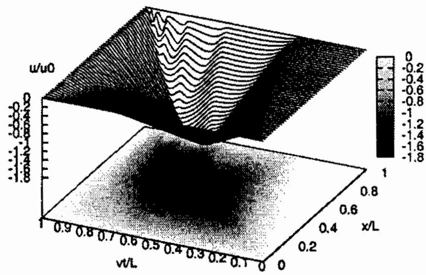
4 Conclusions

In the paper we present a global analytical solution of the vibration problem for the string subjected to a moving mass. The solution is relatively simple and is valid for the whole range of the speed v (sub-critical, critical and over-critical). The problem is reduced to the system of the differential equations of the second order, which finally must be solved numerically. High convergence rate of the solution allows us to apply only few terms of the Fourier series.

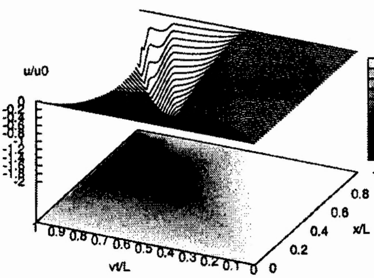
The analysis of results exhibits a jump of the mass in the neighborhood of the end



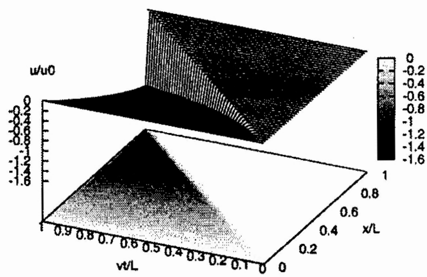
$v = 0.2c$



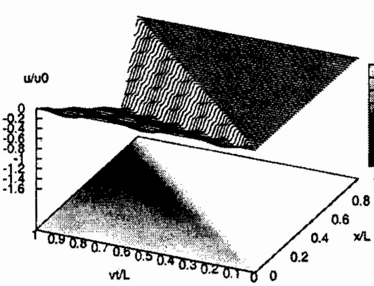
$v = 0.3c$



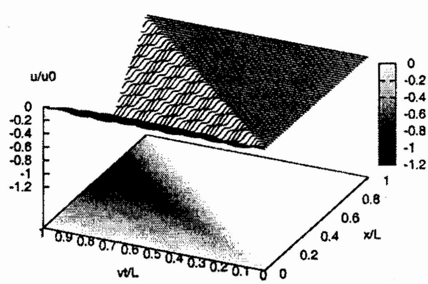
$v = 0.5c$



$v = 1.0c$

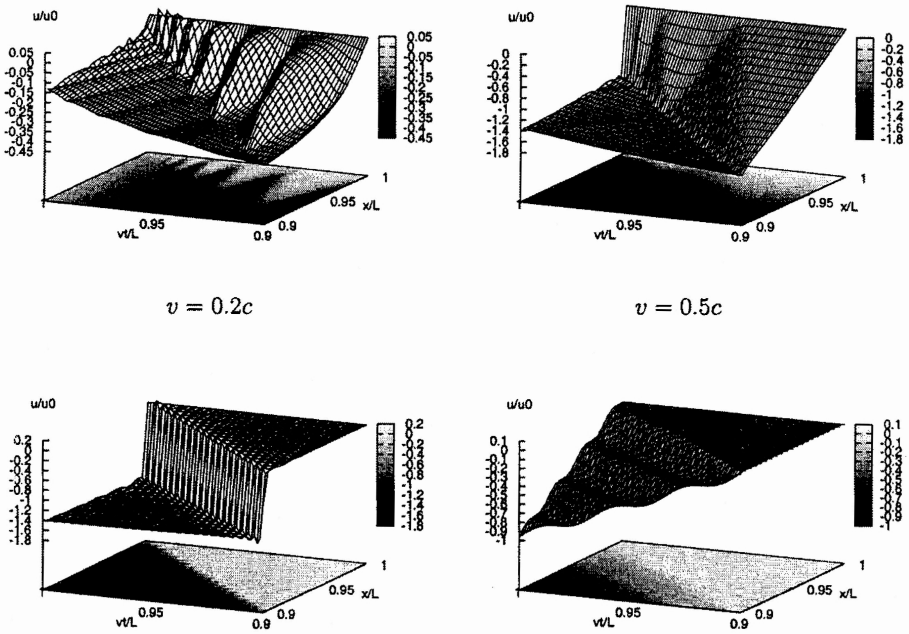


$v = 1.2c$



$v = 1.5c$

Figure 5: Simulation of the string motion under the mass moving at v=0.2c, 0.3c, 0.5c, 1.0c 1.2c and 1.5c.



$v = 0.2c$ $v = 0.5c$

$v = 1.0c$ $v = 1.2c$

Figure 6: Last stage of the mass motion at v=0.2c, 0.5c, 1.0c and 1.2c.

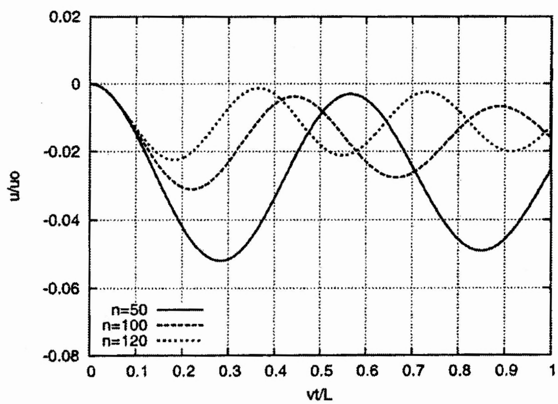


Figure 7: Convergence of displacements under the mass at the speed v=1.05c.

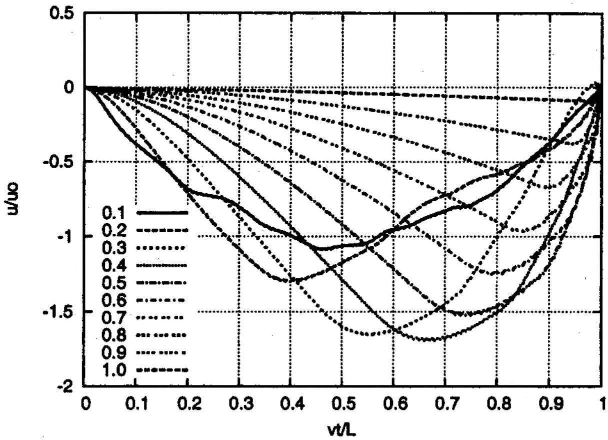


Figure 8: Finite element solution – displacements of the string under the oscillator.

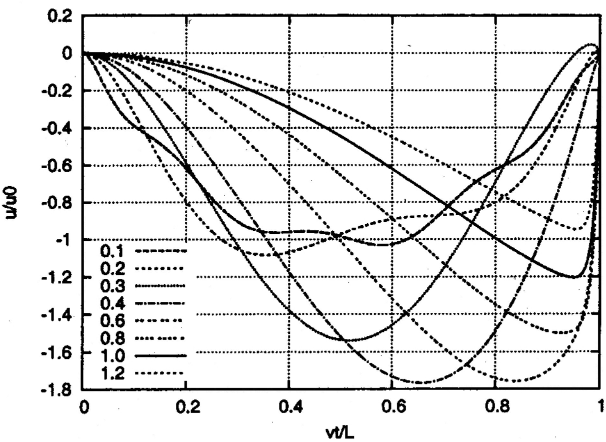


Figure 9: Massless string subjected to a moving mass.

support. The force acting on the mass is, however, limited to the tensile force N. The mass can not be accelerated to the appropriate vertical velocity to arrive directly at the support in a smooth way. Discontinuity of the solution at $x = L$ exists in the case of $v > 0$ (for each non-zero moving speed, *ie.* $0 < v < c$, $v = c$ and $v > c$). It can be proved analytically in the case of massless string.

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