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ADAPTIVE MESH IN DYNAMIC PROBLEMS BY THE SPACE–TIME APPROACH

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Abstract—Stress fields varying in time are typical for dynamic wave problems. Nonclassic problems involve changing of structure properties, especially wave reflection zones or dissipative zones. Stress field propagation requires a variable mesh that allows one to approach the phenomenon with the smallest error in each time step. The space–time approximation of the differential equation of motion enables the modification of the spatial partition into finite elements in a continuous way. Error estimation was the reason to refine and coarsen the spatial partition, moving the nodes towards the zone of higher error. Applying the simplex-shaped space–time elements one can gain the triangular form of coefficient matrix directly in the element matrix assembly process. Consistent characteristic matrices are used. The approach presented was successfully applied for bar, beam and plane strain analysis. The method is more powerful for materially nonlinear cases for which element matrices should be calculated in each time step. Good accuracy of the movable mesh approach was proved in several testing examples.

1. INTRODUCTION

Large scale transient problems of complex geometry involve fine discretization into a multi-degree-of-freedom system. Material and geometrical nonlinearities require the recalculation of the system in successive time steps. Only a relatively short time step enables accurate investigation in the case of impact of shock waves on the structure, explosions, shock interaction or interaction between media of different density. Mesh refinement, if applied to entire structure, results in time-consuming computational schemes with simultaneous increase of the numerical error. The adaptive mesh techniques developed to date can be classified in two groups. In the first one the mesh is locally refined by the addition of new joints. In other regions joints can be removed to coarsen the grid [1, 2]. In this method new nodal parameters must be interpolated for the set of new joints while some information is lost when removing superfluous joints. There is an inconvenience in time-dependent problems, for example contact problems in dynamics. The initial phase is important and each approach accumulates the error. Stress waves in transient problems enforce frequent and total mesh modification so the procedure seems to be expensive. Hierarchical procedure is based on the same principle. In the case of higher order differential equations, considering the time variable, such an approach requires a backward step of recalculation in a general case. Moreover, it does not increase the accuracy in the nearest time of observation after the change. In this kind of algorithm the number of joints can vary in time.

The second group of methods contains algorithms that assume a constant number of joints. Nodal points are directly placed in order to minimize the error giving the refined or coarsened regions of the mesh. Such a method was frequently applied to elliptic or parabolic problems [3–5]. In stress analysis the refined zones can be moved together with the stress field motion or other characteristic line movements. Such problems are rarely mentioned in publications.

The full space–time approximation gives a natural way of mesh modification with a constant pattern of the mesh and unvaried number of nodal points and spatial elements.

In this paper the method of time-continuous mesh adaptation for a vibration structure is described. In successive time steps the spatial partition can be changed with respect to the assumed error estimator. The speed of mesh adaptation is limited by the stability criteria. However, in practical cases, the limitations allow almost arbitrary mesh modeling. The presented technique was applied both to linear and plane structures. An identical approach can be used to 3-D strain.

Space–time approximation with nonstationary representation of the spatial domain was treated in [6–9]. To date no spectacular success was demonstrated. To the contrary, negative opinions were found [10]. The author hopes that some interesting features of the full space–time approximation can be useful in the investigation of selected nonclasic problems. There is a significant interest in the problem presented in [11, 12]. However, hyperbolic problems are rarely treated [13, 14].

2. APPROXIMATION IN SPACE AND TIME

Let us consider a bar split into finite elements. The geometry of a finite element is described by two nodal

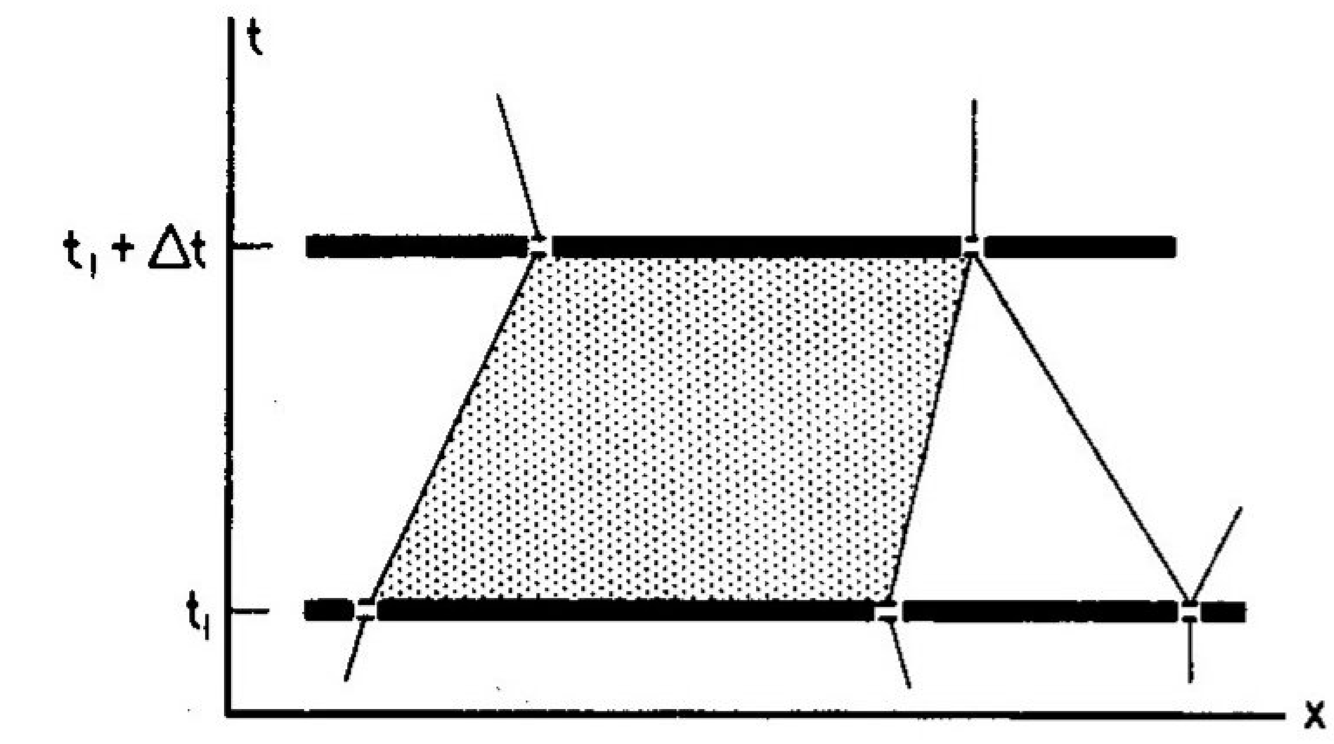

Fig. 1. Example of nonstationary partition of the bar.

coordinates, its displacements by nodal parameters. Let us consider such an element in two successive time steps (Fig. 1). We allow the element geometry modification. The domain between spatial elements in different moments, called the space–time element, is now described by four nodes, being the function of space and time. Displacements in any point of the space–time element area $\mathbf{q}$ are given by the interpolation

$$\mathbf{q} = \mathbf{N}(\mathbf{x}, t)\mathbf{r} \tag{1}$$

where $\mathbf{r}$ is the vector of nodal parameters and $\mathbf{N}$ is the shape function for a given quadrangle. Triangular forms of space–time elements for uni-dimensional structures are also available. Detailed formulae leading to the step-by-step integration scheme can be found in [7]. Triangular, tetrahedral and hypertetrahedral space–time elements (generally called simplex forms) for linear, plane and 3-D structures, respectively, give triangular coefficient matrices of the resulting system of equations directly during the global matrix assembly. This feature causes the mentioned simplex-shaped elements to be chosen in further applications.

2.1. *Space–time formulation*

In the subsequent text only concluding formulae will be presented. The approximation in space and time is described by the shape functions $N(x, t)$. The arbitrary form of the space–time element is available (Fig. 1). The spatial x and time t variables cannot be uncoupled in a general case. It can be done, however, for rectangular, multiplex forms. By integration over the space–time element domain we obtain the characteristic element matrices

$$\mathbf{K} = \int [\mathbb{D}\mathbf{N})^T\mathbf{E}\mathbb{D}\mathbf{N}\,\mathrm{d}V$$

$$\mathbf{M} = \int \left(\frac{\partial \mathbf{N}}{\partial t}\right)^T \mathbf{R} \left(\frac{\partial \mathbf{N}}{\partial t}\right) \mathrm{d}V$$

$$\mathbf{W} = \int (\mathbb{D}\mathbf{N})^T \eta_w \frac{\partial}{\partial t}\,\mathbb{D}\mathbf{N}\,\mathrm{d}V$$

$$\mathbf{Z} = \int \mathbf{N}^T \eta_z \frac{\partial}{\partial t}\,\mathbf{N}\,\mathrm{d}V \tag{2}$$

where $\mathbf{E}$ is the elasticity constant matrix, $\mathbb{D}(\mathbf{x})$ is a specific differential operator, $\mathbf{R}$ is the inertia coefficient matrix and η_z, η_w are damping coefficients or damping matrices. Stiffness matrix $\mathbf{K}$, mass matrix $\mathbf{M}$, internal damping and external damping matrices $\mathbf{W}$ and $\mathbf{Z}$ are of the order equal to the number of degrees of freedom in the space–time element. Then for one time layer we have the equation

$$\sum (\mathbf{K}_i + \mathbf{M}_i + \mathbf{W}_i + \mathbf{Z}_i)\mathbf{r} = \mathbf{F} \quad \text{or} \quad \mathbf{K}^*\mathbf{r} = \mathbf{F}. \tag{3}$$

The sum is extended over all elements in the space–time layer. $\mathbf{r}$ is a vector of nodal impulses. The global matrix is assembled depending on the nodal topology, as in the FEM (Fig. 2). Splitting the matrix $\mathbf{K}^*$ for established layer i into four quarters denoted by $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$, $\mathbf{D}_i$ and considering the next element layer we can write the force equilibrium equation for one space–time layer

$$\mathbf{C}_{i-1}\mathbf{r}_{i-1} + (\mathbf{D}_{i-1} + \mathbf{A}_i)\mathbf{r}_i + \mathbf{B}_i\mathbf{r}_{i+1} = \mathbf{F}_i \tag{4}$$

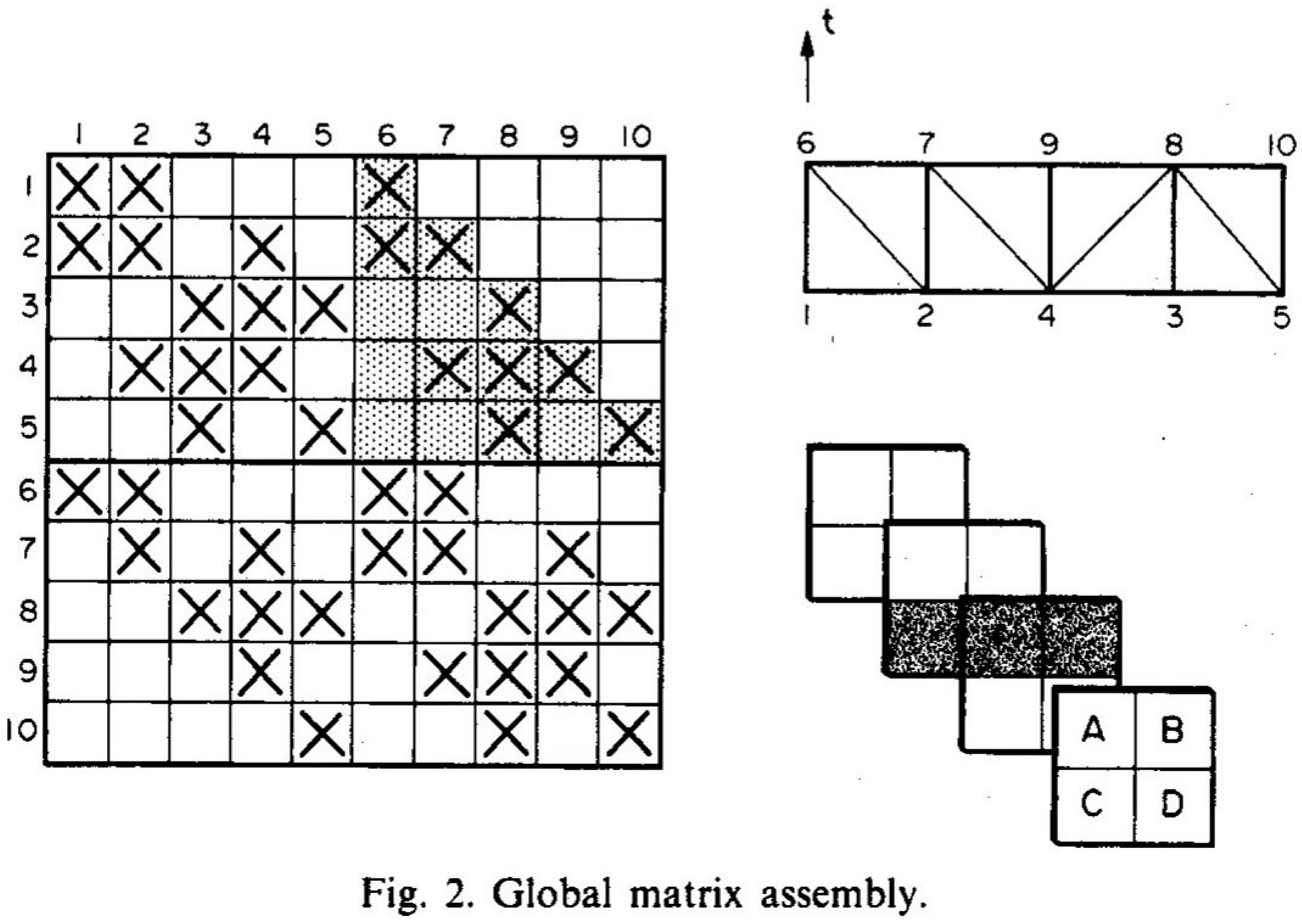

Fig. 2. Global matrix assembly.

in which $\mathbf{r}_{i+1}$ is the only unknown vector. The equation is of the order equal to the total number of d.o.f. in the spatial structure.

2.2. *Velocity formulation*

One can write the two point velocity formula of the integration scheme [15]

$$\mathbf{r}_{i+1} = \mathbf{B}^{-1}[\tfrac{1}{2}(1-\alpha)\mathbf{F}_i + \tfrac{1}{2}\alpha\,\mathbf{F}_{i+1} - \mathbf{A}_i\mathbf{q}_i + \mathbf{M}_{(A)i}\dot{\mathbf{r}}_i]$$

$$\dot{\mathbf{r}}_{i+1} = -\mathbf{M}_{(D)i}^{-1}[\tfrac{1}{2}\alpha\,\mathbf{F}_i - \tfrac{1}{2}(1-\alpha)\mathbf{F}_{i+1} + \mathbf{C}_i\mathbf{q}_i + \mathbf{D}_i\mathbf{r}_{i+1}]. \tag{5}$$

For $\alpha = 0$ we obtain the identical scheme as in (4). The incremental formula has a similar form, with displacement and force impulse increments instead of total values. $\mathbf{M}_{(A)i}$ and $\mathbf{M}_{(D)i}$ are upper left and lower right quarters of mass matrix (2).

2.3. *Solution scheme*

There are two possible numerical algorithms for solution of the equation system with variable coefficients.

The first: we assemble a single-layer equation, holding the remaining matrices to be used in the next step. We keep the following tables:

$$\begin{matrix} \mathbf{C}_i & \mathbf{D}_i + \mathbf{A}_{i+1} & \mathbf{B}_{i+1} \\ & \mathbf{C}_{i+1} & \mathbf{D}_{i+1}. \end{matrix}$$

Three of them are triangular (**B** and **C**) and two quadrangular (**A** and **D**). We obtain $\mathbf{r}_{i+1}$ from the equation

$$\mathbf{B}_i\mathbf{r}_{i+1} = \mathbf{F}_i - \mathbf{C}_i\mathbf{r}_{i+1} - (\mathbf{D}_i + \mathbf{A}_{i+1})\mathbf{r}_i \tag{6}$$

which is rather fast since $\mathbf{B}_i$ is triangular.

The second: we keep only one triangular matrix $\mathbf{B}_i$, one quadrangular $\mathbf{D}_i$ and one temporary vector $\mathbf{t}_i$ which initially depends on the starting conditions. We solve the equation

$$\mathbf{B}_i\mathbf{r}_{i+1} = \mathbf{F}_i - \mathbf{A}_i\mathbf{r}_i - \mathbf{t}_i \tag{7}$$

for $\mathbf{r}_{i+1}$ and prepare $\mathbf{t}_{i+1}$ for the forthcoming step

$$\mathbf{t}_{i+1} = \mathbf{C}_i\mathbf{r}_i + \mathbf{D}_i\mathbf{r}_{i+1}. \tag{8}$$

Products $\mathbf{C}_i\mathbf{r}_i$ and $\mathbf{A}_i\mathbf{r}_i$ can be computed during element matrix calculation. This two-step procedure is similar to the velocity formulation (5). It is more efficient than the first way of solution considering the memory requirements but it needs more arithmetical operations (multiplications) per step.

One characteristic and interesting property of simplex-shaped elements must be mentioned here. That is the limited speed of wave propagation in the direction of slope edges. A regular mesh with slope sides directed identically shows the anisotropy in time, that is the infinite wave speed in one direction and finite speed in the other [7]. It can be useful in some wave propagation problems, i.e. shocks placed to a point. Isotropic propagation can also be achieved by a special partition.

3. MESH ADAPTATION

3.1. *Error indicator*

Many different error estimators have been suggested in the literature [1–3, 16–18]. It is not the purpose of this paper to verify the error measures proposed by the authors. However, we select those satisfying the following requirements:

(i) high speed of error estimation;
(ii) dimensionless form and normalized value, bounded $0 \leqslant e \leqslant 1$;
(iii) possibility of changing the feature that is to be tested; error estimation without necessity of stress calculation, based on the displacement, can occur efficiently in motion investigation of the structure.

In our case we must optimize the mesh distribution for a fixed number of elements. Let us denote H as a mesh size. The distribution of the error is given by the integral over the spatial domain (for example [2]).

$$\int_{\mathrm{A}} H(u_{,xx}^2 + u_{,yy}^2)\,\mathrm{d}x\,\mathrm{d}y = \text{constant}. \tag{9}$$

Modifying (9) due to the interpolation formulae we obtain the error measure for a joint i

$$e_i = \left[\sum_k \left(\int_{\mathrm{A}} \sum_m \sum_j \mathbf{N}^i_{,k}\mathbf{N}^j_{,k} u^j_k \,\mathrm{d}\mathbf{A}\right)^2\right]^{1/2}. \tag{10}$$

j denotes joint numbers in an element m to which node i belongs. u^j_k is the displacement k of the joint j. Since the shape functions $\mathbf{N}_{,k}$ are linear, the derivatives **N** are constant and the form (10) can be given explicitly increasing the efficiency considerably.

3.2. *Stability restrictions*

The time integration scheme (4) can be tested regarding the stability, when one of the joints is moved in time. The following form of (4) is tested

$$\begin{Bmatrix} \mathbf{r}_{i+1} \\ \mathbf{r}_i \end{Bmatrix} = \mathbf{T}\begin{Bmatrix} \mathbf{r}_i \\ \mathbf{r}_{i-1} \end{Bmatrix} \tag{11}$$

where **T** is an amplification matrix. The sufficient condition for (11) to be stable is that any norm of **T** should not be greater than $1 + \alpha\Delta t$, where α is a positive number. In the paper we test the eigenvalues λ_i of **T**, for which we assume the condition

$$|\lambda_i| \leqslant 1, \quad i = 1, \ldots, n. \tag{12}$$

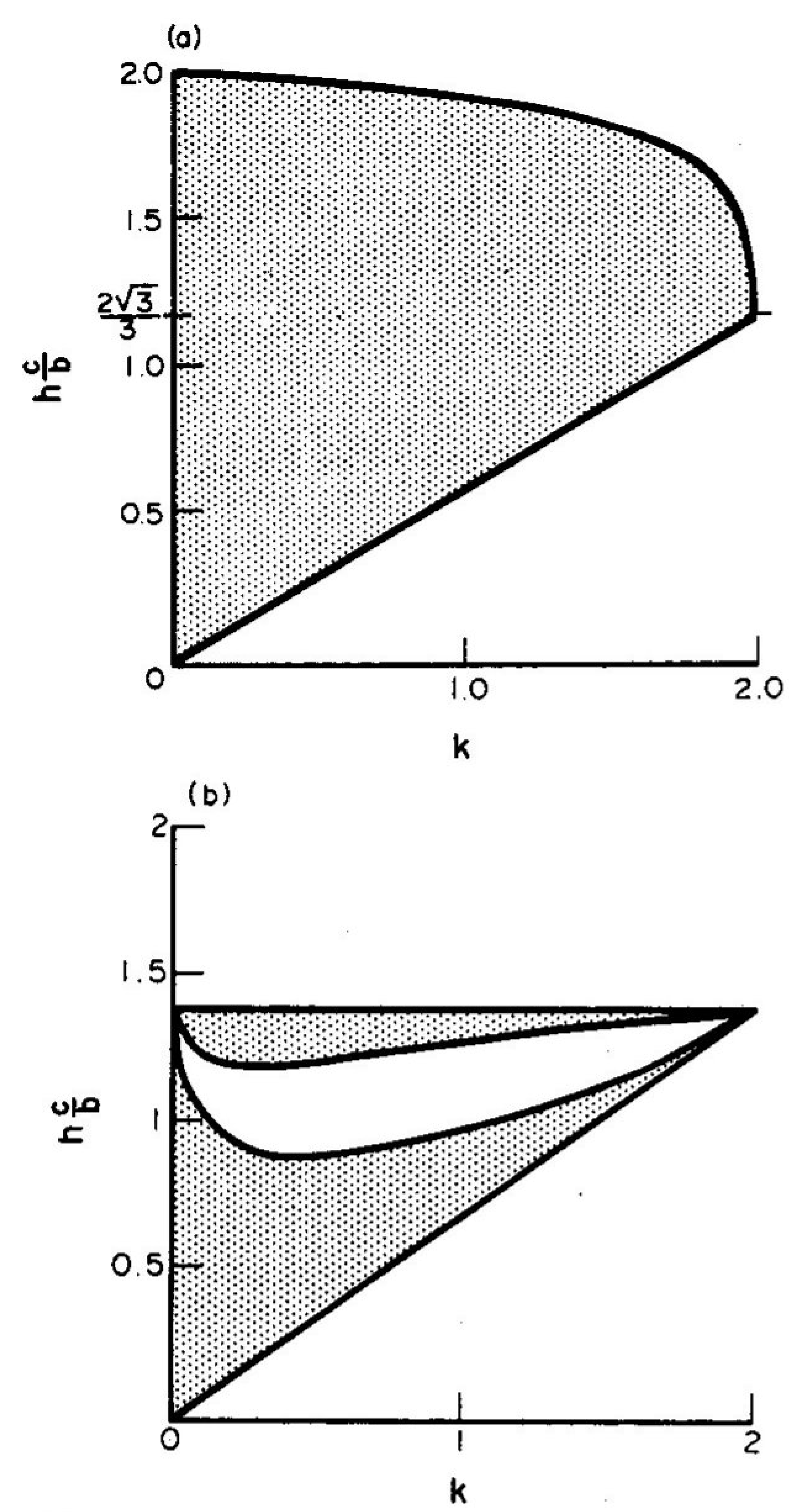

Fig. 3. Stability domain for quadrangular and triangular bar elements.

The first analytical attempt has been done for a uni-dimensional axial vibration bar. In the dimensionless coordinates

$$k = \frac{d}{b}, \qquad K = \frac{ch}{b}, \qquad c^2 = E/\rho, \tag{13}$$

where b = the element length, d = spatial dislocation of a joint in one time step, h = time step, c = wave speed, we can determine the stability domain. k is the Courant number and can be considered as the mesh modification factor. Considering quadrangular elements in space and time the stability area is bounded by the curves

$$\tfrac{1}{3}k^2 \leqslant K^2 \leqslant k^2 \Bigg/ \left(3 - \frac{3k}{\ln\dfrac{2+k}{2-k}}\right). \tag{14}$$

The diagram in Fig. 3(a) presents this area while Fig. 3(b) shows it for triangular elements.

An analogous investigation can be carried out for tetrahedral plane stress/strain elements. The analytical calculation is troublesome and results given in Fig. 4 could be obtained by numerical methods of analysis.

Some other cases being tested can be found in [8]. Presented results of stability investigation are not absolute since they concern selected mesh patterns. They are representative, however, and are helpful in applications.

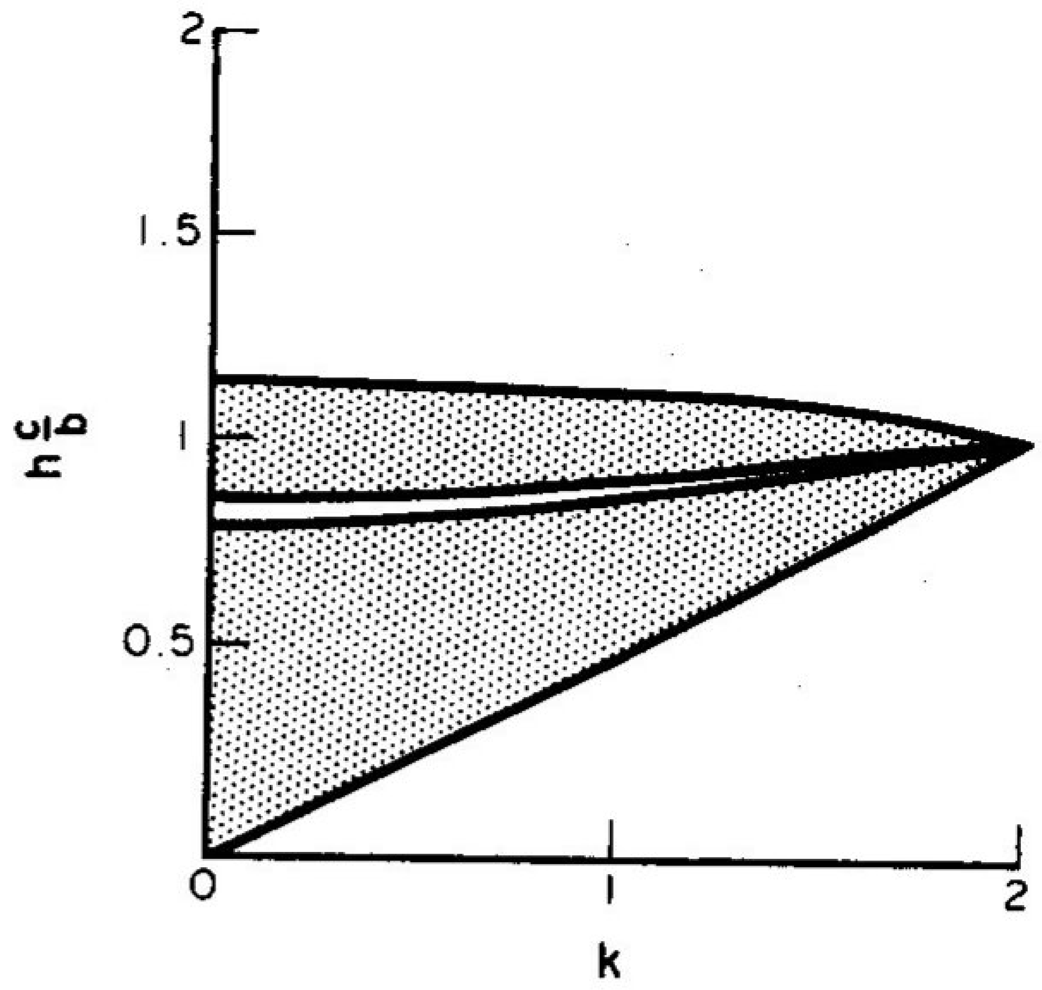

Fig. 4. Stability domain for tetrahedral plane strain elements.

3.3. *Algorithm of mesh modification*

The following steps describe the mesh geometry modification:

1. Calculation of the nodal values of the error using for example (10).
2. Normalization of nodal errors.
3. Calculation of movement components for a joint.

More details should be written about step 3. Concentration of stresses in subjected or supported regions is obvious. Mesh refinement, however, makes the coefficient matrix singular. The condition number, e.g. the quotient of the maximum eigenvalue by the minimum one, becomes large. The solution error grows and forces the successive mesh rearrangement. In a feedback the displacements in some regions (usually in corners) grow dramatically. However, not only elliptic parts of the differential equation generate instabilities. The element size decrease requires time step reduction. If it is not being done or if the element is allowed to be too small, the stability of time marching scheme is lost.

New coordinates are computed as a position of the center of gravity of all joints being in direct connection with the considered joint. The distance of a joint translation in one step counted in step 3 is always shorter than the average element spatial size h in surrounding elements. To avoid the loss of stability we must correlate the joint movement distance d with the time step Δt, wave speed c and element edge b. To ensure the stability the movement distance is multiplied by the parameter k taken from the diagram (Fig. 2 or 3) for the applied time step Δt.

4. NUMERICAL EXAMPLES

Several test problems were solved to estimate the accuracy of the described approach. Below three examples composed of a great number of finite elements are presented to prove the efficiency in the cases of axial vibrations of rods, flexural vibrations of beams and plane strain structures. Uni-dimensional tests are considered since the results can be verified intuitively with ease. However, restrictions are the same as in the case of multidimensional structures.

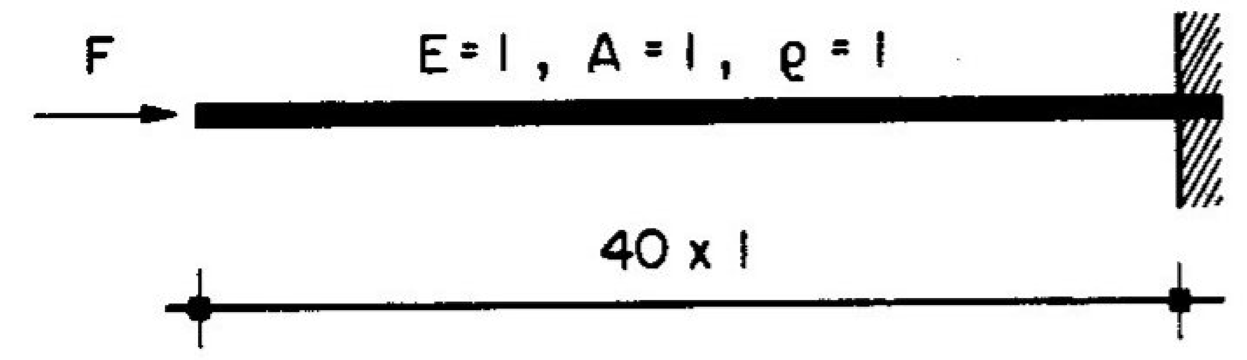

Fig. 5. Axial vibration of a bar—sample problem.

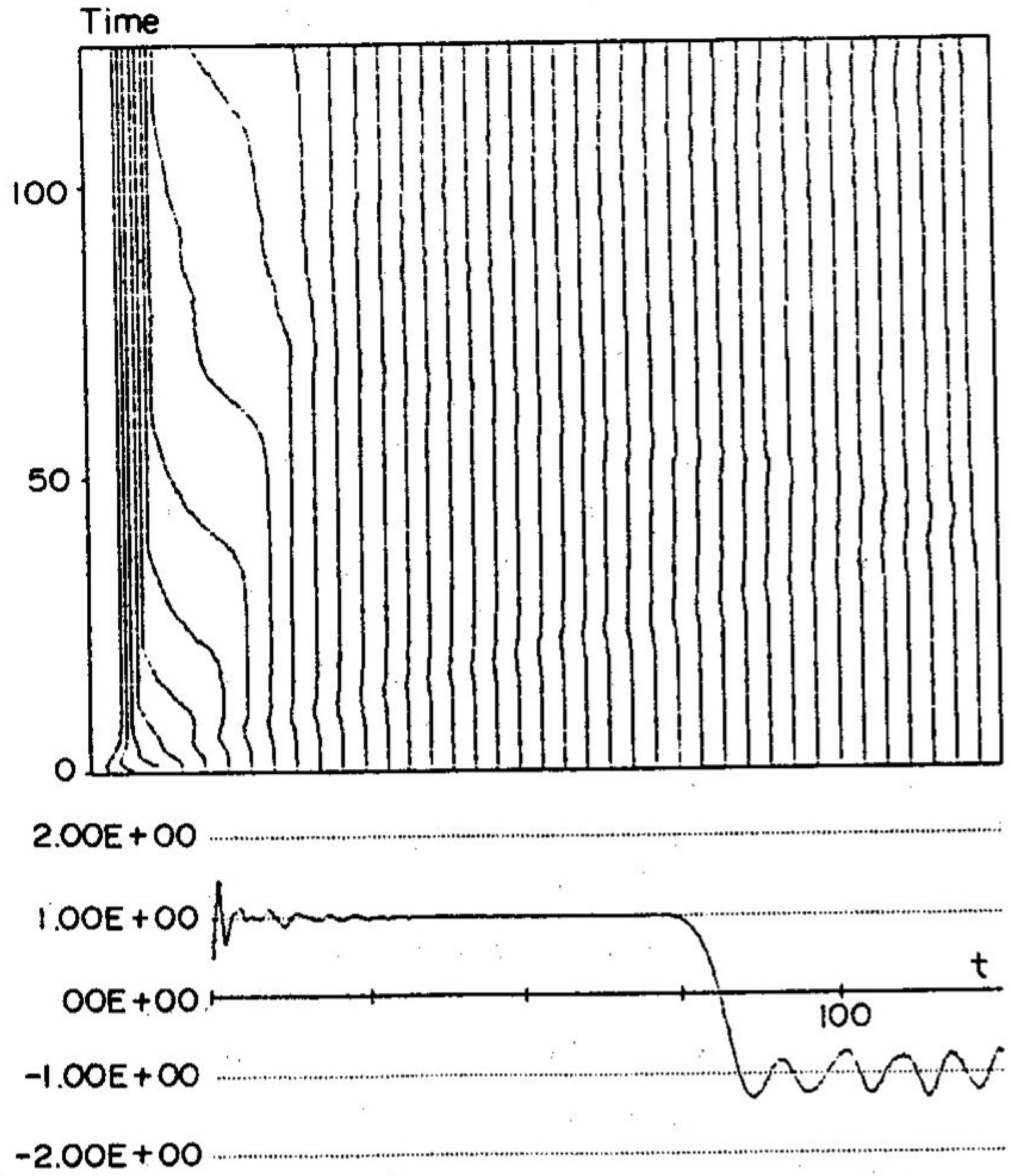

Fig. 6. Mesh evolution in the bar subjected to an impulse placed at the end.

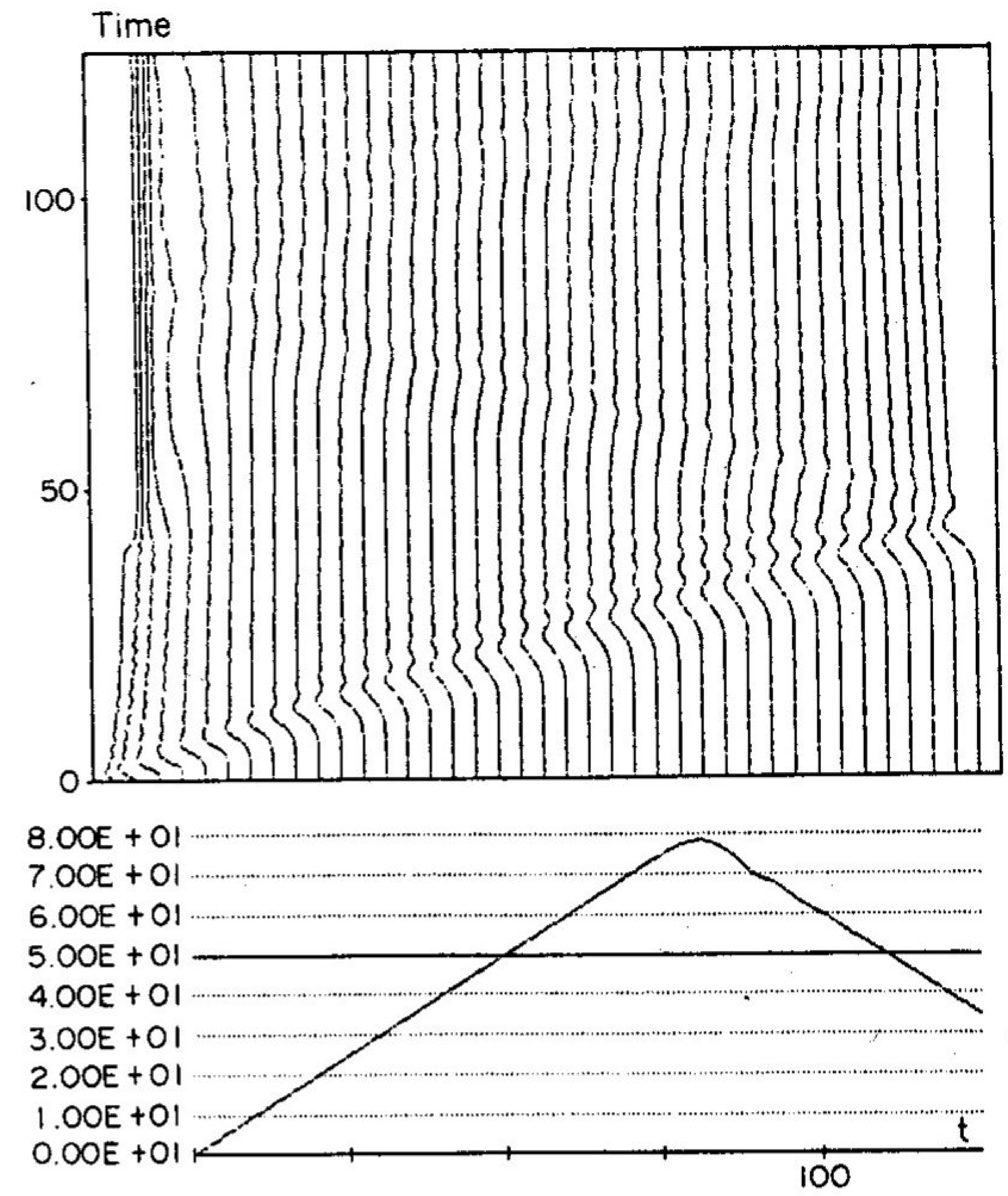

Fig. 7. Mesh evolution in the bar subjected to a Heaviside load placed at the end.

The system of 40 bar elements (Fig. 5) fixed at one end is subjected to an impulse. Location of joints can vary according to the error criterion. A time function of the spatial partitions is depicted in Fig. 6. In the same figure, one can find the displacement in time of the free end. If we put the Heaviside force to the free end point, the displacements of joints in the mesh are as depicted in Fig. 7. In both cases we can notice the wave path and reflections from the ends.

As the second problem, the beam split into 10 finite elements was investigated. It is not trivial to find a correct error estimator. We can choose vertical deflection or rotation as a feature to be tested. Joint path is exhibited in Fig. 8. Figure 9 shows the deflection in time for uniform partition (a), for mesh modification with the deflection as an error feature (b) and for the mesh modified with the rotational error indicator (c). Here we can see, from the full error analysis, that the error estimator should be worked out more precisely to obtain better results. Trajectories of joints can be related to the results described in [19] for diffusion problem.

The third case is the cantilever-shaped plane strain. A rectangular form with two right corners fixed and the lower left one subjected to a Heaviside vertical point force was computed with time step $\Delta t = 0.2$ ($E = 1.0$, $\nu = 0.2$, $\rho = 1.0$, $\eta_z = 0.1$, $\eta_w = 0.1$, mesh modification factor $k = 0.1$). Initial mesh and

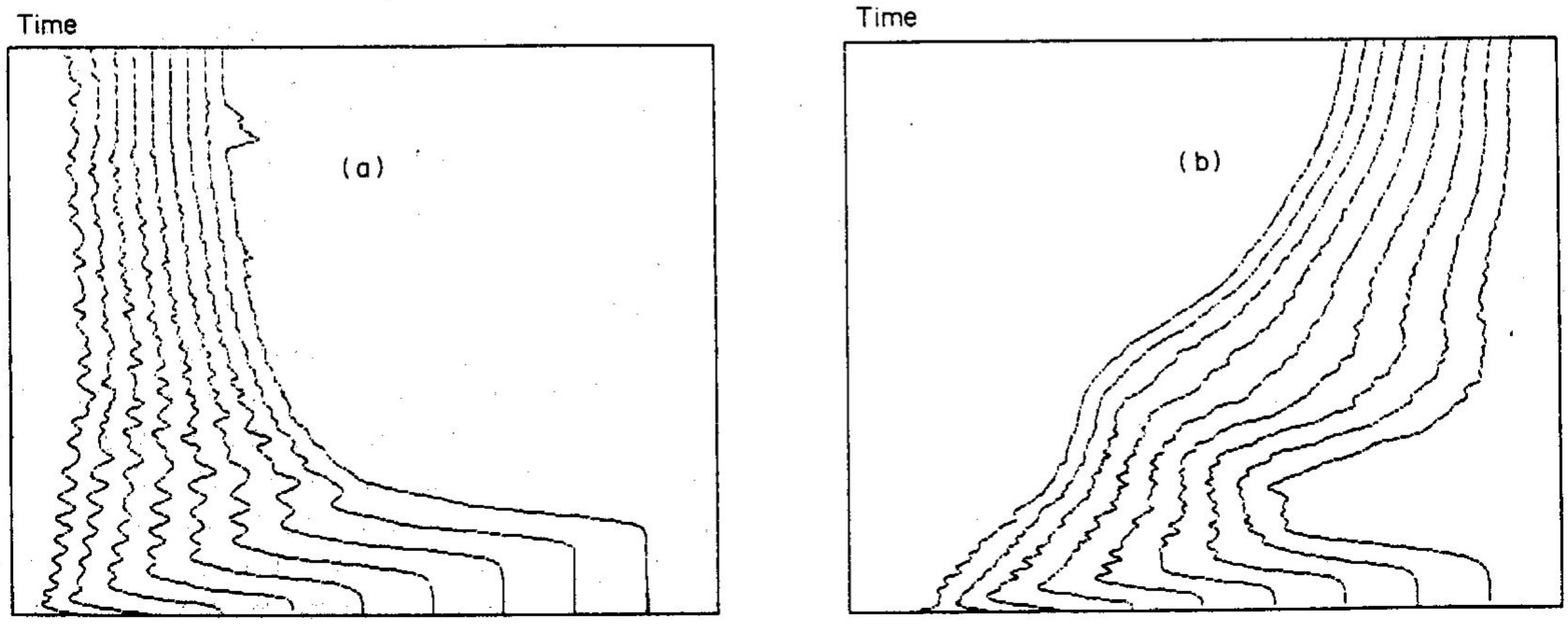

Fig. 8. Mesh evolution in the case of a beam assuming (a) deflections and (b) rotations in the error estimation.

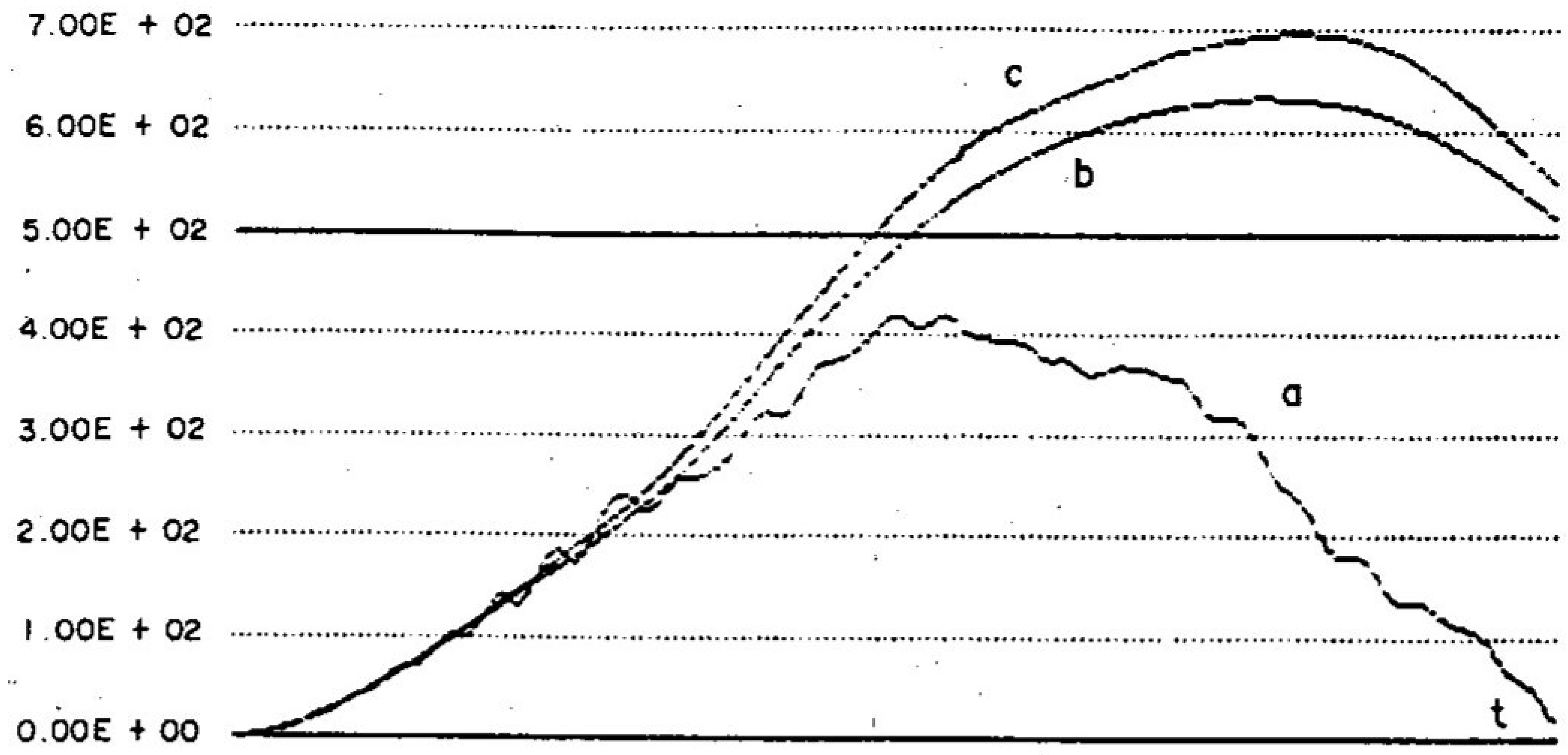

Fig. 9. Displacements in time assuming (a) deflectional error estimator, (b) rotational error estimator, (c) constant spatial partition.

modified forms at times 40.0 and 70.0 are depicted in Fig. 10. The comparison between the solution with the mesh constant in time and the adjusted mesh shown in Fig. 11 exhibits the magnification of the higher mode vibrations.

5. CONCLUSIONS

The most important features of the proposed approach are collected below:

- Relatively higher cost of the element and global matrix formulation. However, this effort is not consumed for nothing. The full approximation in space and time allows us to modify the spatial formulation in time in a natural way. One can notice that in the case of stationary discretization matrices $\mathbf{A}_i$ and $\mathbf{D}_i$ are equal. Also the symmetry of stiffness and mass matrices results in identity of coefficients in $\mathbf{B}_i$ and $\mathbf{C}_i$. Then the cost of matrix formulation is comparative with the cost of other time integration methods although the order of space–time element matrices is equal to the element number of degrees of freedom.
- Low cost of computation of resulting vector per step. Number of arithmetical operations is decreased because of the triangular form of matrices in the system of algebraic equations to be solved.
- Conditional stability, considering time step. In the case of nonlinear problems we must apply short time step ensuring required accuracy of approximation.
- Unconditionally stable schemes are also available. The use of simplex-shaped elements makes the formulation for arbitrary finite element models difficult. However, linear shape functions seem to be good enough and can be successfully applied also for bending structures. Much better element approximation can be achieved when we use multiplex-shaped forms.
- Easiness of nonstationary adaptive mesh usage. Simple algorithms allow one to adjust the mesh edges with some characteristic lines, to move the mesh along with the travelling load etc.

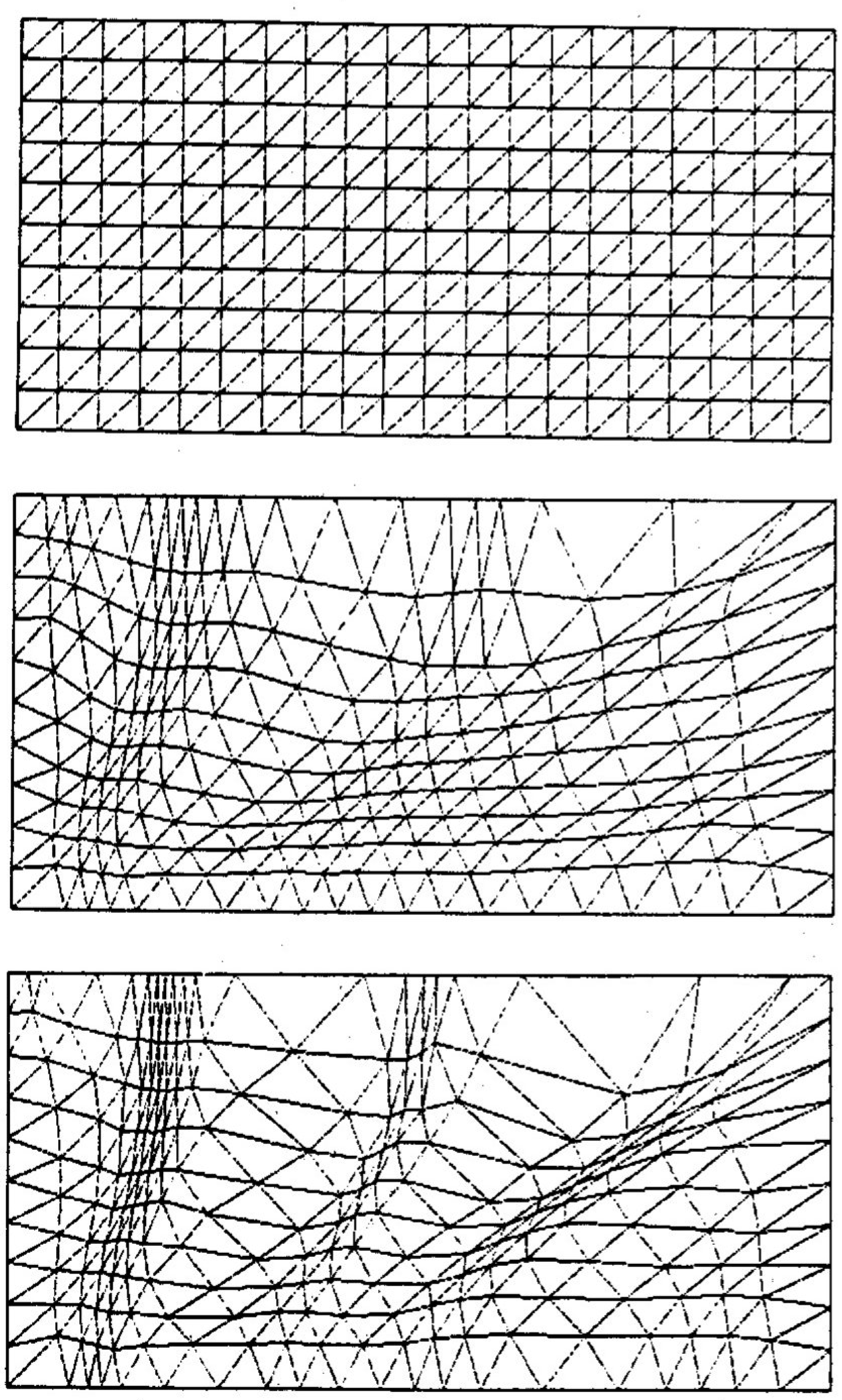

Fig. 10. The mesh after $t = 0.0$ (initial), $t = 40.0$ and $t = 70.0$.

All the examples prove the efficiency of the approach presented. However, the efficiency depends on the assumed error criterion. Assumption of an appropriate error criterion is not a subject of this paper. Some good indications can be found in the literature and successfully applied. The present approach, without increasing the computational cost, can be applied to problems in which recalculation of element matrices must be carried out anyway. In the case of typical linear or even nonlinear problems with the stationary mesh, the space–time element method is expensive because of the greater number of d.o.f. in the space–time element and greater number of

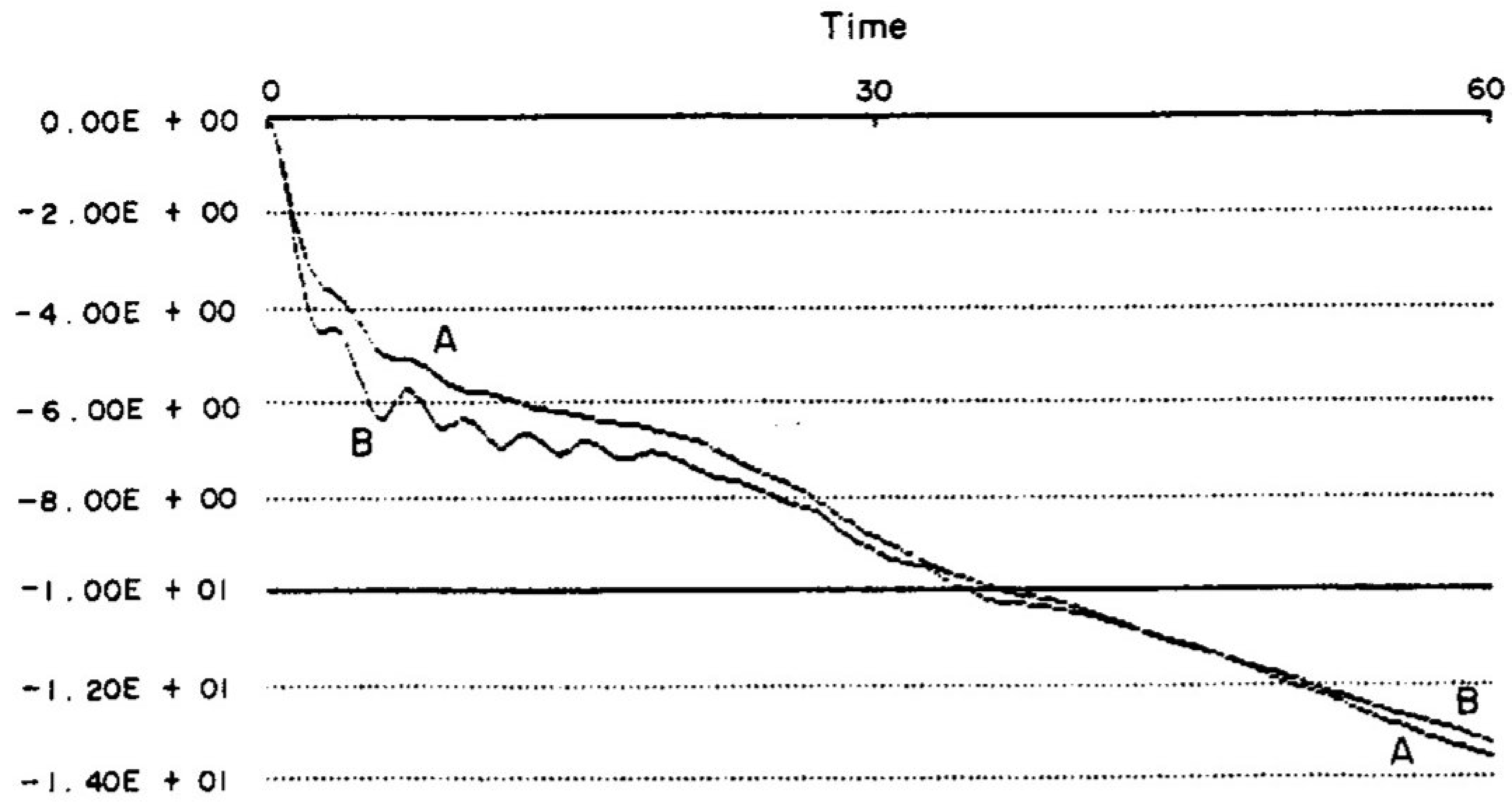

Fig. 11. Vertical displacements in time of the subjected point in the case of constant mesh (A) and modified mesh (B).

simplex-shaped elements used to describe one spatial finite element.

Good error estimators should be used in the case of each type of structure. This is well exposed in the second example in which direct consideration of both deflection and rotation does not allow one to obtain good accuracy. The third example shows that transient higher frequency vibrations can be emphasized by the use of a movable grid technique.

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