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FEATURE ARTICLE: Items of special interest, tutorials, and surveys

STATE-OF-THE-ART IN TRUE SPACE-TIME FINITE ELEMENT METHOD

C.I.Bajer* AND C.G. Bonthoux*

Abstract. The space-time finite element method is rarely used in solutions of engineering dynamic problems. Classic time integration methods are usually included in computational procedures. However, nonstationary spatial discretization cannot be assumed when spatial and time domains are separated. Recent developments of the finite space-time element method allow application of approximation techniques to the spatial and time domains. Special schemes lead to highly efficient algorithms that consider both memory requirements and number of arithmetical operations. Recent work in the field of space-time elements applied to structural dynamics are described in the article. Some comparisons and relations to other computational methods are made.

Recent advances in structural dynamics, supported by the development of faster and larger computers, enable engineers to solve problems of increasingly complex structures. Dynamic behavior of highly discretized three-dimensional objects does not cause any significant problems because stationary solutions are assumed in resolving dynamic problems. Integration of the differential equation of motion is usually performed by simple or selected time integration procedures. Spatial variables are separated from the time variable. Solution in space is searched by any discrete method and a difference scheme is applied to the time domain. The primary design of a structure is considered throughout the entire investigation.

Such an approach limits possibilities of arbitrary modeling of geometry and the application of boundary conditions or other spatial data in time. If problems with variable coefficients, material nonlinearities, and contact effects are considered computational effort increases rapidly. On the other hand, new technological problems require more complex solutions. Because microcomputers are now in common use, the numerical efficiency of algorithms must be carefully examined.

New possibilities are obtained by the true space-time finite element method. The time domain is regarded in the same way as are spatial variables. Recent achievements in space-time technology are described in this article. Some examples prove the efficiency of the method in solutions of atypical problems.

HISTORICAL BACKGROUND

Early approaches to the space-time modeling of physical problems were published in 1969 by Oden [1], Fried [2], and Argyris and Sharpf [3]. In the first paper readers can find an attempt of generalization of the finite element method over the time domain. Unfortunately, the suggestion of arbitrary partition of time space was not continued. More papers appeared on the simple time integration of differential equations based on the stationary partition. Fried [2] indicated simplifications that can be gained when noncontinuous time boundary conditions are considered. Formulations were supported only by the example of a one-degree-of-freedom vibrating system and a heat conduction example in which the time variable was still independent of space. At the same time Argyris and Sharpf [3] formulated a similar approach. Although time is included in derivations of basic quantities, all of the papers treat the problem by Hamilton's principle. Alturi [4] gave a more general approach in 1973.

More advanced description in space and time was given by Bonnerot and Jamet [5,6] for heat transfer problems. New ideas in modeling a movable edge should be noted. In successive work [7,8] time integration schemes were still called by the term space-time element. Both spatial and time variables were approximated by different and separated functions. The paper by Kujawski and Desai [9] is a good example of such an approach. Three time level procedures described can be unconditionally stable when some constant parameters are properly chosen.

Simultaneously considerable progress in numerical methods of direct integration of differential equations for stationary problems was made. Fundamental work was published for stationary problems [10-20]. Faster, more efficient, and unconditionally stable methods

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were competing for accuracy 21-27. In each case a space discretizing method was applied to the structure and then a difference method was used for time integration. In structural dynamics static mass and stiffness matrices allowed consideration of continuous variation in time boundary conditions or spatial domain. Some review papers show high performance in stability and accuracy analysis [28-33]. However, in many systems one or a few concentrated masses were used to prove the efficiency of the methods. Continuous objects are rarely found in these papers.

The first complete and clear space-time approximation for structural vibrations was made by Koczkowski and Langer [34-36]. Strains and stresses are approximated in the space-time domain. There is no difference between spatial and time variables. Initial conditions can be regarded as boundary conditions on the constant time layer. Only such a formulation can be understood as an extension of the finite element method. A similar approach was presented by Riff and Baruch [37,38]. Although the early papers describe uni-dimensional structures and although a conditional stability is still the main disadvantage of the formulation, the space-time finite element method quickly improved. Space-time elements of non-rectangular shapes were introduced [39,40]. A triangular beam element of arbitrary joint location was used to present a space-time element approach to contact problems. A decrease in the integration time step for some spatial regions with higher speed of displacement is possible with the use of triangles. A formulation of an unconditionally stable variant of the method for multiplex-shaped space-time elements [41] can be regarded as a further improvement. However, in this case the method is a time integration scheme because separation of spatial and time variables is possible. Recently developed simplex-shaped elements lead to highly efficient procedures.

Some recent applications of the space-time finite element method are described below. However, a general formulation is restrained by stability limitations. Arbitrary node location in time space cannot be successfully applied. Stability criteria are given for selected structures. Numerical examples prove the efficiency of the approach in different problems of structural dynamics.

FUNDAMENTAL CONCEPTS

In the space time finite element method (STFEM), interpolation functions are applied both to spatial and time domains. That is why, for uni-dimensional structures, there are two dimensional space-time elements and three-dimensional objects for two-dimensional problems.

Some sample space-time elements are depicted in Figure 1. Note that shape functions have one more variable -- the time variable -- than that found in the classical finite element method (FEM). Element matrices are larger because in STFEM a greater number of joints are considered in the element. The assembly of the global matrix [44] is performed in the same was as in the FEM. Element matrix $\mathbf{K}^*$ is composed of stiffness $\mathbf{K}$, inertia $\mathbf{M}$, and damping matrices $\mathbf{W}$ and $\mathbf{Z}$, depending on the rheological model of the body

$$\mathbf{K}^* = \mathbf{K} + \mathbf{M} + \mathbf{Z} + \mathbf{W} \tag{1}$$

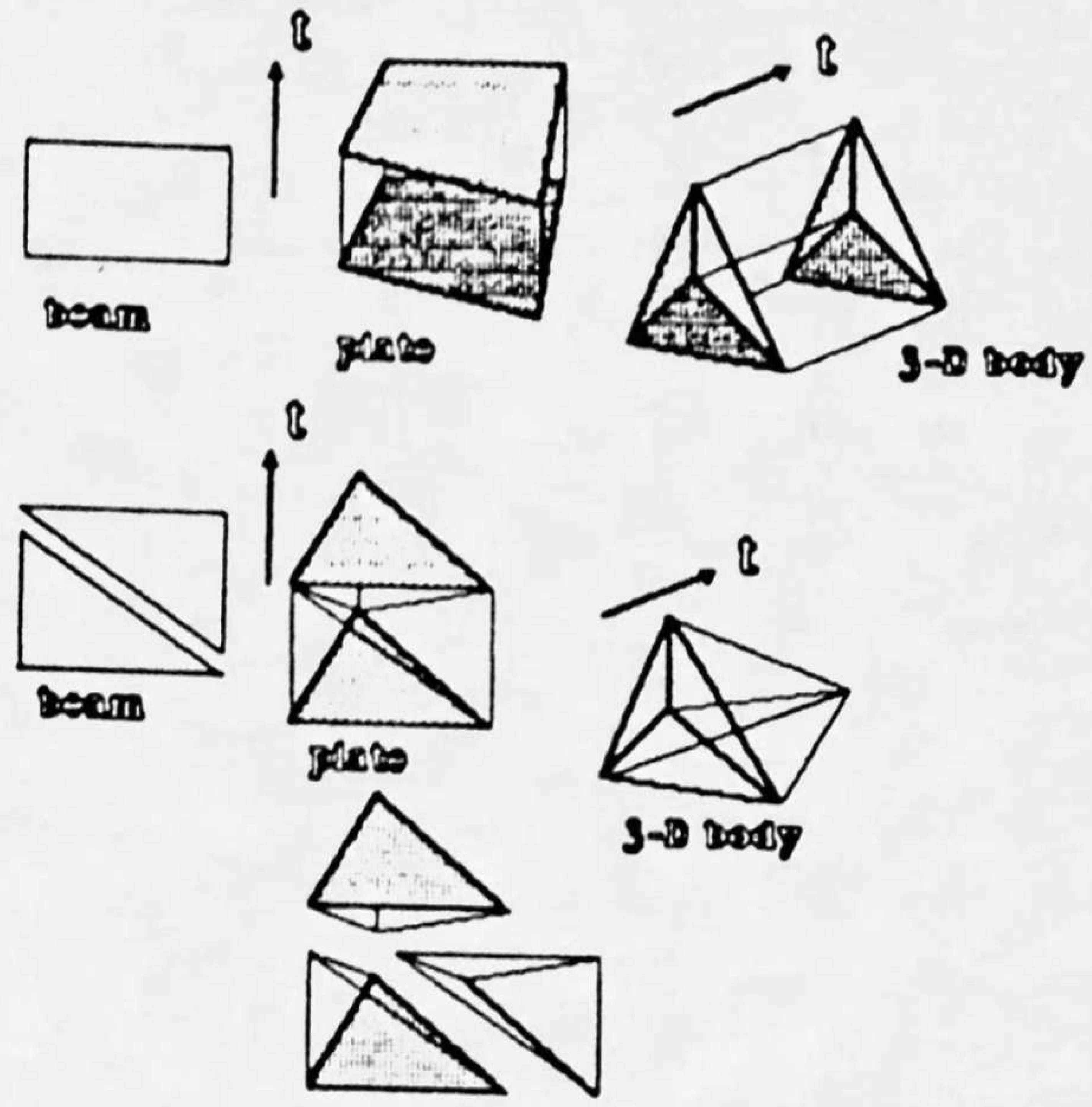

Figure 1. Examples of Space-Time Elements.

The assembly of global matrix leads to the half-infinite form

$$\begin{bmatrix} A & B & & & & \\ C & D & B & (0) & & \\ & C & D & B & & \\ (0) & & C & D & B & \\ & & & \cdots & \cdots & \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \vdots \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \end{Bmatrix} \tag{2}$$

that can be solved stepwise when the initial displacements $\delta_1 = \delta_{10}$ are known:

$$C\delta_{i-1} + D\delta_i + B\delta_{i+1} = F_i \tag{3}$$

In equation (3) all matrices have dimensions equal to the total number of degrees of freedom in a structure. Initial speed can be included, assuming for example

$$\dot{\delta}_0 = \frac{1}{h}\left[(1-\beta)\delta_0 + \beta\delta_1\right] \qquad 0 \le \beta \le 1 \tag{4}$$

In early formulations space-time finite elements were of multiplex (quadrangular) shape. Interpolation functions were constructed by multiplying spatial and time terms

$$N_i(x,t) = M_i(x) \cdot T_i(t) \tag{5}$$

$M_i(x)$ were the same functions as in FEM. $T_i(t)$ were assumed as

$$T_1(t) = {}^1/_h\,(t_2 - t)$$

$$T_2(t) = {}^1/_h\,(t - t_1) \tag{6}$$

or in local coordinates

$$T_i(\tau) = {}^1/_2\,(1 + \tau\tau_i) \tag{7}$$

Apply equation (7) both to real and virtual displacements to obtain a conditionally stable formula. Modification of equation (7) by third power terms [41] leads to the unconditionally stable variant of the method

$$T_i(\tau) = {}^1/_2\,(1 + \tau\tau_i) + \alpha\tau_i(\tau^3 - \tau) \tag{8}$$

The system described with the use of equation (8) is unconditionally stable for $\geq$ 1.25. However, in the case of linear problems conditional stability does not cause any substantial difficulty. Each second layer of joints can be eliminated; the result is a space-time super element [42]. Then the system of equations has the following coefficients:

$$\begin{bmatrix} A-BD^{-1}B & -BD^{-1}B & & \\ -CD^{-1}C & D-CD^{-1}B-BD^{-1}C & -BD^{-1}B & \\ & -CD^{-1}C & D-CD^{-1}B-BD^{-1}C & -BD^{-1}B \end{bmatrix} \begin{Bmatrix} \delta_0 \\ \delta_2 \\ \delta_4 \end{Bmatrix} = \begin{Bmatrix} F_0^* \\ F_2^* \\ F_4^* \end{Bmatrix} \tag{9}$$

$$F_0^* = F_0 - BD^{-1}F_1$$

$$F_i^* = F_i - CD^{-1}F_{i-1} - BD^{-1}F_{i+1} \qquad i = 2,4,6,\ldots$$

Successive eliminations allow solutions for each 2^n instants. Although equation (9) is expressed in even moments, it is possible to eliminate odd layers so that the displacement vector is in even moments. Such a technique passes stability limitations for a super element of sufficiently high order. Unfortunately, this procedure cannot be applied to problems with variable coefficients. The super element technique can be applied to each explicit time integration method, not only to the space-time element method.

A range of possibilities for the simultaneous discretization in space and time was utilized in the following work. Nonstationary spatial discretization was used in a detailed investigation of the stability problem in which not time stability but instability caused by changeable joint location in successive time layers was considered.

EXAMPLES OF RESOLUTION WITH NONSTATIONARY PARTITION OF A STRUCTURE

The method is especially useful when the geometry of a structure is changed in time, as in the following group of problems:

- in robotics, where large dislocation of masses and changing geometry are assumed
- in jet plane design when changeable wing geometry is allowed
- in satellites while opening arms of the antennas

The second group of problems for STFEM includes problems of infinite structure subjected to a traveling load:

- rolling problems
- contact wheel-rail problem
- stress wave propagation under moving load

Consider space-time elements with nodes placed in different coordinates in each time step (Figure 2). Different time steps can be applied to chosen spatial parts of a structure. When both the stiffness of the system in the zone of interaction and the speed of displacements of two elastic bodies is high, the time partition in such a region can be condensed (Figure 3).

Approximation of the contact area with the element either entirely in or out of contact was used to investigate displacements in time of a beam placed on a unilateral Winkler foundation. The unilateral case was solved with the use of quadrangular beam elements supported by triangles for time condensation. The bilateral case was solved with the use of only triangular elements; constant joint location was assumed. Results can be seen in Figure 4.

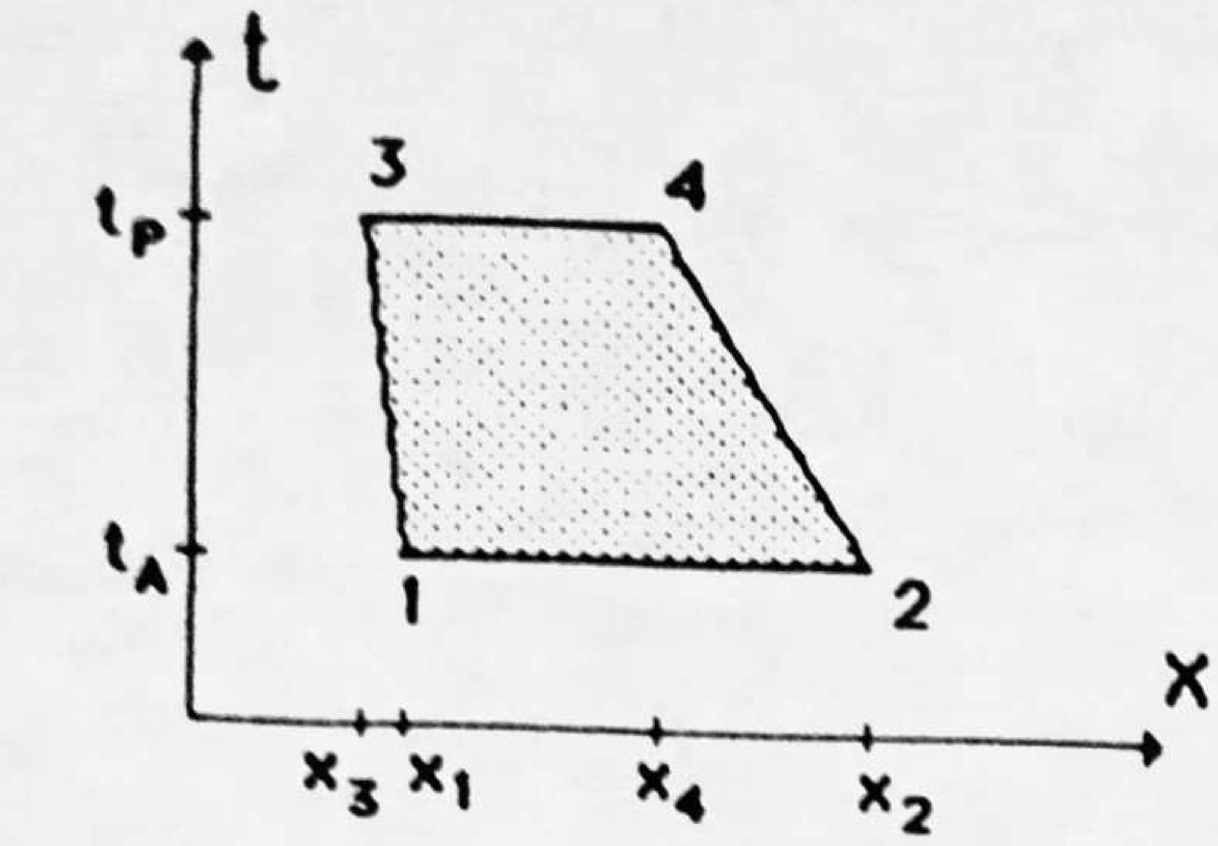

Figure 2. Nonstationary Division at Each Time Step.

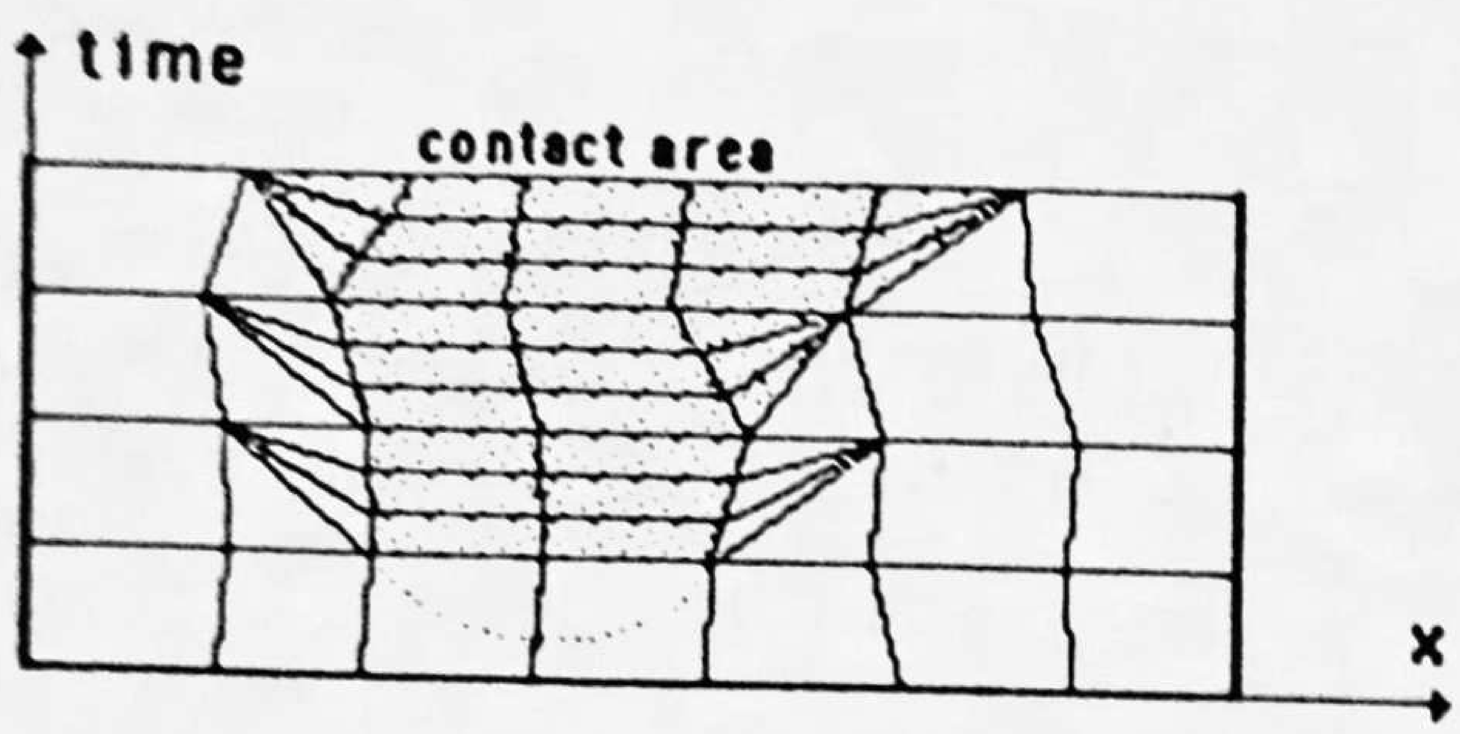

Figure 3. Time Division Condensation in Regions of Higher Stiffness.

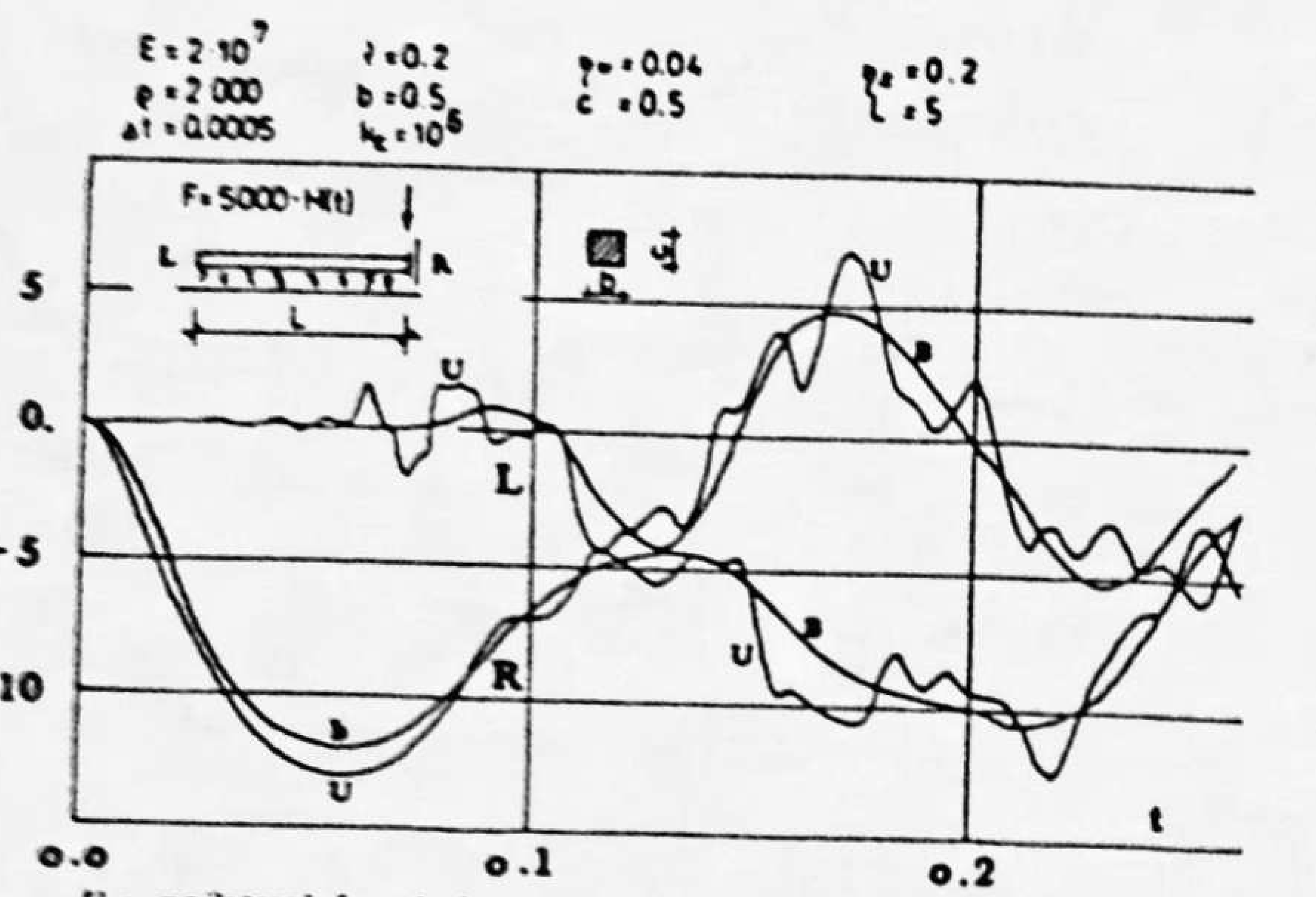

Figure 4. Vibration of a Beam Placed on Unilateral and Bilateral Foundations.

The next test problem was concerned with axial vibrations. The length of the bar was changed periodically along with the function $l(t) = 2.5-1.5 \cos (0.5t)$. Motion was excited by the inertia effects. A in Figure 5 presents displacements in time of selected points and was made for the initial period. B represents later time. The initial disorders were damped by a small damping coefficient. Note the regularity of displacements.

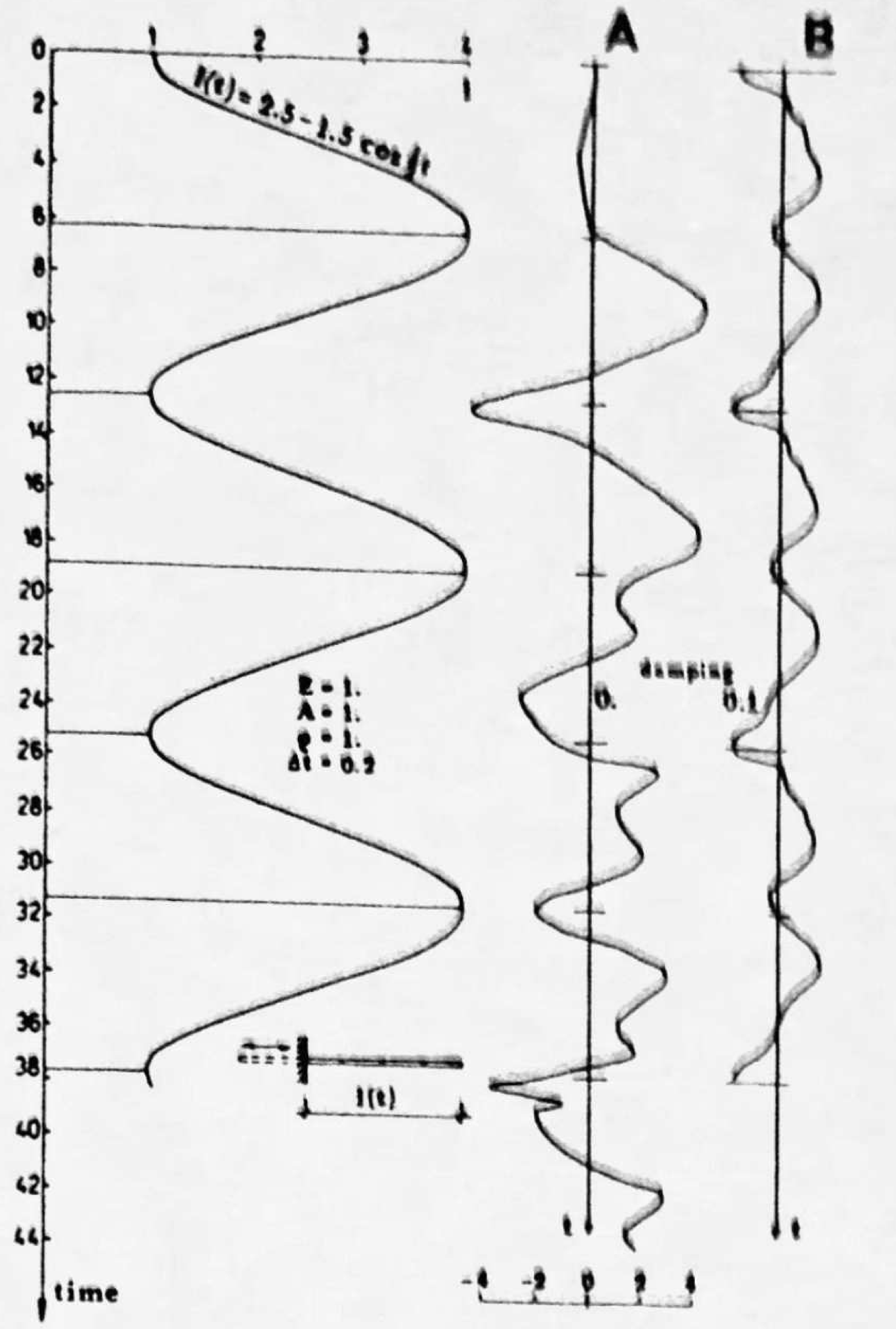

Figure 5. Displacement of the End of a Bar with Changeable Length.

The greatest savings can be achieved when movable load or contact zone is considered. The mesh is refined in the region of significant influence of forces, leaving the rough net in remote parts. Infinite ends of the band can be cut off (Figure 6). All nodal points of the mesh assumed only in limited distance from the load are displaced with the traveling force. The local traveling coordinate system allows the mesh geometry to be described independently of time. Up to now, when the stationary mesh has been given, the identical partition in the entire range of contact had to be assumed. In such a case some spurious effects can be observed when the body in contact goes from one finite element to another one. The method proposed in this article simplifies the calculations. Wave phenomena can be observed more easily.

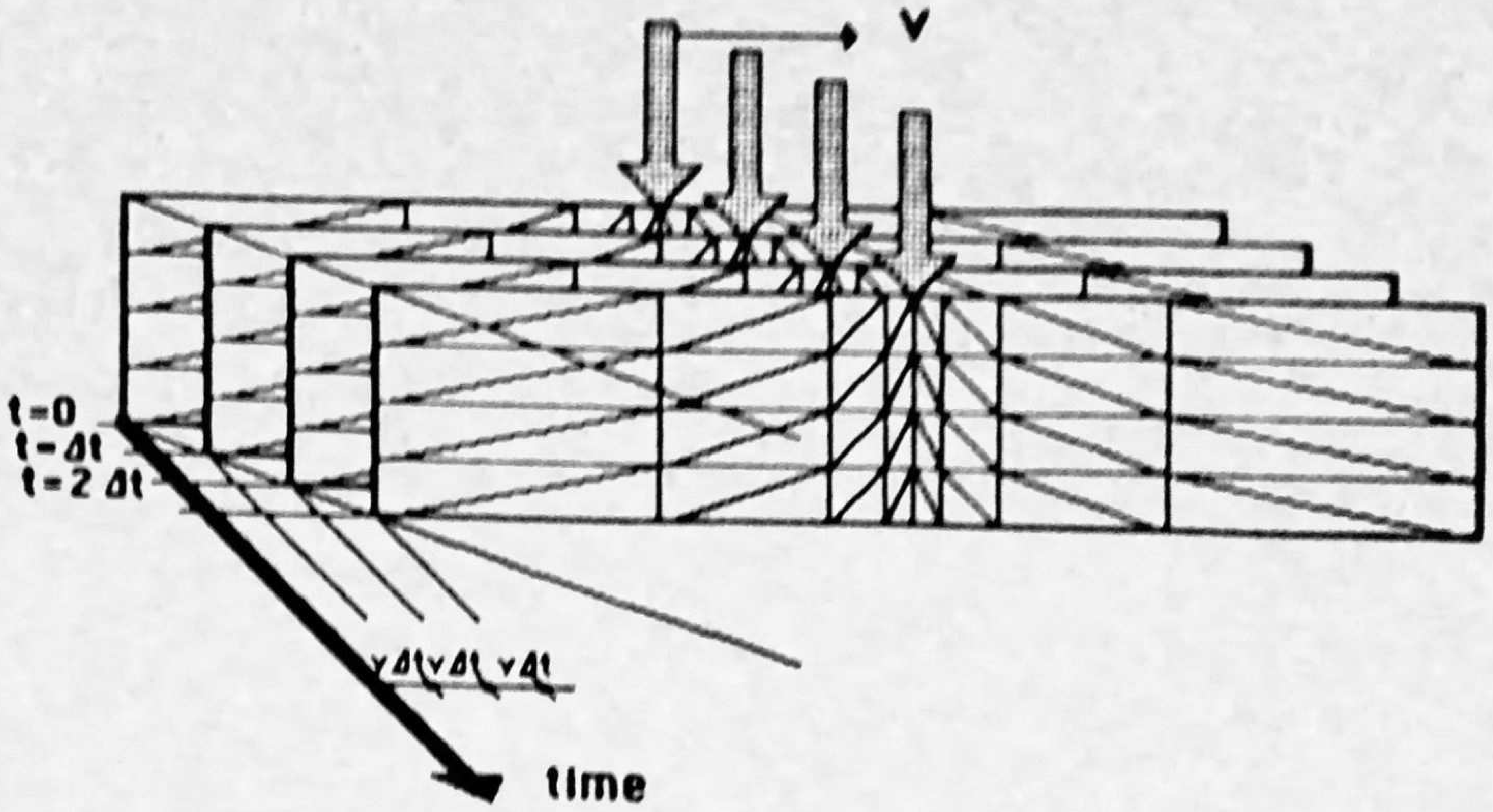

Figure 6. Movable Mesh in Surface Problems.

However, the arbitrary joint location cannot be taken without danger of loss of stability. Analytical calculations and experimental observations proved that limitations are imposed on the coefficient ζ, which is the speed of joint displacement related to the wave speed c

$$\zeta = {}^{d}/_{h} \cdot {}^{1}/_{c} \tag{10}$$

d = displacement of the joint position in successive time steps (Figure 7)
h = time step
c = wave speed, $c^2 = {}^{E}/_{\rho}$

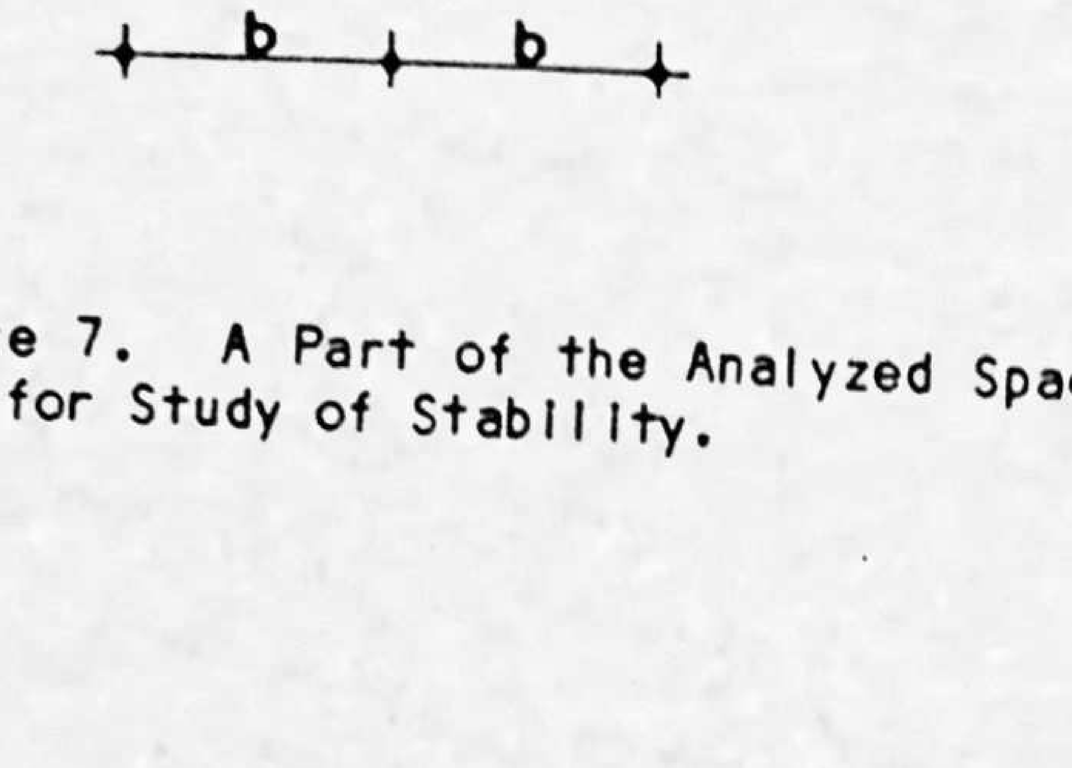

Figure 7. A Part of the Analyzed Space-Time Band for Study of Stability.

investigations were done only for selected structures. The maximum values of parameter ζ for selected structures are listed below:

- a bar in longitudinal vibrations, modeled by quadrangular elements, $\zeta<\sqrt{3}$
- longitudinally vibrating bar modeled by triangular elements, $\zeta<\sqrt{2}$
- beam element (quadrangular and triangular) $\zeta<1.5$

These values can serve as indicators in other cases. In engineering practice the rate ζ is not reached, even in wave problems.

Unfortunately, in general the simplex-shaped (triangular) elements lead to time conditionally stable schemes. The virtual shape function cannot be improved in a simple way. Analytical considerations are complicated and were successfully done only for the case of an axially vibrating bar element. Unconditional stability can be reached with the following functions of area coordinates of triangle L_i:

$$N_i = L_i^3 - {}^{3}/_{2}L_i^2 + {}^{3}/_{2}L_i - L_1L_2L_3 \tag{11}$$

$$\tilde{N}_i = L_i + \alpha(-2L_i^3 + 3L_i^2 - L_i + L_1L_2L_3) \tag{12}$$

where α is a function of the square of the Courant number $K = ch/b$ [10] and **b** is the spatial length of the element. For the greatest accuracy we assume [40]

$$\alpha(K) = \begin{cases} 0, & \text{for } K \leq \sqrt{2} \\ 30(K^2-2)/(3K^2-4), & \text{for } K < \sqrt{2} \end{cases} \tag{13}$$

BENEFITS OF SIMPLEX-SHAPED ELEMENTS

If the space-time elements have the shapes of triangles, tetrahedrons, or hyper-tetrahedrons, considerable savings of computational effort can be gained. Efficiency is significantly increased for several reasons: simplex-shaped elements lead to triangular forms of matrices (Figure 8) with smaller number of coefficients; second, triangulation of the coefficient matrix is unnecessary; so the solution can thus be carried out joint by joint [43,44]. The number of arithmetic operations per time step is reduced. A short analysis of efficiency for linear and nonlinear solutions is given below.

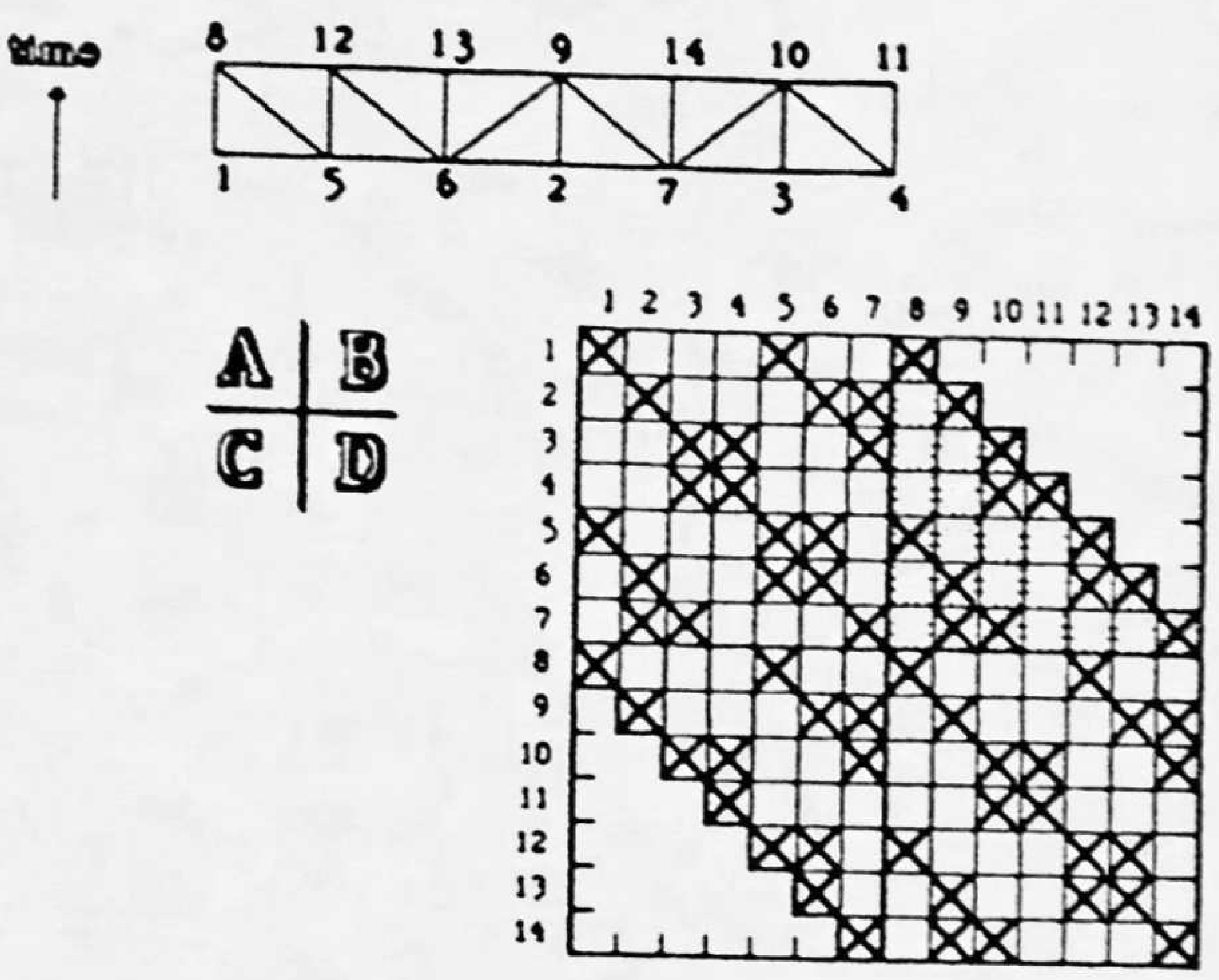

Figure 8. Space-Time Net for One Space-Time Layer and Its Global Matrix.

Number of multiplications M per step both for the linear and nonlinear variants of the method is described by the inequality

$$M < 2\ sN(p+1) \tag{14}$$

- N = total number of degrees of freedom
- s = nodal number of degrees of freedom
- p = number of joints connected with any given joint

For instance, consider a regular square mesh composed of 10x10 joints with one degree of freedom in a joint. The Newmark method requires 4,700 arithmetical operations. The present method needs only 1,500 or three times less.

Another reason for the increase in efficiency is that, because of the triangular forms of matrices and possibility of direct solution of the system of equations, only nonzero coefficients must be held. In other methods applied to band matrices some zero coefficients below the threshold become nonzero during solution, increasing the capacity requirements. In the present approach a large reduction of consumed memory and computational time is gained.

Storage requirements in the case with constant coefficients L_c and in the case with variable coefficients L_v are, respectively:

$$L_c < 2\ sN(p+1) + sN \tag{15}$$

$$L_v < {}^{7}/_{2}\ sN(p+1) + {}^{3}/_{2}\ sN \tag{16}$$

For these reasons small computers can be used for calculations, especially microcomputers.

Finally, this interesting simplex-shaped element mesh has other advantages in nonlinear problems but must be used carefully. An investigation of the form of triangular matrices shows the time anisotropy of the solution scheme [44]. When a regular triangular mesh is used and the force is applied in a joint, the speed of the information flow is different in both directions in successive time layers (Figure 9). The unique speed of wave propagation is reached when special partition of the space-time layer is assumed (Figure 10). It should be emphasized that the application of simplex-shaped elements does not affect the accuracy of results, but may be used if the anisotropy of propagation is to be conspicuous.

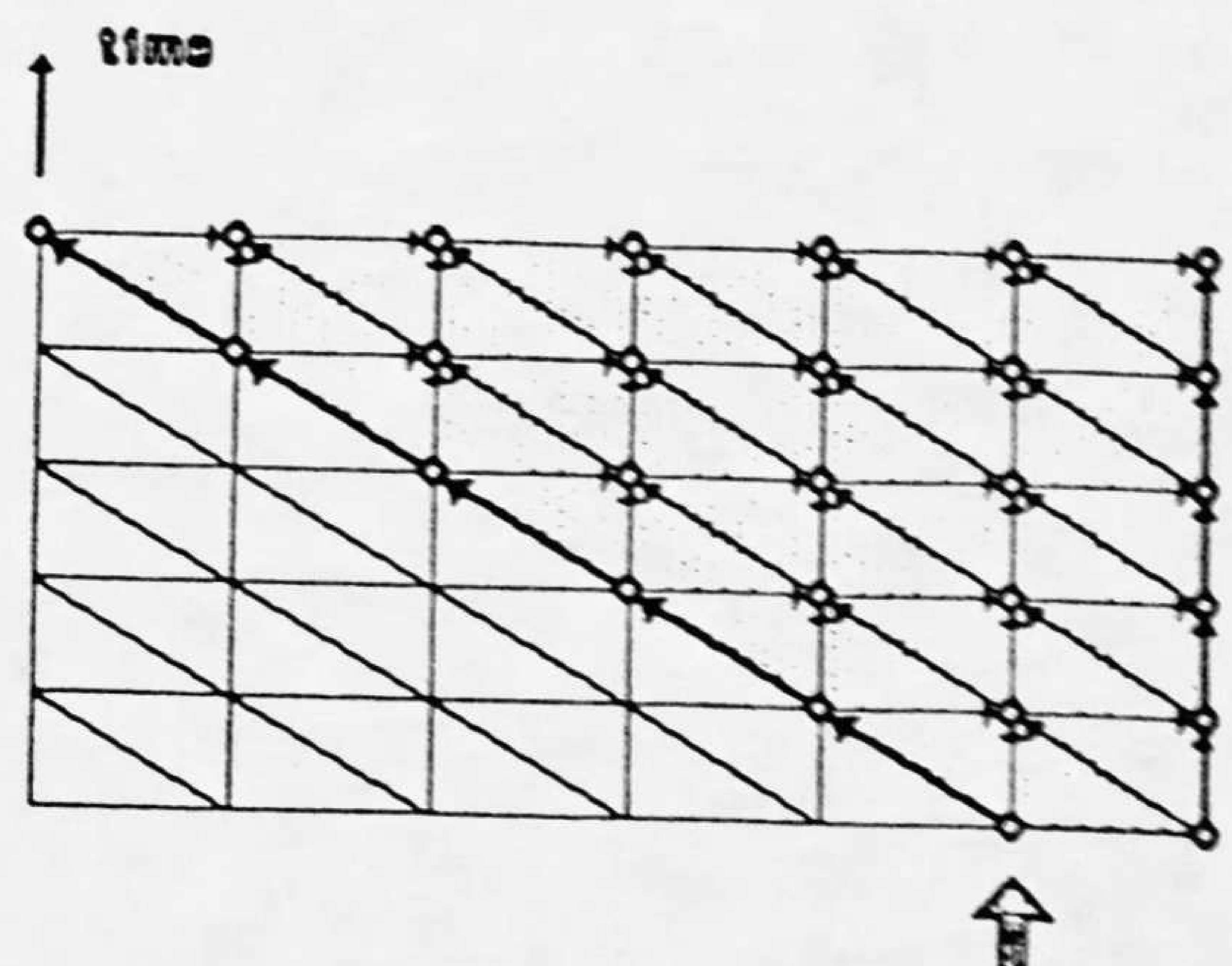

Figure 9. Information Flow in the Case of Successive Numbering.

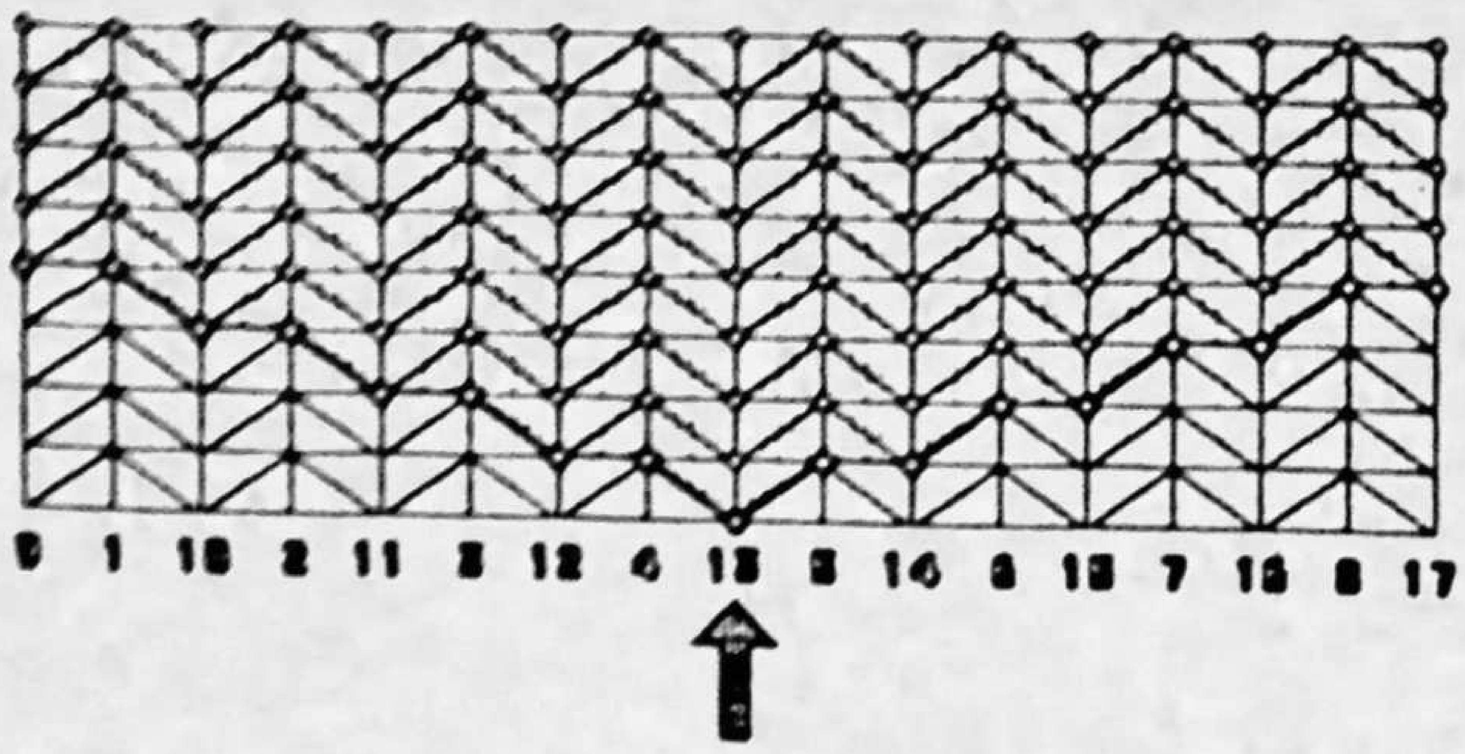

Figure 10. Isotropic Information Propagation.

A static three-dimensional problem described by Zienkiewicz [45] can be simply resolved in dynamics with this method. A scheme of the test problem is given in Figure 11. The system consists of 125 joints and 384 tetrahedral spatial elements. It is subjected by a heaviside force of the value 1. In Figure 12 displacements of joints under the load are drawn. Note the coincidence with the static results given by Zienkiewicz. Calculations were carried out on a microcomputer compatible with IBM-PC (8MHz, 8087). After an assembly of global matrices that lasted about five minutes, one step took four seconds of the computational time.

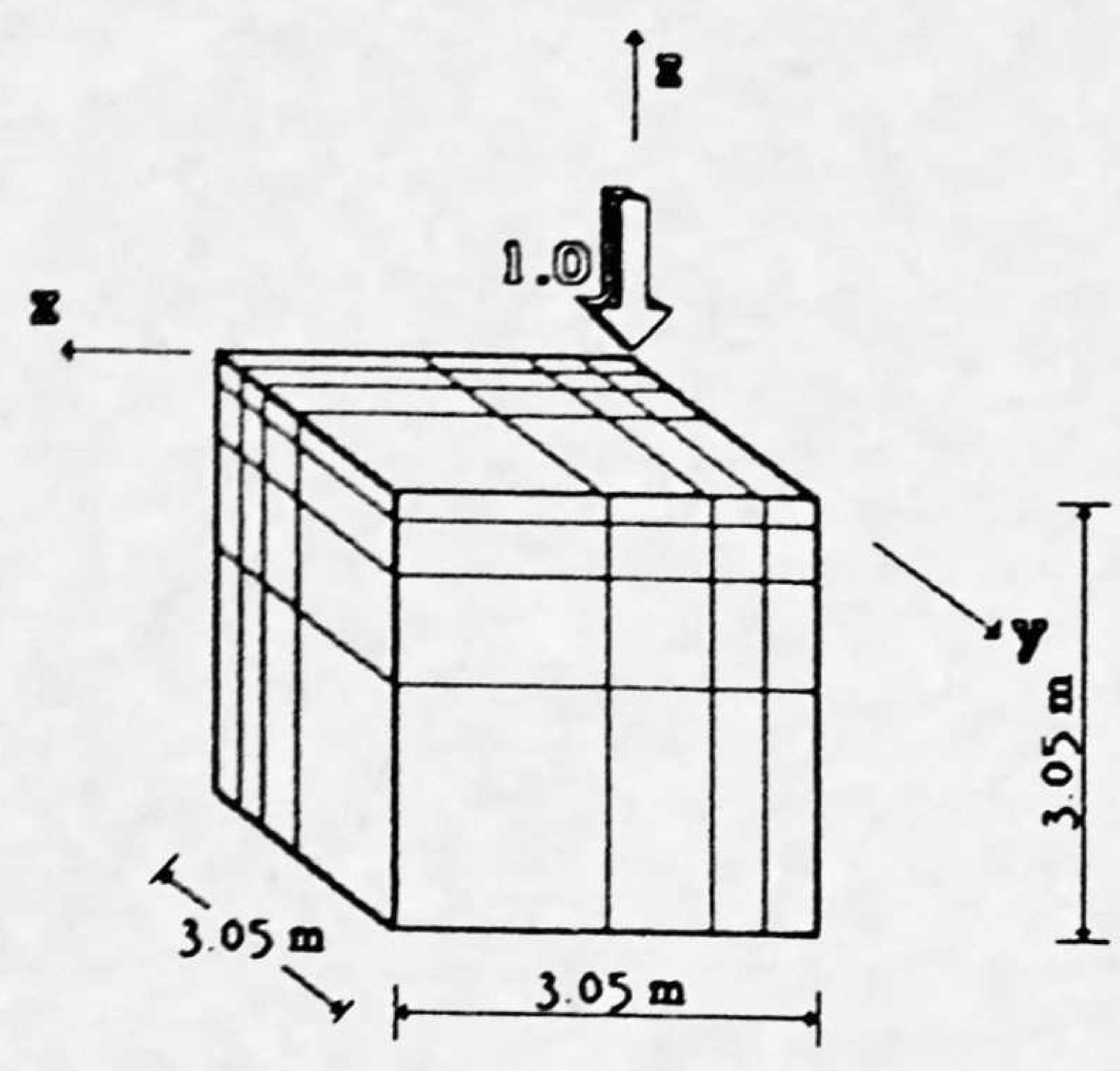

Figure 11. Static Three-Dimensional Problem of Zienkiewicz [4].

FUTURE DIRECTIONS

Several directions can be indicated for the future. Higher order simplex space-time elements, with higher order approximation both in space and in time can be developed. Space-time element models of simplex shapes can be improved to obtain unconditionally stable schemes with respect to the time step. Element models can be developed to allow the arbitrary partition of the space-time layer. Parameters can be introduced dependent on the time step, element geometry, material constants, and others as has been shown [15]. A good approach to that kind of problem is available [16]. Broad development of element models for different dynamic problems; e.g., robotics, material processing, should be undertaken. Large numerical problems should be solved, and error analysis should be developed. The method should be applied to the analysis of infinite structures (movable mesh, infinite space-time elements).

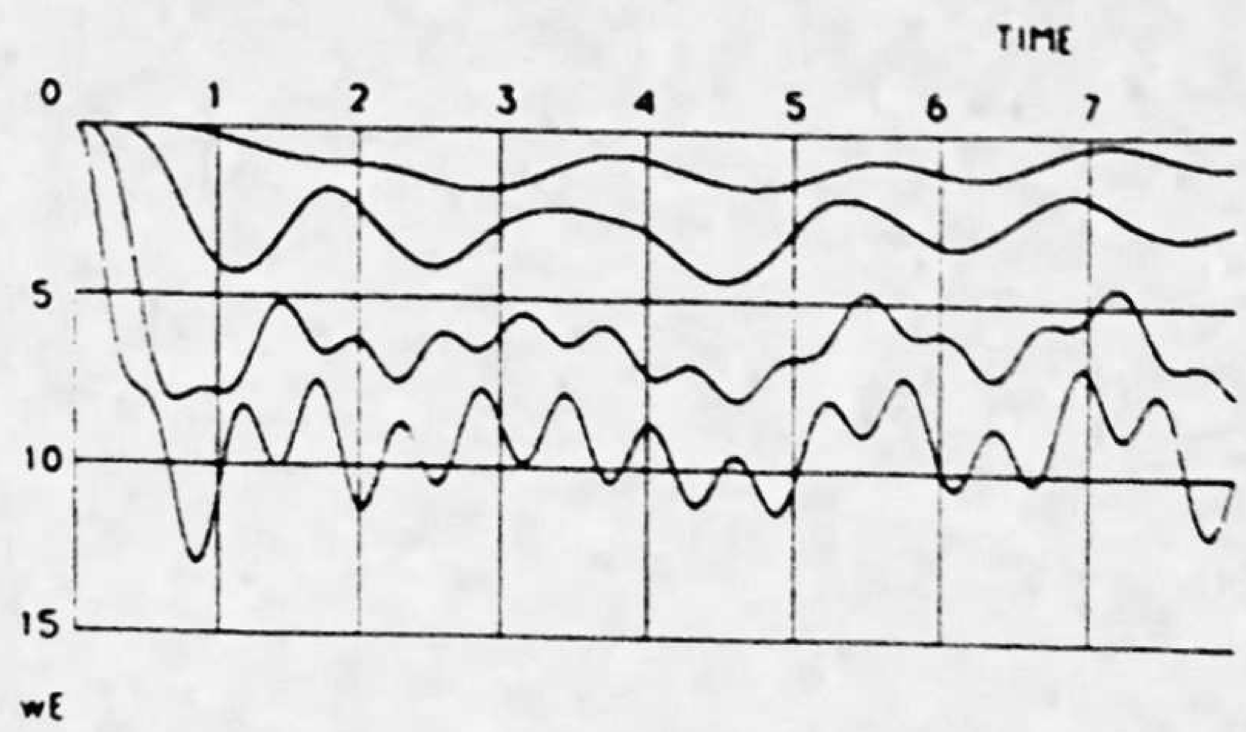

Figure 12. Response of Static Three-Dimensional Problem of Zienkiewicz [4].

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