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THE SHOCK AND VIBRATION DIGEST

A Publication of the

VIBRATION INSTITUTE

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THE SHOCK AND VIBRATION DIGEST

Volume 23, No. 5

May 1991

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FEATURE ARTICLE

tutorials and items of special interest

State-of-the-Art in the Space-Time Element Method

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***Abstract.** The space-time element method (STEM), a computational discrete method for the space-time modeling of physical problems, is described. The method has many practical advantages and complements existing solution methods.*

In recent years computational mechanics has developed considerably because of the progress in the field of computational tools. Fast supercomputers, matrix, and parallel processors enable engineers to solve problems of increasingly complex structures. Static solutions, both linear and nonlinear, do not cause any significant difficulties. Eigenvalue problems are solved also in a single-stage process, and in this case even a highly discretized domain can be treated.

On the basis of the estimation of approximation error, a new direction of research has been initiated. All the so-called "adaptive processes" are to minimize the error resulting from the replacement of the continuous problem by a discrete one. Such a treatment mainly concerns non-evolutionary processes.

True computational difficulties appear when the integration of the differential equation of motion is to be performed. Time of calculation is the main restriction, especially when nonlinearities are included. Nonlinear problems necessitate more realistic material behavior, consideration of contact between two deformable bodies, existence of multi-phase media, and other advanced technological methods.

Many different computational discrete methods are in use. To this variety, the space-time element method (STEM) can be added. It can be considered as an extension and generalization of the finite element method (FEM). Although it was developed for dynamic problems of structures, it can be easily applied to every time-dependent problem. Below a short description of characteristics of the method is given. Advantages and disadvantages are exhibited. Further considerations, supported by examples of testing and real problems, are provided to enable engineers to determine if the method is suitable for their own professional practices.

HISTORICAL BACKGROUND

Early approaches to the space-time modeling of physical problems were published in 1964 by Gurtin [1,2] and Herrera and Bielack [3]. Special definition of the functional to be minimized, based on the convolution theory, allowed one to derive the dependence between spatial and time variables in objects that could be called "space-time elements." Later, in 1969 Oden [4] proposed a generalization of the finite element method. He extended the interpretation of the structure for the time domain. Unfortunately, nonstationary partition of the structure into discrete subdomains as suggested in his work was not continued. Ainsworthy [5], Argyris, Scharpf, and Chan [6-8] started to treat both spatial and time variables equally during formulation of the problem. However, even in the work of Kuang and Alturi [9], the final discrete treatment was performed separately in space and time. Still, the dynamic problems were solved by methods in which spatial and time variables were decoupled. Spatial domain was discretized by a selected discrete method (finite element, finite difference) while the time derivatives were integrated separately by another numerical tool.

Numerous papers appeared on the direct time integration of the differential equation based on the stationary partition [10,11]. Unconditional stability in time, damping of high-frequency modes, and efficiency in applications are the features required commonly in practice. Classical time integration methods have such advantages. However, there are problems in which the existing methods fail, for example, in problems with a moving boundary. In such a case the approach to heat transmission was described by Bonnerot and Jamet [12,13]. In

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successive works [14] time integration schemes were still called by the term "space-time element" although the distribution of spatial and time variables was performed by different functions. The paper by Kujawski and Dessai [15] is a good example of such an approach.

The first complete and clear space-time approximation for structural dynamics was made by Kaczkowski [16,17]. Since then many papers have been written on the subject of vibration analysis in the space-time domain. New ideas replaced the old as researchers found ways to better solve difficulties. However, only some of these ideas have made a contribution to the present form of the method. Only these will be cited.

Kaczkowski and Langer [18] expressed strains and stresses in terms of spatial and time variables. Initial conditions can be regarded as boundary conditions placed on the layer of constant time. The interpolation (shape) functions are then written as $N(x,t)$, and generally variables cannot be uncoupled. Only such a formulation can be understood as a full STEM approach. A similar description was presented by Riff and Baruch [19,20]. New constitutive equations were considered by Podhorecki [21]. Accuracy investigations were also performed [22].

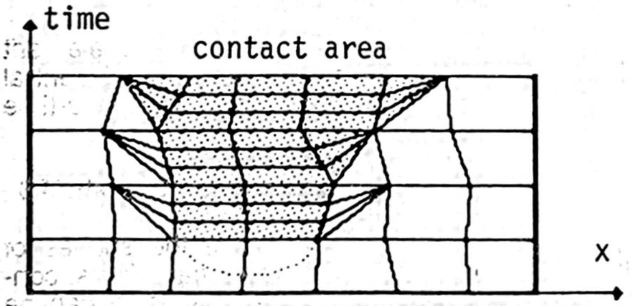

Figure 1. Subdivision of time step in the contact region

Although the early papers concern uni-dimensional structures, and conditional stability is still the main disadvantage of the formulation, the space-time finite element method improved quickly. Space-time elements of non-rectangular shapes were introduced [23,24]. A triangular beam element of arbitrary joint location was used to express a space-time element approach to contact problems. A decrease in the integration time step for some spatial regions with higher speed of displacement is possible with the use of triangles (Figure 1). A formulation of an unconditionally stable variant of the method for multiplex-shaped, space-time elements [25] can be regarded as a further improvement. Multiplex-shaped elements applied to constant spatial mesh leads to one of the time integration methods because separation of spatial and time variables can be completed. Shape functions can be written then as $N(x,t) = X(x) \cdot T(t)$. However, additional facilities are gained when variable mesh is applied. Recently developed simplex-shaped elements lead to highly efficient numerical procedures.

The space-time element approach, which is in the scope of this paper, can be considered as a time-continuous Galerkin method. It differs from the time-discontinuous Galerkin method [26,27], which leads to A-stable higher order accurate O.D.E. solvers, but in turn, gives uneconomically large matrices. Moreover, the generalization of the time-continuous Galerkin method to elastodynamics seems possible only in some circumstances.

In this paper only the continuous in time space-time finite element formulation is discussed. However, a general approach is restrained by stability limitations. Arbitrary node location in time space cannot be applied successfully (this feature results from the properties of the hyperbolic equation of motion). Stability criteria are given for selected structures [28]. Numerical examples prove the efficiency of the approach in different problems of structural dynamics [29].

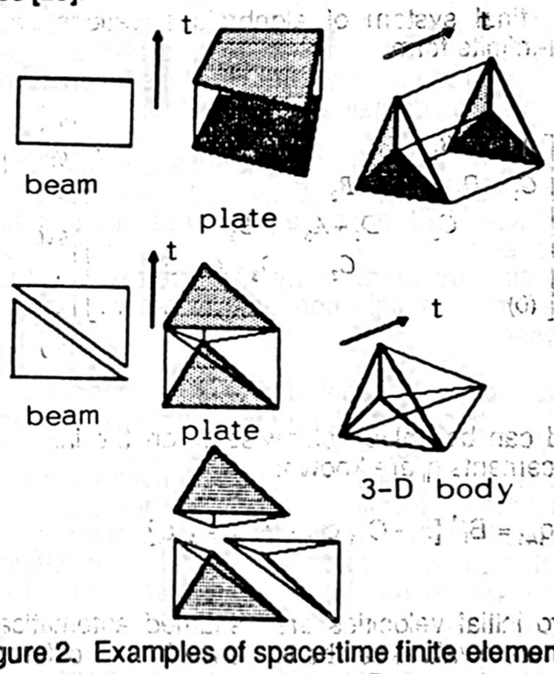

Figure 2. Examples of space-time finite elements

FUNDAMENTAL CONCEPTS

In the space-time element method interpolation functions are applied both to spatial and time domains. Time variable is considered as an additional variable of the coordinate system [16,24]. The object to be analyzed gains one additional coordinate axis. That is why for uni-dimensional structures there are two-dimensional space-time elements and three-dimensional objects for two-dimensional problems. Some sample space-time elements are depicted in Figure 2. The shape functions N_i that are used to interpolate displacements u inside the space-time domain with nodal values q_i, are functions of x and t:

$$u(x,t) = \sum N_i(x,t)\, q_i \tag{1}$$

Detailed derivation of the fundamental relations can be found in works by Kaczkowski [17], Kaczkowski and Langer [18], and Bajer [24]. The resulting element matrices have dimensions equal to the number of degrees of freedom in an element. The assembly of the global matrix K* is performed in the same way as in the FEM, considering the topology of the space-time mesh. The resulting matrix is composed of stiffness K, inertia M, and damping Z contributions

$$K^* = \sum^{NE} (K_i + M_i + Z_i) \tag{2}$$

The final system of algebraic equations has a half-infinite form:

$$\begin{bmatrix} A_1 & B_1 & & & (0) \\ C_1 & D_1+A_2 & B_2 & & \\ & C_2 & D_2+A_3 & B_3 & \\ & & C_3 & D_3+A_4 & B_4 \\ (0) & & & \ldots & \ldots \quad \ldots \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ \cdot \\ \cdot \\ \cdot \end{Bmatrix} \tag{3}$$

and can be solved stepwise when the initial displacements q_i are known:

$$q_{i+1} = B_i^{-1} [F_i - C_{i-1}\, q_{i-1} - (C_{i-1} + A_i) q_i] \tag{4}$$

Zero initial velocities are assumed automatically. Non-zero values can be expressed by the difference $(q_2 - q_1)/\Delta t$. In Equation 4 all the matrices have dimensions equal to the total number of degrees of freedom in a real (spatial) structure.

The STEM enables the mesh modification in each step in a natural way [30]. It means that joints of the spatial mesh taken in two successive intervals of time are connected, and in this way the space-time subdomains can be determined. Since the joints have different coordinates in different time intervals limiting the time layer at the top and the bottom, the space-time elements have non-rectangular forms.

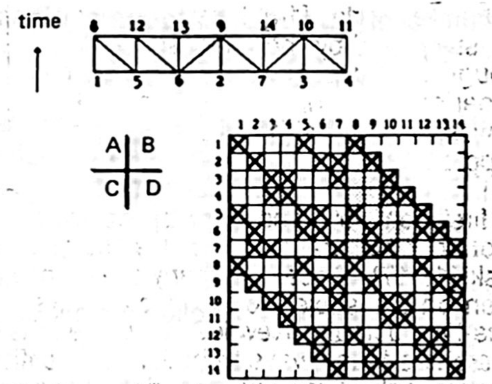

Figure 3. Assembly of a global matrix for triangular space-time elements

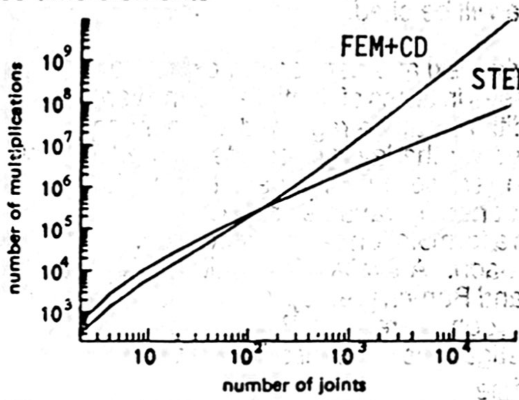

Figure 4. The cost of the space-time element method: the finite element method and central difference method (FEM+CD) and the space-time element method (STEM)

BENEFITS OF SIMPLEX-SHAPED ELEMENTS

If the space-time elements have the shapes of triangles, tetrahedrons, or hypertetrahedrons, considerable savings of computational effort can be gained. The efficiency is increased significantly for several reasons. Simplex-shaped elements lead to triangular forms of coefficient matrix B (Figure 3) with reduced number of coefficients. Second, the solution of the algebraic system of equations can be carried out joint-by-joint directly, since the resulting coefficient matrix is triangular [24,31]. Thus the number of arithmetic operations per time step is reduced. This is essential particularly in nonlinear problems, when the system of equations has to be solved in every step. The number of multiplications per step M is given by the relation

$$M = 2sN(c+1) \tag{5}$$

where:

s - nodal number of degrees of freedom

N - total number of degrees of freedom

c -- number of joints adjacent to one joint in a spatial mesh

It must be emphasized that the cost is independent of the numbering of nodes, and terms such as the bandwidth do not exist. A cost comparison between the simplex-shaped STEM and the central difference method is depicted in Figure 4. For larger structures the STEM is more economical than other methods of direct integration of the motion equation.

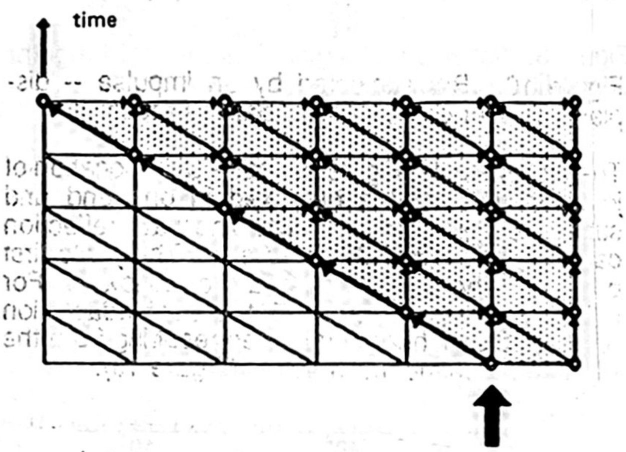

Figure 5. Propagation of information in the triangular mesh

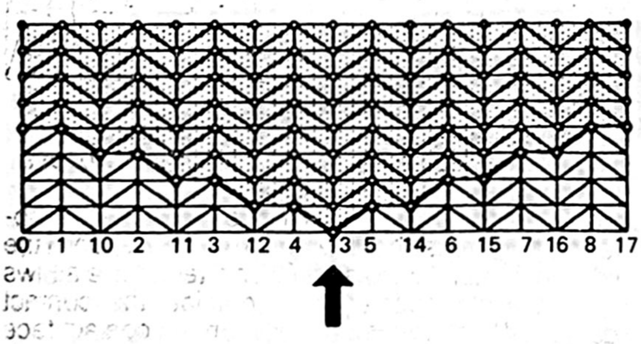

Figure 6. Symmetric wave speed for triangular mesh

The next advantage concerns the physical property of the simplex-based mesh. In such cases the limited speed of information flow can be determined according to the slope edges (Figures 5,6). This feature is an improvement of the numerical solution of the hyperbolic differential equation. In practice the wave speed should limit the speed of propagation of the numerical information in the discrete system.

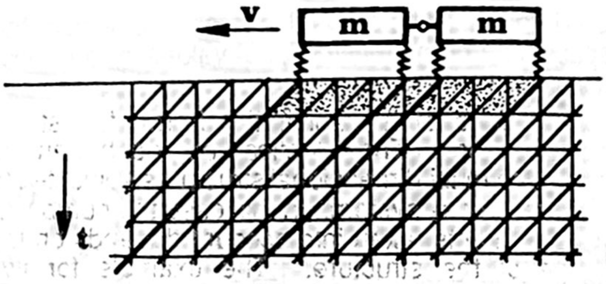

Figure 7. Infinite beam subjected to a moving mass system with several points of contact

The described property seems to be helpful in the solution of problems with traveling force. Consider an infinitely long string or beam on which the system of masses moves with the speed close to the wave speed (Figure 7). In a typical numerical solution a limited number of elements in a discrete model must be considered. In such a case the reflection of waves from the ends of the structure disturbs the solution. The STEM allows the problem to be reduced to only a few degrees of freedom and avoids the influence of reflection on the vibrating masses. Even a limited speed of information flow makes the wave unable to precede the mass system, and the reflection from the right end will never reach the moving mass [32]. Different dimensions between contact points do not state the problem since the varying spatial partition allows one to adjust the nodal points of the infinite linear object to the traveling vehicle.

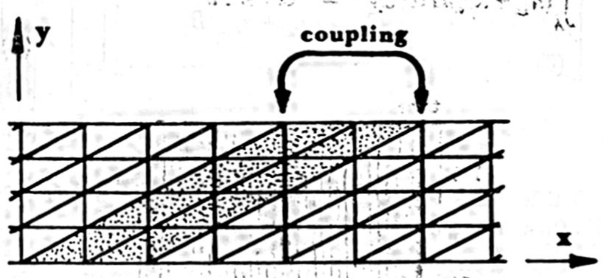

Figure 8. Domain of the load influence for a two-dimensional object

The problem reduced to one with a small number of joints is convenient for stability testing. The coupling of contact points through the mass moving in a supersonic range generates growing amplitudes. The stepping scheme can be written in a form:

$$\begin{Bmatrix} q_{i+1} \\ q_i \end{Bmatrix} = \mathbf{T} \begin{Bmatrix} q_i \\ q_{i-1} \end{Bmatrix} \tag{6}$$

and the stability investigation is reduced to the calculation of the spectral radius ρ of T. The system is unstable when the following condition occurs:

$$\rho(T) > 1 \tag{7}$$

Eigenvectors of T exhibit the form of unstable deformations. The identical approach can be applied to multi-dimensional structures subjected to a moving mass system. In this case the number of finite elements taken into account depends on the shape of the structure. The example for two dimensions is presented in Figure 8.

MESH ADAPTATION

Adaptive techniques were developed to reduce the interpolation error when the computational cost forces a limited number of finite elements on which the domain can be divided. Three general directions can be seen: h-adaptation, in which a single element is divided into smaller elements, reducing the mesh dimension H; p-adaptation, in which the accuracy increasing the polynomial order and r-adaptation is improved; and when nodal points are relocated from regions of lower error estimation to regions of higher error values. Below, the third kind of adaptive technique is exhibited since the joint relocation is very natural in the STEM [30,33-34].

Various error estimators can be applied to determine the error distribution [35-37]. Basically the rule is fulfilled below:

$$\int_A (u_{xx}^2 + u_{yy}^2) H \,\mathrm{d}x\mathrm{d}y = \text{constant} \tag{8}$$

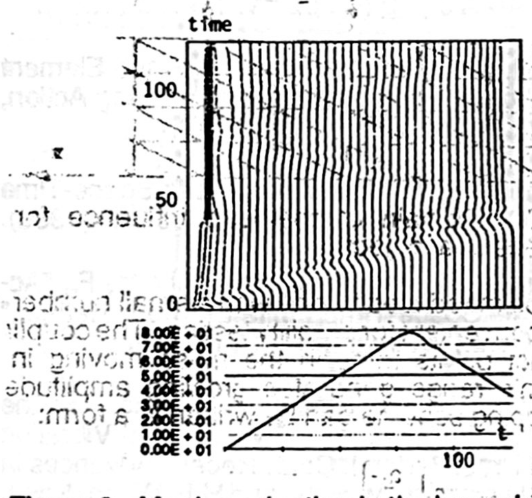

Figure 9. Mesh evaluation in the bar subjected to a Heaviside force

All the existing error estimators show the error distribution in a single static state. In dynamic problems the estimation is carried on globally, in the time interval in which the structure is observed. Since quite arbitrary mesh modification is not allowed, the exact adaptation strategy for evolutionary problems should not be derived from simple error estimation, based on a state in a given moment.

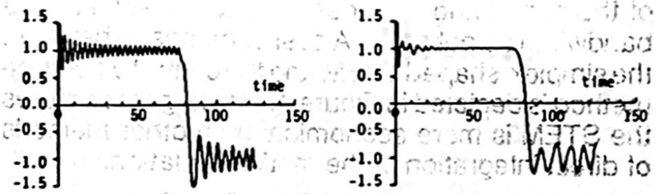

Figure 10. Bar subjected by an impulse -- displacement for constant and adaptive mesh

The first example (Figure 9) shows the relocation of joints in time for the bar fixed in one end and subjected to a Heaviside force. The wave reflection can be noticed from the fixed end. The joints first go toward the wave force and then follow it. For uni-dimensional problems [30] the mesh adaptation removes the higher order modes resulting from the reflection from the mesh nodes (Figure 10).

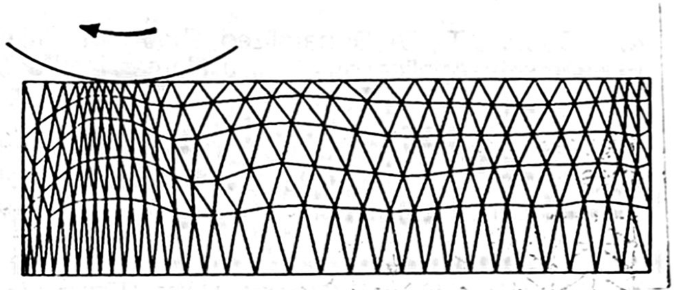

Figure 11. Mesh adaptation in the rolling process

Another example presents the rolling contact problem [38,39] and shows the mesh condensation in the contact region (Figure 11). Such a technique allows one to apply coarse mesh outside the contact domain. Phenomena occurring on the contact face (i.e. stresses, friction distribution) can be investigated with a good accuracy while outer parts do not require fine partitioning. In applying other methods one should refine the mesh on the entire path of the roller, increasing the computational cost highly.

FUTURE DIRECTIONS

The STEM described above has many practical advantages. It complements the existing solution methods. Future developments should remove restrictions in applications of the method. Conditional stability for simplex-shaped elements is one of them. Higher order approximation of both space and time is not investigated sufficiently. The limitation of the wave speed by algebraic means shows the opportunity to introduce the method into infinite structure analysis. The whole range of h-adaptive techniques can be developed and applied successfully to wave problems. Mixed h-r-adaptivity could allow physically multiphase mediums to be treated.

In general the space-time element method is the same convenient tool for dynamics as the finite element method is for statics. For vibrations it is the most educational numerical method available based on the domain discretization.

REFERENCES

1. Gurtin, M.E., "Variational Principles for Linear Elastodynamics," Arch. Rat. Mech. Anal., 16, pp 34-50 (1964).

2. Gurtin, M.E., "Variational Principles for Linear Initial-Value Problems," Quart. Appl. Math., 22, pp 252-256 (1964).

3. Herrera, I. and Bielak, J., "A Simplified Version of Gurtin's Variational Principles," Arch. Rat. Mech. Anal., 53, pp 131-149 (1974).

4. Oden, J.T., "A Generalized Theory of Finite Elements, II. Applications," Intl. J. Numer. Methods Engrg., 1, pp 247-259 (1969).

5. Ainsworth, M., Zhu, J.Z., Craig, A.W., and Zienkiewicz, O.C., " Finite Element Analysis of Time-Dependent Phenomena," AIAA J., 7, pp 1170-1173 (1989).

6. Argyris, J.H. and Scharpf, D.W., "Finite Elements in Time and Space," Aeronaut. J., 73, pp 1041-1044 (1969).

7. Argyris, J.H. and Scharpf, D.W., "Finite Elements in Space and Time," Nucl. Engrg. Des., 10, pp 456-469 (1969).

8. Argyris, J.H. and Chan, A.S.L., "Application of the Finite Elements in Space and Time," Ing.-Arch., 41, pp 235-257 (1972).

9. Kuang, Z.B. and Alturi, S.N., "Temperature Field Due to a Moving Heat Source," J. Appl. Mech., Trans. ASME, 52, pp 274-280 (1985).

10. Park, K.C., "Practical Aspects of Numerical Time Integration," Computers Struc., 7, pp 343-353 (1977).

11. Zienkiewicz, O.C., "Finite Elements in the Time Domain," A.K. Noor and W.D. Pilkey (Eds.) *State-of-the-Art Surveys on Finite Element Technology*, ASME (1983).

12. Bonnerot, R. and Jamet, P., "A Second Order Finite Element Method for the One-Dimensional Stefan Problem," Intl. J. Numer. Methods Engrg., 8, pp 811-820 (1974).

13. Bonnerot, R. and Jamet, P., "Numerical Computation of the Free Boundary for the Two-Dimensional Stefan Problem by Space-Time Finite Elements," J. Comp. Phys., 25, pp 163-181 (1977).

14. Kohler, W. and Pittr, J., "Calculation of Transient Temperature Fields with Finite Elements in Space and Time Dimensions," Intl. J. Numer. Methods Engrg., 8, pp 625-631 (1974).

15. Kujawski, J. and Dessai, C.S., "Generalized Time Finite Element Algorithm for Non-Linear Dynamic Problems," Engrg. Computers, 1 pp 247-251 (1984).

16. Kaczkowski, Z., "The Method of Finite Space-Time Elements in Dynamics of Structures," J. Tech. Phys., 16 (1), pp 69-84 (1975).

17. Kaczkowski, Z., "General Formulation of the Stiffness Matrix for the Space-Time Finite Elements," Arch. Inz. Lad., 25 (3), pp 351-357 (1979).

18. Kaczkowski, Z. and Langer, J., "Synthesis of the Space-Time Finite Element Method," Arch. Inz. Lad., 26 (1), pp 11-17 (1980).

19. Riff, R. and Baruch, M., "Stability of Time Finite Element," AIAA J., 22 (8), pp 1171-1172 (1984).

20. Riff, R. and Baruch, M., "Time Finite Element Discretization of Hamilton's Law of Varying Action, AIAA J., 22 (9), pp 1310-1318 (1984).

21. Podhorecki, A., "The Viscoelastic Space-Time Element," Computers Struc., 23, pp 535-544 (1986).

22. Bajer, C., Burkhard, G., and Taltello, F., "Accuracy of the Space-Time Finite Element Method," 11th IKM Conf., Weimar, pp 9-12 (1987).

23. Bajer, C.I., "Non-Stationary Division by the Space-Time Finite Element Method in Vibration Analysis," Proc. 2nd Intl. Conf. Recent Advances in Structural Dynamics, M. Petyt and H.F. Wolfe (Eds.), Southampton, pp 161-170 (1984).

24. Bajer, C.I., "Triangular and Tetrahedral Space-Time Finite Elements in Vibration Analysis," Intl. J. Numer. Methods Engrg., 23, pp 2031-2048 (1986).

25. Kacprzyk, Z. and Lewinski, T., "Comparison of Some Numerical Integration Methods for the Equations of Motion of Systems with a Finite Number of Degrees of Freedom," Engrg. Trans., 31 (2), pp 213-240 (1983).

26. Hughes, T.J.R. and Hulbert, G.M., "Space-Time Element Methods for Elastodynamics: Formulations and Error Estimates," Comp. Meth. Appl. Mech. Engrg., 66, pp 339-363 (1988).

27. Hulbert, G.M., "Space-Time Finite Element Method for Second-Order Hyperbolic Equation," Ph.D. Thesis, Stanford Univ. (Aug 1989).

28. Bajer, C.I., "Notes on the Stability of Non-Rectangular Space-Time Finite Elements," Intl. J. Numer. Methods Engrg., 24, pp 1721-1739 (1987).

29. Bajer, C.I., and Bonthoux, C.G., "State-of-the-art in True Space Time Finite Element Method," Shock Vib. Dig., 20, pp 3-11 (1988).

30. Bajer, C.I., "Adaptive Mesh in Dynamic Problem by the Space-Time Approach," Computers Struc., 33 (2), pp 319-325 (1989).

31. Bajer, C.I., "Microcomputer Applications of Non-Rectangular Space-Time Finite Elements in Vibration Analysis -- Direct Joint-by-Joint Procedure," *Engineering Software IV*, R.A. Adey (Ed.), Proc. 4th Intl. Conf., Springer-Verlag, London (1985).

32. Bajer, C. and Bogacz, R., "Stability of the Mass Motion on the String in a Supersonic Range," (to be published).

33. Bajer, C., "Movable Grid Approach by the Space-Time Element Method," 11th IKM Conf., Weimar, pp 5-8 (1987).

34. Bajer, C., Bogacz, R., and Bonthoux, C., "Adaptive Space-Time Elements in the Dynamic Elastic-Viscoplastic Problems," Computers Struc. (to be published).

35. Zienkiewicz, O.C. and Zhu, J.Z., "A Simple Error Estimator and Adaptive Procedure for Practical Engineering Analysis," Intl. J. Numer. Methods Engrg., 24, pp 337-357 (1987).

36. Zienkiewicz, O.C., Zhu, J.Z., and Gong, N.G., "Effective and Practical h-p-version Adaptive Analysis Procedures for the Finite Element Method," Intl. J. Numer. Methods Engrg., 28, pp 879-891 (1989).

37. Ainsworth, M., Zhu, J.Z., Craig, A.W., and Zienkiewicz, O.C., "Analysis of the Zienkiewicz-Zhu A-Posteriori Error Estimator in the Finite Element Method," Intl. J. Numer. Methods Engrg., 28, pp 2161-2174 (1989).

38. Bajer, C.I. and Bonthoux, C.G., "A New Numerical Approach for Dynamic Problems of Contact," Euromech 209, Vibrations with Unilateral Constraints, H. Cabannes and C. Citrini (Eds.), Como, pp 80-89 (1986).

39. Bajer, C. and Bogacz, R., "Dynamic Contact Problem by Means of the Space-Time Element Method," Proc. 5th Intl. Symp. Numer. Methods in Engrg., R. Gruber, J. Periaux, and R.P. Shaw (Eds.), Springer-Verlag, Lausanne, pp 313-318 (Sept 1989).