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New feature of the solution of a Timoshenko beam carrying the moving mass particle

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The paper deals with the problem of vibrations of a Timoshenko beam loaded by a travelling mass particle. Such problems occur in a vehicle/track interaction or a power collector in railways. Increasing speed involves wave phenomena with significant increase of amplitudes. The travelling speed approaches critical values. The moving point mass attached to a structure in some cases can exceed the mass of the structure, i.e. a string or a beam, locally engaged in vibrations. In the literature, the travelling inertial load is often replaced by massless forces or oscillators. Classical solution of the motion equation may involve discussion concerning the contribution of the Dirac delta term, multiplied by the acceleration of the beam in a moving point in the differential equation. Although the solution scheme is classical and successfully applied to numerous problems, in the paper the Lagrange equation of the second kind applied to the problem allows us to obtain the final solution with new features, not reported in the literature. In the case of a string or the Timoshenko beam, the inertial particle trajectory exhibits discontinuity and this phenomenon can be demonstrated or proved mathematically in a particular case. In practice, large jumps of the travelling inertial load is observed.

Key words: moving mass, travelling inertial load, Timoshenko beam, Lagrange equation.

1. Introduction

In engineering practice, problems with travelling masses are of special interest. The influence of the mass attached locally to the structure can not be neglected (Fig. 1). We can only mention that the mass of a single train wheel is 500 kg and a wheelset has a mass equal to 1500 kg. A similar case occurs in the problem with power collectors in railways. The speed of a rail vehicle in certain circumstances can reach the critical speed. In such a case, the wave phenomena significantly differ from the responses of systems subjected to massless loads.

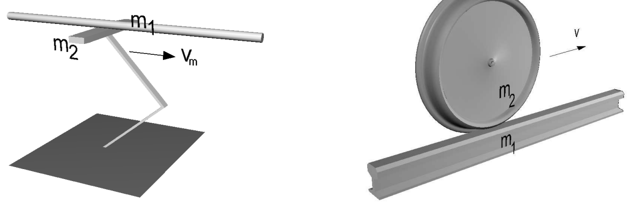

FIG. 1. Examples of problems with a mass m_2 travelling over a string or a beam m_1.

There are two types of the problems with a travelling load: a moving massless force and a moving inertial force (Fig. 2). In the second case, the moving force is accompanied by the mass placed directly on a structure: a string, a beam or a plate. The analysis of the moving massless force is relatively simple and has been treated in numerous papers [1, 2]. We include in this group all the papers devoted to the travelling oscillator, i.e. a mass particle joined to the base with a spring [3]. Although the authors call this type of load an inertial one, we consider it as a massless force generated only by the particle's inertia. The inertial force moving over the structure is widely reported in the literature [4–12]. These are mainly semi-analytical solutions or, as in the case of the mass particle moving along a massless string, we know the full analytical solution [13].

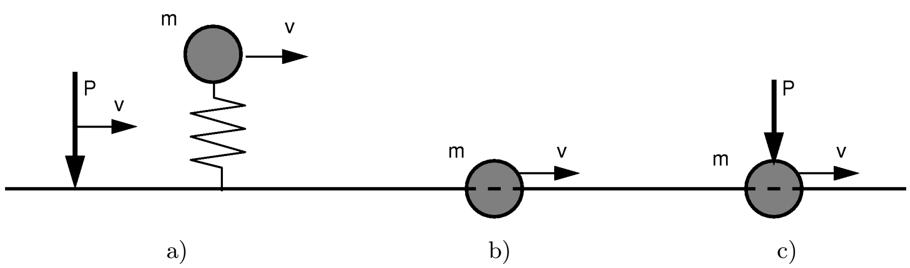

FIG. 2. a) Massless load, b) inertial load, c) inertial load with a massless force.

In the paper we present the semi-analytical solution of the problem with a mass travelling on the simply supported Timoshenko beam, by using of the Lagrange equation of the second kind. The alternative type of the solution gives the Fourier transformation in a finite domain [7], or integro-differential solution [8]. Unfortunately, the Bernoulli-Euler beam considered in the paper does not exhibit the discontinuity of the mass trajectory. In [9] the authors formulated and solved an integro-differential equation. They found a continuous function that satisfied the given equation. However, they did not consider the Timoshenko beam. The classical method of the Fourier transformation with respective comments is only mentioned below. For details, the reader is addressed to reference papers.

Two coupled motion equations of the Timoshenko beam under a moving mass particle are given by the equation

$$
\begin{cases}
\rho A \dfrac{\partial^2 w(x,t)}{\partial t^2} - \dfrac{GA}{k}\left(\dfrac{\partial^2 w(x,t)}{\partial x^2} - \dfrac{\partial \psi(x,t)}{\partial x}\right) = q(x,t), \\[2ex]
\rho I \dfrac{\partial^2 \psi(x,t)}{\partial t^2} - EI \dfrac{\partial^2 \psi(x,t)}{\partial x^2} - \dfrac{GA}{k}\left(\dfrac{\partial w(x,t)}{\partial x} - \psi(x,t)\right) = 0,
\end{cases}
\tag{1.1}
$$

where

$$
q(x,t) = \delta(x-vt)m\left(g - \frac{d^2 w(vt,t)}{dt^2}\right). \tag{1.2}
$$

The acceleration of the moving mass particle at a constant speed v is called the Renaudot formulation

$$
\frac{d^2 w(vt,t)}{dt^2} = \left.\frac{\partial^2 w(x,t)}{\partial t^2}\right|_{x=vt} + 2v \left.\frac{\partial^2 w(x,t)}{\partial x \partial t}\right|_{x=vt} + v^2 \left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=vt}. \tag{1.3}
$$

The solution given by the Fourier transformation starts from the partial differential equation (1.1), and by the direct mathematical transformation is reduced to the second-order matrix ordinary differential equation. The last stage is performed numerically and for this reason we call this solution semi-analytical. This solution has one disadvantage. The formulation (1.2) contains the term $\delta(x-vt)m\, d^2 w(vt,t)/dt^2$ which defines the inertial force of the mass particle in space x and time t. The Dirac delta contributes discontinuities to the formulation. Although the solution can be defined, we can not prove that it verifies the equation of the problem. That is why we intend to apply another method which avoids the discontinuous formulation. In such a case we could eliminate a weak point of the investigation.

The problem of the moving mass is important since in the case of a string and a simply supported Timoshenko beam, the results exhibit discontinuity of the mass trajectory at the end support. This phenomenon in the case of a string was presented and discussed for the first time in our former paper [14] and in the case of a massless string it was mathematically proved. It can be also noticed in engineering practice. In railway traction systems the cables are broken just before the end support. Also the road plates are destroyed at the end parts.

Travelling loads are generally unlikely to be solved by commercial codes. Most of the existing systems for dynamic simulations usually do not allow us to solve even simple problems comprising travelling massless point forces, travelling distributed non-inertial loads and even the travelling and elastically joined moving substructures. Inertial moving loads are not completely implemented in computer systems. The intuitive approach to the discrete analysis with the ad hoc lumping of forces and masses to neighbouring nodes always fails. Sometimes, especially in the case of beams, numerical solutions are limited, but significantly inaccurate. We emphasise here that the travelling mass problem is not trivial, even if at the first sight it seems to be such a problem.

2. Mathematical model of a travelling mass particle

The Dirac delta term which describes the point distribution of the quantity which is analysed in partial differential equations (1.1), results in solutions which are not solutions in the classical sense. We must here extend the meaning of the solution. We assume that each limit of the almost uniformly convergent sequence of the classical solutions will be considered as a general solution (distributive solution). Distributions are defined then as limits of sequences of continues functions. It is a base of the sequential theory of distributions [15]. Another known theory is called functional [16]. For each Schwartz distribution (functional), exactly one sequential distribution exists in the Mikusiński–Sikorski sense, and *vice versa*. Bijection exists [17]. Distributions are then considered as generalised functions. Distributions are introduced to give mathematical creations, for example the Dirac delta $\delta(x)$. A correct sense. The important feature of distributions is the differentiability, which not always occurs in the case of functions.

Continuous functions in a constant interval $A < x < B$ $(-\infty \leq A < B \leq \infty)$ are the starting point of the sequential theory of distributions. If the sequence $f_n(x)$ of continuous functions is almost uniformly convergent to a function $f(x)$, it is also distributively convergent to $f(x)$ [15]. Each convergent sequence of distributions can be differentiated term by term, analogously to a series. Finally, each sequence uniformly convergent is almost uniformly convergent. It enables differentiation of an arbitrary function in a distributive sense, change of the order of differentiation and computation of limits without restrictions. In a classical analysis, such a statement is generally false or requires supplementary assumptions. Thus the series uniformly convergent is also distributively convergent.

Energetic description of the issue is removed as a weak point of analysis. The kinetic energy of a moving mass particle m travelling with a constant speed v is described by the equation

$$T_m = \frac{1}{2}m\left(\frac{dw(vt,t)}{dt}\right)^2 + \frac{1}{2}mv^2. \tag{2.1}$$

Effect of the moving force of gravity mg can be written as the potential energy:

$$U_m = mg\, w(vt,t). \tag{2.2}$$

A moving mass is always in a pure rigid contact with the beam. The displacement of the point of a beam, being in contact with the mass particle, is described by the same relation as a travelling point mass motion.

3. Analytical formulation

Let us consider a simply supported Timoshenko beam with the constant cross-sectional area A, mass density ρ and moment of area I. The examined beam has a finite length l. The kinetic and potential energy of the beam take a form

$$T = \frac{1}{2}\rho A \int_0^l \left(\frac{\partial w(x,t)}{\partial t}\right)^2 dx + \frac{1}{2}\rho I \int_0^l \left(\frac{\partial \psi(x,t)}{\partial t}\right)^2 dx, \tag{3.1}$$

$$U = \frac{1}{2}EI \int_0^l \left(\frac{\partial \psi(x,t)}{\partial x}\right)^2 dx + \frac{1}{2}\frac{GA}{k} \int_0^l \left(\frac{\partial w(x,t)}{\partial x} - \psi(x,t)\right)^2 dx. \tag{3.2}$$

Here E is the Young's modulus, G is the shear modulus and k is the shear coefficient, which depends on the shape of the beam cross-section.

We impose the boundary conditions

$$\begin{aligned} &w(x,t)\big|_{x=0} = 0, && w(x,t)\big|_{x=l} = 0, \\ &\left.\frac{\partial \psi(x,t)}{\partial x}\right|_{x=0} = 0, && \left.\frac{\partial \psi(x,t)}{\partial x}\right|_{x=l} = 0, \end{aligned} \tag{3.3}$$

and initial conditions

$$\begin{aligned} &w(x,t)\big|_{t=0} = 0, && \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0} = 0, \\ &\psi(x,t)\big|_{t=0} = 0, && \left.\frac{\partial \psi(x,t)}{\partial t}\right|_{t=0} = 0. \end{aligned} \tag{3.4}$$

We assume the general solution in the following form:

$$w(x,t) = \sum_{j=1}^{n} X_{1j}(x)\xi_j(t), \qquad \psi(x,t) = \sum_{j=1}^{n} X_{2j}(x)\gamma_j(t), \tag{3.5}$$

where $X_{1j}(x)$ and $X_{2j}(x)$ are orthogonal functions which fulfil the boundary conditions (3.3):

$$X_{1j}(x) = \sin\frac{j\pi x}{l}, \qquad X_{2j}(x) = \cos\frac{j\pi x}{l}. \tag{3.6}$$

The displacement of the beam in a contact point with a travelling mass is expressed by the equation

$$w(vt,t)=\sum_{j=1}^{n}\xi_j(t)\sin\frac{j\pi vt}{l}. \tag{3.7}$$

According to the differentiation rule, we obtain the following formula:

$$\frac{dw(vt,t)}{dt}=\sum_{j=1}^{n}\dot{\xi}_j(t)\sin\frac{j\pi vt}{l}+\sum_{j=1}^{n}\xi_j(t)\frac{j\pi v}{l}\cos\frac{j\pi vt}{l}. \tag{3.8}$$

The kinetic energy of the moving inertial point (2.1) is expressed as a function of both the generalised coordinates and the derivative of generalised coordinates with respect to time:

$$T_m=f(\xi,\dot{\xi}). \tag{3.9}$$

Required derivation of the above quantity results in essential consequences. According to the Hamilton's principle for a conservative system, we can write the well-known law

$$\int_{t_1}^{t_2}\delta(T-U)dt=0. \tag{3.10}$$

Equation (3.10) with respect to (3.9) is transformed to the following form:

$$\int_{t_1}^{t_2}\left(\frac{\partial T}{\partial\dot{\xi}}\delta\dot{\xi}+\frac{\partial T}{\partial\xi}\delta\xi-\frac{\partial U}{\partial\xi}\delta\xi\right)dt=0. \tag{3.11}$$

Integration by parts with the assumption of $\delta\xi(t_1)=\delta\xi(t_2)=0$ results in the Lagrange equation of the second kind. The general form of it is given below

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\xi}_j}\right)-\frac{\partial T}{\partial\xi_j}+\frac{\partial U}{\partial\xi_j}=0. \tag{3.12}$$

According to the formula for kinetic and potential energy of the beam, we obtain two coupled motion equations. These two equations can be reduced to two uncoupled equations with ξ and γ. Let us focus our attention on a displacement case:

$$
\begin{aligned}
(3.13)\quad & \ddddot{\xi}_j(t)+\beta\sum_{k=1}^{n} f_1(j,k,t)\ddddot{\xi}_k(t)+2\beta\sum_{k=1}^{n}(\omega_j f_2(j,k,t)+2f_3(j,k,t))\dddot{\xi}_k(t)\\
&+\frac{A}{I}c_1^2\ddot{\xi}_j(t)+\frac{\omega_j^2}{v^2}(c_1^2+c_2^2)\ddot{\xi}_j(t)\\
&+\sum_{k=1}^{n}[g(j)f_1(j,k,t)+6\beta(\omega_j\omega_k f_4(j,k,t)-\omega_k^2 f_1(j,k,t))]\ddot{\xi}_k(t)\\
&+2\sum_{k=1}^{n}[g(j)\omega_k f_3(j,k,t)-\beta(3\omega_j\omega_k^2 f_2(j,k,t)+2\omega_k^3 f_3(j,k,t))]\dot{\xi}_k(t)\\
&+\frac{\omega_j^4}{v^4}c_1^2c_2^2\xi_j(t)-\sum_{k=1}^{n}[g(j)\omega_k^2 f_1(j,k,t)+\beta(2\omega_j\omega_k^3 f_4(j,k,t)-\omega_k^4 f_1(j,k,t))]\xi_k(t)\\
&=\frac{P}{\rho A\beta}g(j)\sin\omega_j t,
\end{aligned}
$$

where

$$
(3.14)\qquad c_1=\sqrt{\frac{G}{k\rho}},\quad c_2=\sqrt{\frac{E}{\rho}},\quad \beta=\frac{2m}{\rho A l},\quad \omega_k=\frac{k\pi v}{l},\quad \omega_j=\frac{j\pi v}{l},
$$

$$
(3.15)\qquad
\begin{aligned}
f_1(j,k,t)&=\sin\omega_j t\sin\omega_k t,\\
f_2(j,k,t)&=\cos\omega_j t\sin\omega_k t,\\
f_3(j,k,t)&=\sin\omega_j t\cos\omega_k t,\\
f_4(j,k,t)&=\cos\omega_j t\cos\omega_k t,
\end{aligned}
$$

$$
(3.16)\qquad g(j)=\beta\left(\frac{A}{I}c_1^2+\omega_j^2\left(\frac{c_2^2}{v^2}-1\right)\right).
$$

Coefficients c_1 and c_2 are the shear and bending wave velocity in a Timoshenko beam, respectively. Lagrange methods lead us to the system of differential equations (3.13) with variable coefficients. This system of equations can not be easily solved in an analytical way and we must integrate it numerically. We perform this integration by means of the Runge–Kutta method. Equation (3.13) can be written in a short form

$$
(3.17)\qquad \mathbf{\Gamma}\ddddot{\xi}+\mathbf{U}\dddot{\xi}+\mathbf{M}\ddot{\xi}+\mathbf{C}\dot{\xi}+\mathbf{K}\xi=\mathbf{P}.
$$

Matrices $\mathbf{\Gamma}$, $\mathbf{U}$, $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ and a vector $\mathbf{P}$ are given in the Appendix. Formula (3.17) constitutes a system of ordinary differential equations of the 4th order with respect to time, hence we need two additional initial conditions [18]:

$$\left.\frac{\partial^2 w(x,t)}{\partial t^2}\right|_{t=0} = \left.\frac{1}{\rho A} q(x,t)\right|_{t=0}, \qquad \left.\frac{\partial^3 w(x,t)}{\partial t^3}\right|_{t=0} = \left.\frac{1}{\rho A}\frac{\partial q(x,t)}{\partial t}\right|_{t=0}, \tag{3.18}$$

where $q(x,t)$ is given by the equation (1.2). According to the Fourier sine transformation in a finite range of the initial conditions (3.4) and (3.18), we can write initial subvectors for displacements in the following form:

$$\xi_j(t)\big|_{t=0} = 0, \qquad \dot{\xi}_j(t)\big|_{t=0} = 0, \qquad \ddot{\xi}_j(t)\big|_{t=0} = 0, \qquad \dddot{\xi}_j(t)\big|_{t=0} = \frac{P\omega_j}{\rho A}. \tag{3.19}$$

Finally, the displacements of the arbitrary point of the beam can be determined from the following relation (see Eq. 3.5):

$$w(x,t) = \sum_{i=1}^{n} \xi_i(t) \sin\left(\frac{i\pi x}{l}\right). \tag{3.20}$$

4. Examples

We use dimensionless data $L = 1$, $\rho = 1$, $A = 1$, $I = 0.01$, $E = 1$, $G = 0.4$ and $k = 1$. These data result in the shear wave speed $c_1 = 0.63$ and the bending wave speed $c_2 = 1.0$ (Eq. (3.14)). Results of the semi-analytical solution are depicted in Fig. 3. Displacements are related to the amplitude of the quasi-static displacement of the mid-point beam w_0. A more detailed presentation of the Timoshenko beam motion is given in Fig. 4. Both types of waves are noticeable. We emphasise the sharp edge of the wave and the reflection from the

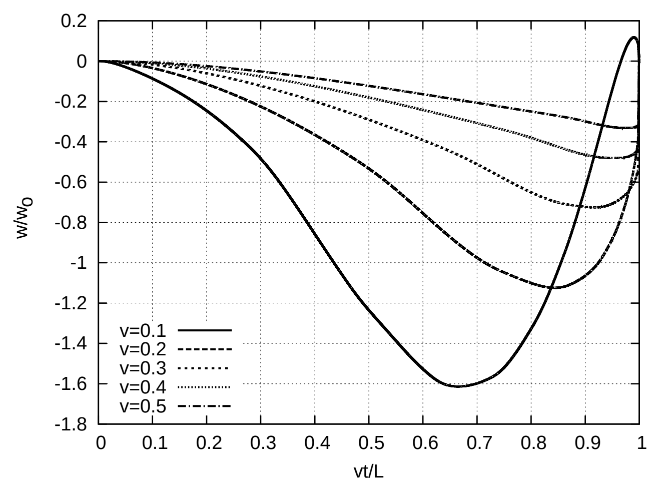

FIG. 3. Semi-analytical solution of the mass trajectory moving along the Timoshenko beam at various velocities ($c_1 = 0.63$, $c_2 = 1.00$).

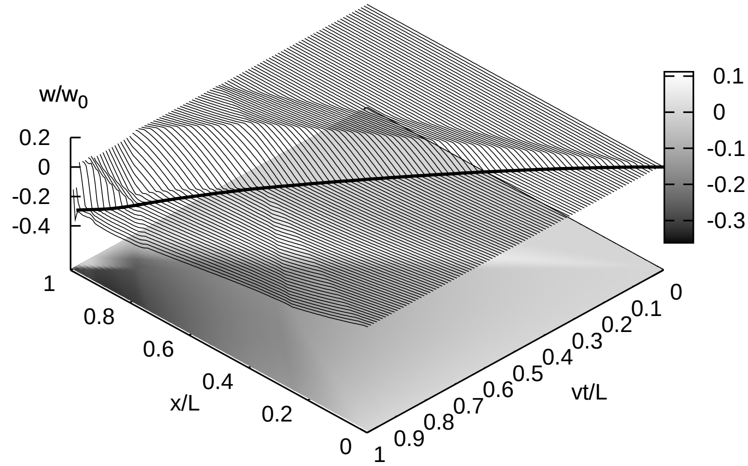

Fig. 4. Deflection in time of the Timoshenko beam under the load moving at the speed $v{=}0.5c_2$.

support and from the moving mass point. The velocity $v = 0.5c_2$ is characteristic in our example, since the discontinuity of the mass trajectory is well visible. Further tests will be performed with this velocity. The convergence rate is low and we examined it in relation to the number of terms (Fig. 5), taken in the Eq. (3.20). The plot with low number of terms is smooth in the neighbourhood of the support. The increasing number of terms makes the plot of the last 1 per cent of the trajectory sharp. It can be compared with the same phenomenon obtained for a string [14].

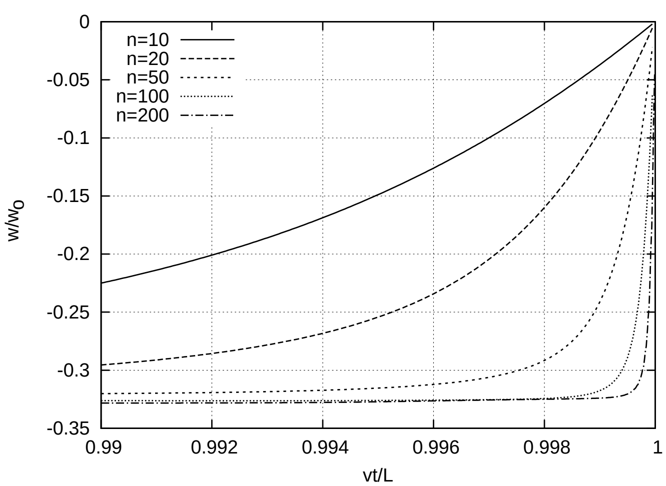

Fig. 5. The convergence of the mass trajectory travelling with $v{=}\ 0.5c_2$ near the end point, for various numbers of terms $(10, 20, \ldots, 200)$ in Eq. (3.20).

Examples show the same type of discontinuity of the solution in the case of the Timoshenko beam. Although we can not prove mathematically this feature in the case of the inertial Timoshenko beam matter, we can say that for practical use, the differential equation of the Timoshenko beam motion under

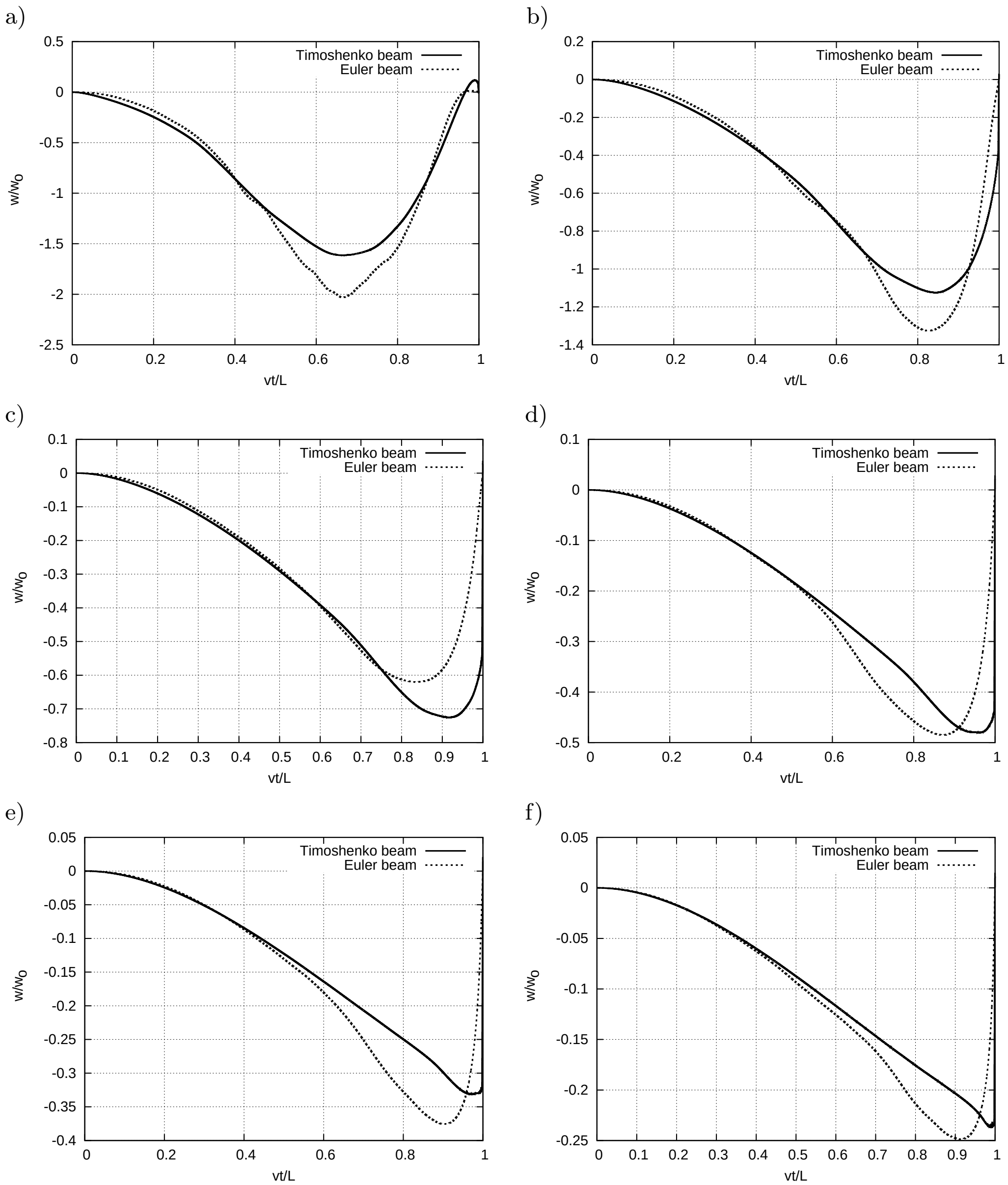

Fig. 6. Trajectory of a mass travelling at the speed: a) $v = 0.1$, b) $v = 0.2$, c) $v = 0.3$, d) $v = 0.4$, e) $v = 0.5$ and f) $v = 0.6$ ($c_1 = 0.63$, $c_2 = 1.00$).

the assumption of small displacements, involves the discontinuity of the structure in the neighbourhood of the support. Such a phenomenon is observed in real structures (a track or bridge plates) in a form of high value impacts. The Bernoulli–Euler beam does not exhibit the discussed discontinuity of the solution. Comparison of trajectories of the moving inertial point travelling along the Euler beam and the Timoshenko beam illustrates Fig. 6. Figure 7 depicts the deflection of the Timoshenko beam in time and reflections of the transverse wave c_1, and longitudinal wave c_2 at the subcritical and critical speed.

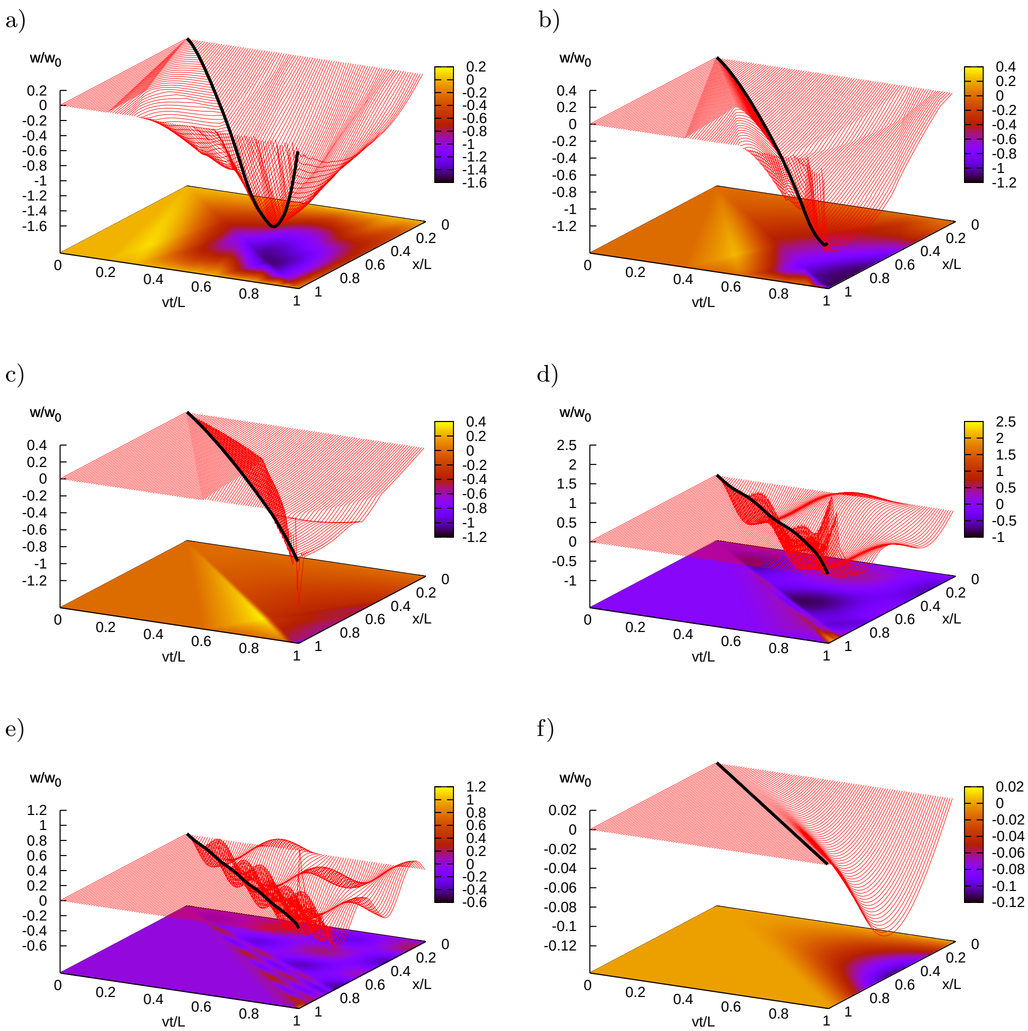

Fig. 7. Simulation of the Timoshenko beam under the moving inertial point at the speed: a) $v = 0.2$, b) $v = 0.4$, c) $v = 0.6$, d) $v = 0.7$, e) $v = 0.9$ and f) $v = 1.0$ ($c_1 = 0.63$, $c_2 = 1.00$).

5. Conclusions and discussion

In the paper we have derived the solution of the mass particle travelling on the Timoshenko beam. The problem is complex, since the product of the Dirac delta function with the acceleration commonly used in literature to problems with moving point mass, contributes certain discontinuities to the governing differential equation. The Lagrange equation of the second kind allowed us to solve the problem and to prove correctness of the results. They are identical with the direct transformation of differential equations of motion.

The solution of the problem discussed here can not be simply applied to complex problems, for example strings, beams, or three-dimensional bodies, subjected to a system of masses or composed of segments with variable rigidity. In such cases, discrete methods should be applied. However, it enables us to exhibit qualitative features or to validate numerical solutions. The existing numerical approaches fail in the case of inertial loads. Although solutions converge in some cases, the error in the case of the mass motion, compared with the critical speed, is significant. A numerical solution with the space-time finite element method was elaborated in [19]. The space-time finite element carrying the mass particle enables us to incorporate the local mass effect to classical general codes.

Appendix

$$\mathbf{\Gamma} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} + \beta \begin{bmatrix} f_1(1,1,t) & f_1(1,2,t) & \cdots & f_1(1,n,t) \\ f_1(2,1,t) & f_1(2,2,t) & \cdots & f_1(2,n,t) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(n,1,t) & f_1(n,2,t) & \cdots & f_1(n,n,t) \end{bmatrix},$$

$$\mathbf{U} = 2\beta \begin{bmatrix} \omega_1 f_2(1,1,t) & \omega_1 f_2(1,2,t) & \cdots & \omega_1 f_2(1,n,t) \\ \omega_2 f_2(2,1,t) & \omega_2 f_2(2,2,t) & \cdots & \omega_2 f_2(2,n,t) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_n f_2(n,1,t) & \omega_n f_2(n,2,t) & \cdots & \omega_n f_2(n,n,t) \end{bmatrix} + 4\beta \begin{bmatrix} f_3(1,1,t) & f_3(1,2,t) & \cdots & f_3(1,n,t) \\ f_3(2,1,t) & f_3(2,2,t) & \cdots & f_3(2,n,t) \\ \vdots & \vdots & \ddots & \vdots \\ f_3(n,1,t) & f_3(n,2,t) & \cdots & f_3(n,n,t) \end{bmatrix},$$

$$\mathbf{M} = \frac{A}{I}c_1^2 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} + \frac{c_1^2 + c_2^2}{v^2} \begin{bmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \omega_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_n \end{bmatrix}$$

$$+ \begin{bmatrix} g(1)f_1(1,1,t) & g(1)f_1(1,2,t) & \cdots & g(1)f_1(1,n,t) \\ g(2)f_1(2,1,t) & g(2)f_1(2,2,t) & \cdots & g(2)f_1(2,n,t) \\ \vdots & \vdots & \ddots & \vdots \\ g(n)f_1(n,1,t) & g(n)f_1(n,2,t) & \cdots & g(n)f_1(n,n,t) \end{bmatrix}$$

$$+ 6\beta \begin{bmatrix} \omega_1\omega_1 f_4(1,1,t) & \omega_1\omega_2 f_4(1,2,t) & \cdots & \omega_1\omega_n f_4(1,n,t) \\ \omega_2\omega_1 f_4(2,1,t) & \omega_2\omega_2 f_4(2,2,t) & \cdots & \omega_2\omega_n f_4(2,n,t) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_n\omega_1 f_4(2,n,t) & \omega_n\omega_2 f_4(n,2,t) & \cdots & \omega_n\omega_n f_4(n,n,t) \end{bmatrix}$$

$$- \begin{bmatrix} \omega_1^2 f_1(1,1,t) & \omega_2^2 f_1(1,2,t) & \cdots & \omega_n^2 f_1(1,n,t) \\ \omega_1^2 f_1(2,1,t) & \omega_2^2 f_1(2,2,t) & \cdots & \omega_n^2 f_1(2,n,t) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^2 f_1(n,1,t) & \omega_2^2 f_1(n,2,t) & \cdots & \omega_n^2 f_1(n,n,t) \end{bmatrix},$$

$$\mathbf{C} = 2 \begin{bmatrix} g(1)\omega_1 f_3(1,1,t) & g(1)\omega_2 f_3(1,2,t) & \cdots & g(1)\omega_n f_3(1,n,t) \\ g(2)\omega_1 f_3(2,1,t) & g(2)\omega_2 f_3(2,2,t) & \cdots & g(2)\omega_n f_3(2,n,t) \\ \vdots & \vdots & \ddots & \vdots \\ g(n)\omega_1 f_3(n,1,t) & g(n)\omega_2 f_3(n,2,t) & \cdots & g(n)\omega_n f_3(n,n,t) \end{bmatrix}$$

$$- 6\beta \begin{bmatrix} \omega_1\omega_1^2 f_2(1,1,t) & \omega_1\omega_2^2 f_2(1,2,t) & \cdots & \omega_1\omega_n^2 f_2(1,n,t) \\ \omega_2\omega_1^2 f_2(2,1,t) & \omega_2\omega_2^2 f_2(2,2,t) & \cdots & \omega_2\omega_n^2 f_2(2,n,t) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_n\omega_1^2 f_2(n,1,t) & \omega_n\omega_2^2 f_2(n,2,t) & \cdots & \omega_n\omega_n^2 f_2(n,n,t) \end{bmatrix}$$

$$+ 4 \begin{bmatrix} \omega_1^3 f_3(1,1,t) & \omega_2^3 f_3(1,2,t) & \cdots & \omega_n^3 f_3(1,n,t) \\ \omega_1^3 f_3(2,1,t) & \omega_2^3 f_3(2,2,t) & \cdots & \omega_n^3 f_3(2,n,t) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^3 f_3(n,1,t) & \omega_2^3 f_3(n,2,t) & \cdots & \omega_n^3 f_3(n,n,t) \end{bmatrix},$$

$$\mathbf{K} = \frac{c_1^2 c_2^2}{v^4} \begin{bmatrix} \omega_1^4 & 0 & \cdots & 0 \\ 0 & \omega_2^4 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_n^4 \end{bmatrix}$$

$$- \begin{bmatrix} g(1)\omega_1^2 f_1(1,1,t) & g(1)\omega_2^2 f_1(1,2,t) & \cdots & g(1)\omega_n^2 f_1(1,n,t) \\ g(2)\omega_1^2 f_1(2,1,t) & g(2)\omega_2^2 f_1(2,2,t) & \cdots & g(2)\omega_n^2 f_1(2,n,t) \\ \vdots & \vdots & \ddots & \vdots \\ g(n)\omega_1^2 f_1(n,1,t) & g(n)\omega_2^2 f_1(n,2,t) & \cdots & g(n)\omega_n^2 f_1(n,n,t) \end{bmatrix}$$

$$+ \beta \begin{bmatrix} \omega_1\omega_1^3 f_4(1,1,t) & \omega_1\omega_2^3 f_4(1,2,t) & \cdots & \omega_1\omega_n^3 f_4(1,n,t) \\ \omega_2\omega_1^3 f_4(2,1,t) & \omega_2\omega_2^3 f_4(2,2,t) & \cdots & \omega_2\omega_n^3 f_4(2,n,t) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_n\omega_1^3 f_4(n,1,t) & \omega_n\omega_2^3 f_4(n,2,t) & \cdots & \omega_n\omega_n^3 f_4(n,n,t) \end{bmatrix}$$

$$- \begin{bmatrix} \omega_1^4 f_1(1,1,t) & \omega_2^4 f_1(1,2,t) & \cdots & \omega_n^4 f_1(1,n,t) \\ \omega_1^4 f_1(2,1,t) & \omega_2^4 f_1(2,2,t) & \cdots & \omega_n^4 f_1(2,n,t) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^4 f_1(n,1,t) & \omega_2^4 f_1(n,2,t) & \cdots & \omega_n^4 f_1(n,n,t) \end{bmatrix},$$

$$\mathbf{P} = \frac{mg}{\rho A \beta} \begin{bmatrix} g(1)\sin\frac{1\pi vt}{l} \\ g(2)\sin\frac{2\pi vt}{l} \\ \vdots \\ g(n)\sin\frac{n\pi vt}{l} \end{bmatrix}.$$

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