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Finite zeros of positive linear continuous-time systems

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Abstract. The notion of finite zeros of continuous-time positive linear systems is introduced. It is shown that such zeros are real numbers. It is also shown that a square positive strictly proper or proper system of uniform rank with observability matrix of full column rank has no finite zeros. The problem of zeroing the system output for positive systems is defined. It is shown that a square positive strictly proper or proper system of uniform rank with observability matrix of full column rank has no nontrivial output-zeroing inputs. The obtained results remain valid for non-square positive systems with the first nonzero Markov parameter of full column rank.

Key words: finite zeros, output-zeroing problem, positive continuous-time linear systems.

1. Introduction

In positive systems inputs, state variables and outputs take merely non-negative values. Examples of positive systems can be found in industrial processes involving chemical reactors, distillation columns, storage systems, water and atmospheric pollution models. A number of mathematical models with positive linear behaviour can be found in management science, economics, biology and medicine, social sciences, etc. A number of topics in the area of positive systems is widely discussed in the literature, in particular: positive realization problems, state space properties (e.g. stability, controllability, observability), behavioral approach, positive 2-D systems, positive systems and related disciplines. An overview of the state of the art in positive linear systems can be found in [1-3] and the references therein. Some new results concerning stability of positive systems with delays and of positive fractional order systems can be found in [4, 5].

Unfortunately, the notions of zeros and poles of positive systems are not extensively discussed in the literature. The notions of decoupling zeros of positive discrete-time systems are introduced in [6] and the relationship between decoupling zeros of standard and positive discrete-time systems are analyzed. The presented approach is based on the notions of reachability and observability for positive discrete-time systems and on a canonical decomposition of the pairs of matrices (A, B) and (A, C) of a linear discrete-time positive system. In [7] the output-zeroing problem and finite zeros in positive discrete-time linear systems are analyzed. It is shown that the zeros are real nonnegative numbers. It is also shown that a square positive strictly proper or proper discrete-time system of uniform rank with observability matrix of full column rank has no nontrivial output-zeroing inputs nor finite zeros.

In the present paper we extend the results obtained in [7] to positive continuous-time systems. In particular, we introduce the notion of finite zeros for continuous-time linear positive systems. This notion is based on state-zero and input-zero directions (comp. [8, 9]) and uses the additional assumption concerning positivity of inputs and solutions generated by such zeros. In this way, the finite zeros of positive systems constitute a counterpart of the notion of invariant zeros for standard continuous-time systems [9].

The paper is organized as follows. In Sec. 2 the basic definitions and theorems concerning positive systems are recalled and the definition of the output-zeroing problem is introduced. Moreover, some basic facts concerning invariant zeros of standard continuous-time systems that are necessary for further discussion have been also recalled. The main results of the paper are presented in Sec. 3. Section 4 contains simple numerical examples and concluding remarks are given in Sec. 5.

2. Preliminary results

The set of all $n \times m$ complex (real) matrices is denoted by $C^{n\times m}$ $(R^{n\times m})$ respectively and by definition $C^{n\times 1} := C^n$ $(R^{n\times 1} := R^n)$. The set of all $n \times m$ real matrices with nonnegative entries is denoted by $R_+^{n\times m}$ and $R_+^{n\times 1} := R_+^n$.

Consider a linear continuous – time system of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \qquad t \geq 0, \tag{1}$$

where $x(t) \in R^n$, $\mathbf{u(t)} \in R^m$, $y(t) \in R^r$ are the state, input and output vectors and $A \in R^{n\times n}$, $0 \neq B \in R^{n\times m}$, $0 \neq C \in R^{r\times n}$, $D \in R^{r\times m}$. Throughout the paper we assume that the inputs u(t) are continuous vector-functions of t, $t \in [0, +\infty)$. System (1) is called proper if $D \neq 0$; otherwise the system is called strictly proper. The matrices D, CB, CAB, ..., CA^lB, ... are called the *Markov parameters* of (1). By

$$\begin{bmatrix} C \\ CA \\ . \\ CA^{n-1} \end{bmatrix}$$

we denote the observability matrix for (1).

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Definition 1. [1, 2] The system (1) is called (internally) positive if and only if $x(t) \in R_+^n$ and $y(t) \in R_+^r$ for every initial condition $x_o \in R_+^n$ and any input vector-function $u(t) \in R_+^m$, $t \geq 0$.

Remark 1. Recall [1, 2] that a square matrix with real entries is called the *Metzler matrix* if and only if all its off-diagonal entries are nonnegative. Moreover, $e^{tA} \in R_+^{n\times n}$, $t \geq 0$, if and only if $A \in R^{n\times n}$ is a Metzler matrix.

Theorem 1. [1, 2] The system (1) is (internally) positive if and only if $A \in R^{n\times n}$ is a Metzler matrix, $B \in R_+^{n\times m}$, $C \in R_+^{r\times n}$, $D \in R_+^{r\times m}$.

By analogy to the standard case, for positive continuous-time systems we can consider the problem of zeroing the system output (comp. [9–11]). To this end we will use the following formulation of the *output-zeroing problem* for the positive system (1). Find all pairs $(x_o, u_o(t))$ consisting of an admissible initial state $x_o \in R_+^n$ and an admissible input $u_o(t) \in R_+^m$, $t \geq 0$, such that the corresponding output is identically zero, i.e., $y(t) = 0$ for all $t \geq 0$. Any nontrivial pair of this kind (i.e., such that $x_o \neq 0$ or $u_o(t)$ is not identically zero) will be called the *output-zeroing input*. Of course, by virtue of Theorem 1 and Remark 1, for the corresponding solution

$$x_o(t) = e^{tA}x_o + \int_0^t e^{(t-\tau)A}Bu_o(\tau)d\tau \tag{2}$$

of the state equation the condition $x_o(t) \in R_+^n$ for all $t \geq 0$ will be satisfied (i.e., $x_o(t)$ will be admissible). In each output-zeroing input $(x_o, u_o(t))$, $u_o(t)$ should be understood as an open-loop real-valued control signal which, when applied to the positive system (1) exactly at the initial state $x(0) = x_o$ yields the solution $x_o(t)$ of the form (2) and the system response $y(t) = 0$ for all $t \geq 0$. Naturally, the trivial pair $(x_o = 0, u_o(t) \equiv 0)$ also yields $y(t) \equiv 0$; this pair will be called the *trivial output-zeroing input*.

Remark 2. In this remark we recall some basic facts concerning zeros of the standard system (1). For such system, the most commonly used notion of zeros are the Smith zeros [8–10]. These zeros are defined on the basis of the Smith canonical form of the system (Rosenbrock) matrix

$$P(s) = \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}. \tag{3}$$

With the aid of elementary row and column operations (3) is transformed to the Smith diagonal form. The product of diagonal polynomials is called the zero polynomial and its roots are the Smith zeros of (1). The Smith zeros can be equivalently defined as those points of the complex plane for which the pencil (3) loses its normal (determinantal) rank.

In [12] it is shown that a more general concept of zeros of (1) than the Smith zeros can be derived from the generalized eigenvalue problem for the matrix (3) when the latter is written as

$$P(s) = sN - M, \tag{4}$$

where

$$N = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} A & B \\ -C & -D \end{bmatrix}.$$

In the generalized eigenvalue problem for (4) we are looking for all complex numbers $\lambda \in C$ such that $\lambda Nw = Mw$ for some vector $0 \neq w \in C^{n+m}$. Each $\lambda \in C$ with the above property is called the generalized eigenvalue of the pair (N, M) and the corresponding to it vector w is called the generalized eigenvector. It is clear that λ is a generalized eigenvalue of (N, M) if and only if rank of $\lambda N - M$ is smaller than $n + m$. The set of all generalized eigenvalues of (N, M) we denote as $\sigma(N, M) := \{\lambda \in C : \ rank(\lambda N - M) < n + m\}$. It is important to note that the generalized eigenvalues of (N, M) are not only those complex numbers λ for which rank of $\lambda N - M$ is smaller than normal rank of $sN-M$. In general case, $\sigma(N, M)$ may be empty, finite or equal to the whole complex plane. The last case takes place if for example $n+r < n+m$ or, more generally, if rank of $sN - M$ is smaller than $\min\{n + r, \ n + m\}$. Immediately from the definition of Smith zeros, it follows that each Smith zero of (1) is also a generalized eigenvalue of (N, M). In fact, if λ is a Smith zero of (1), then

$$rank(\lambda N - M) < normal\ rank(sN - M) \leq$$
$$\leq \min\{n + r, \ n + m\} \leq n + m$$

and consequently, $\lambda \in \sigma(N, M)$.

The definition of generalized eigenvalues for the pair (N, M) can be expressed in the form:

a number $\lambda \in C$ is a generalized eigenvalue of P(s) (4), i.e., $\lambda \in \sigma(N, M)$, if and only if there exists a nonzero vector $\begin{bmatrix} x^o \\ g \end{bmatrix} \in C^{n+m}$ such that $P(\lambda) \begin{bmatrix} x^o \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, i.e.,

$$\lambda x^o - Ax^o = Bg, \qquad Cx^o + Dg = 0. \tag{5}$$

As a zero of the standard system (1) we take any generalized eigenvalue $\lambda \in \sigma(N, M)$ which satisfies the following two conditions:

a) λ generates a nontrivial solution of the state equation in (1) (i.e., a solution which is not the identically zero solution),

b) this solution yields the identically zero system response, i.e., $y(t) = 0$ for all $t \geq 0$.

It is easy to note (see (5)) that if $\lambda \in \sigma(N, M)$, then $e^{\lambda t}x^o$ is a solution of the state equation in (1) which corresponds to the initial condition x^o and to the input $e^{\lambda t}g$. The term $e^{\lambda t}x^o$ is called the solution of the state equation in (1) generated by λ. Of course, λ, x^o and g may in general case be complex. Note that if λ, x^o, g satisfy (5), then the triple $\overline{\lambda}$, $\overline{x}^o$, $\overline{g}$ consisted of complex conjugates also satisfies (5). On the other hand, we are interested only in real initial conditions, inputs and solutions. The latter we obtain (see [10]) by taking $\mathrm{Re}\, x^o$ as the initial condition and $\mathrm{Re}\,(e^{\lambda t}g)$, $\mathrm{Re}\,(e^{\lambda t}x^o)$ as the corresponding input and solution or $\mathrm{Im}\, x^o$ as the initial condition and $\mathrm{Im}\,(e^{\lambda t}g)$, $\mathrm{Im}\,(e^{\lambda t}x^o)$ as the corresponding input and solution (Re and Im denote real and imaginary part of a complex value).

As is shown in [12], the set of zeros of the standard system (1) consists of all those generalized eigenvalues of (N, M) for which there exists a generalized eigenvector $\begin{bmatrix} x^o \\ g \end{bmatrix}$ with the property $x^o \neq 0$. The elements of this set we call, in order to distinguish from the Smith zeros, the invariant zeros of (1). In this way we obtain the following definition [9, 10, 12]:

a number $\lambda \in C$ is an invariant zero of the standard system (1) if and only if there exist vectors $0 \neq x^o \in C^n$ and $g \in C^m$ such that the triple λ, x^o, g satisfies (5).

As is known [9, 10], for the standard system (1) the set of its invariant zeros is an extension of the set of Smith zeros (i.e., each Smith zero is also an invariant zero).

Let $\lambda \in C$ be an invariant zero of the standard system (1), i.e., let a triple λ, $x^o \neq 0$, g satisfy (5). Denote $\lambda = \mathrm{Re}\lambda + j\,\mathrm{Im}\lambda$, $x^o = \mathrm{Re}\,x^o + j\,\mathrm{Im}x^o$, $g = \mathrm{Re}\,g + j\,\mathrm{Im}\,g$. Then (5) takes the form

$$\begin{aligned} \mathrm{Re}\,\lambda\,\mathrm{Re}\,x^o - \mathrm{Im}\,\lambda\,\mathrm{Im}\,x^o - A\,\mathrm{Re}\,x^o &= B\,\mathrm{Re}\,g, \\ \mathrm{Im}\,\lambda\,\mathrm{Re}\,x^o + \mathrm{Re}\,\lambda\,\mathrm{Im}\,x^o - A\,\mathrm{Im}\,x^o &= B\,\mathrm{Im}\,g \end{aligned} \tag{6}$$

and

$$\begin{aligned} C\,\mathrm{Re}\,x^o + D\,\mathrm{Re}\,g &= 0, \\ C\,\mathrm{Im}\,x^o + D\,\mathrm{Im}\,g &= 0, \end{aligned} \tag{7}$$

while the real valued initial conditions (x_o), inputs $(u_o(t))$ and solutions $(x_o(t))$ generated by $\lambda = \sigma + j\omega$ are of the form (comp. [9]):

$$\begin{aligned} x_o &= \mathrm{Re}\,x^o, \\ u_o(t) &= \mathrm{Re}\,(e^{\lambda t} g) = e^{\sigma t}(\mathrm{Re}\,g\,\cos\omega t - \mathrm{Im}\,g\,\sin\omega t), \\ x_o(t) &= \mathrm{Re}\,(e^{\lambda t} x^o) = e^{\sigma t}(\mathrm{Re}\,x^o\,\cos\omega t - \mathrm{Im}\,x^o\,\sin\omega t) \end{aligned} \tag{8}$$

and

$$\begin{aligned} x_o &= \mathrm{Im}\,x^o, \\ u_o(t) &= \mathrm{Im}\,(e^{\lambda t} g) = e^{\sigma t}(\mathrm{Re}\,g\,\sin\omega t + \mathrm{Im}\,g\,\cos\omega t), \\ x_o(t) &= \mathrm{Im}\,(e^{\lambda t} x^o) = e^{\sigma t}(\mathrm{Re}\,x^o\,\sin\omega t + \mathrm{Im}\,x^o\,\cos\omega t), \end{aligned} \tag{9}$$

Remark 3. Note that if $\lambda \in C$ such that $\mathrm{Im}\,\lambda \neq 0$ is an invariant zero of the standard system (1), i.e., a triple λ, $x^o \neq 0$, g satisfies (5), then $\mathrm{Im}\,g \neq 0$ or $\mathrm{Im}\,x^o \neq 0$. In fact, suppose that $\mathrm{Im}\,g = 0$ and $\mathrm{Im}\,x^o = 0$. Then, from the second equality in (6), it follows that $\mathrm{Im}\,\lambda\,\mathrm{Re}\,x^o = 0$, and consequently, $\mathrm{Re}\,x^o = 0$. Hence, $x^o = 0$, contrary to the assumption.

Remark 4. As is known [9], for the standard observable (strictly proper or proper) system (1) the property of being an invariant zero is equivalent to the property of generating output-zeroing inputs. More precisely, a triple $\lambda \in C$, $0 \neq x^o \in C^n$, $g \in C^m$ satisfies (5) if and only if the input $e^{\lambda t}g$, $t \geq 0$, and the initial condition $x^o \neq 0$ yield $y(t) = 0$ for all $t \geq 0$. Moreover, in the considered triple is $g \neq 0$ and the solution corresponding to x^o and $e^{\lambda t}g$ has the form $e^{\lambda t}x^o$.

3. Main results

For the positive system (1) we adopt the following definition of zeros.

Definition 2. A number $\lambda \in C$ is a *finite zero* of the strictly proper or proper positive continuous-time system (1) if and only if there exist vectors $0 \neq x^o \in C^n$ and $g \in C^m$ such that the triple λ, $x^o \neq 0$, g satisfies (5) and λ generates an admissible (i.e., nonnegative) real valued input and an admissible (i.e., nonnegative) real valued solution of the state equation.

Theorem 2. If λ is a finite zero of the strictly proper or proper positive continuous-time system (1) and λ, $x^o \neq 0$, g satisfy (5), then λ is a real number. Moreover, $x^o \in R^n_+$ and $g \in R^m_+$.

Proof. For the proof of the first assertion of the theorem it is enough to show that if $\lambda \in C$ satisfies (5) and λ is such that $\mathrm{Im}\,\lambda \neq 0$, then the conditions of Definition 2 are not fulfilled, i.e., λ does not generate admissible inputs and solutions.

Suppose first that a triple λ, $x^o \neq 0$, g satisfies (5) and $\mathrm{Im}\,\lambda \neq 0$. We consider the following two disjoint cases: $g = 0$ and $g \neq 0$.

In the first case, i.e., $g = 0$, we have, by virtue of Remark 3, $\mathrm{Im}\,x^o \neq 0$. Let the j-th component $(\mathrm{Im}\,x^o)_j$ of the vector $\mathrm{Im}\,x^o$ be nonzero. Then the j-th component $(x_o(t))_j$ of the solution $x_o(t)$ in (8) takes, for $t \geq 0$, the form

$$(x_o(t))_j = ce^{\sigma t}\cos(\omega t + \alpha),$$

where

$$c = \sqrt{(\mathrm{Re}\,x^o)_j^2 + (\mathrm{Im}\,x^o)_j^2}$$

and

$$\sin\alpha = \frac{(\mathrm{Im}\,x^o)_j}{c}.$$

It means that this component changes sign and consequently, $x_o(t)$ can not remain in R^n_+. The same reasoning applies to $x_o(t)$ in (9), where the j-th component takes, for $t \geq 0$, the form $(x_o(t))_j = ce^{\sigma t}\sin(\omega t + \alpha)$.

In the second case, i.e., $g \neq 0$, we have $\mathrm{Re}\,g \neq 0$ or $\mathrm{Im}\,g \neq 0$. Suppose that for the j-th component of g is $(\mathrm{Re}\,g)_j \neq 0$ or $(\mathrm{Im}\,g)_j \neq 0$. Then the j-th component $(u_o(t))_j$ of the input $u_o(t)$ in (8) takes, for $t \geq 0$, the form

$$(u_o(t))_j = de^{\sigma t}\cos(\omega t + \beta),$$

where

$$d = \sqrt{(\mathrm{Re}\,g)_j^2 + (\mathrm{Im}\,g)_j^2}$$

and

$$\sin\beta = \frac{(\mathrm{Im}\,g)_j}{d}.$$

It means that this component changes sign when t changes and consequently, $u_o(t)$ can not remain in R^m_+. In the same way we analyze the j-th component of $u_o(t)$ in (9). Then, for $t \geq 0$, we have $(u_o(t))_j = de^{\sigma t}\sin(\omega t + \beta)$ and $u_o(t)$ is not contained in R^m_+.

In this way we have shown that if $\lambda \in C$ satisfies Definition 2, then $\mathrm{Im}\,\lambda = 0$, i.e., λ is a real number.

The last assertion of the theorem follows directly from the above and Definition 2. In fact, since λ is real, we take as x^o and g real vectors and consequently, $u_o(t) = e^{\lambda t}g$ and $x_o(t) = e^{\lambda t}x^o$ are real and, by Definition 2, they remain, for all $t \geq 0$, in R^m_+ and R^n_+ respectively. In particular, this holds also for $t = 0$. Hence the theorem follows.

From Definition 2 and Theorem 2 we obtain the following equivalent characterization of finite zeros of the positive system (1).

Theorem 3. A number λ is a finite zero of the strictly proper or proper positive system (1) if and only if $\lambda \in R$ and there exist vectors $0 \neq x^o \in R_+^n$ and $g \in R_+^m$ such that the triple $\lambda,\ x^o,\ g$ satisfies (5).

Proof. If λ is a finite zero of the positive system (1), then the assertion of the theorem follows directly from Theorem 2. Conversely, if a triple $\lambda \in R$, $0 \neq x^o \in R_+^n$, $g \in R_+^m$ satisfies (5), then λ satisfies Definition 2. Hence the theorem follows.

In the remaining part of this section we consider a positive proper or strictly proper system (1) of *uniform rank* which means that the system is square (the number of inputs equals the number of outputs, i.e., $m = r$) and the first nonzero Markov parameter is nonsingular. For strictly proper systems the first nonzero Markov parameter is denoted by $CA^{\nu}B$, where $0 \leq \nu \leq n - 1$.

3.1. Output-zeroing inputs in positive systems of uniform rank.

Theorem 4. Suppose that in the positive proper system (1) of uniform rank the observability matrix has full column rank. Then the system has only the trivial output-zeroing input.

Proof. Let $(x_o,\ u_o(t))$ be a nontrivial output-zeroing input and let $x_o(t)$ denote the corresponding solution of the state equation. At this assumption for each $t \geq 0$ the following equalities hold

$$\begin{aligned} \dot{x}_o(t) = Ax_o(t) + Bu_o(t), \qquad x_o(0) = x_o, \\ 0 = Cx_o(t) + Du_o(t). \end{aligned} \tag{10}$$

By virtue of Theorem 1 and the definition of the output-zeroing problem for positive systems, we obtain from the second equality in (10) the relation $Du_o(t) = -Cx_o(t)$ and consequently, $Cx_o(t) = 0$ and $Du_o(t) = 0$ for all $t \geq 0$. The last equality yields $u_o(t) \equiv 0$ and consequently, $x_o(t) = e^{tA}x_o$. Hence $Cx_o(t) = Ce^{tA}x_o = 0$ for all $t \geq 0$. Differentiating both sides of the equality $Ce^{tA}x_o = 0$ an appropriate number of times and taking $t = 0$, we can write $Cx_o = 0$, $CAx_o = 0, \ldots,\ CA^{n-1}x_o = 0$. In view of the assumption concerning the observability matrix, we obtain $x_o = 0$. This contradicts the assumption that $(x_o,\ u_o(t))$ is non-trivial.

Theorem 5. Suppose that in a positive strictly proper system (1) of uniform rank the observability matrix has the full column rank n. Then the system has only the trivial output-zeroing input.

Proof. It is enough to show that each output-zeroing input is trivial. To this end, let $(x_o,\ u_o(t))$ be a nontrivial output-zeroing input and let $x_o(t)$ denote the corresponding solution of the state equation. At this assumption we have the following equalities

$$\begin{aligned} \dot{x}_o(t) = Ax_o(t) + Bu_o(t), \qquad x_o(0) = x_o, \\ Cx_o(t) = 0 \end{aligned} \tag{11}$$

which are valid for each $t \geq 0$. From the second equality in (11) and from (2) we obtain

$$Ce^{tA}x_o + \int_0^t Ce^{(t-\tau)A}Bu_o(\tau)d\tau = 0. \tag{12}$$

Since both vectors on the left-hand side of (12) are nonnegative, we have, for all $t \geq 0$ and $0 \leq \tau \leq t$, the equalities

$$Ce^{tA}x_o = 0 \quad \text{and} \quad \int_0^t Ce^{(t-\tau)A}Bu_o(\tau)d\tau = 0. \tag{13}$$

The first equality in (13) yields, via the observability assumption, $x_o = 0$. In the second equality in (13) the term $Ce^{(t-\tau)A}Bu_o(\tau)$ is continuous and nonnegative in the interval $0 \leq \tau \leq t$. Hence, by virtue of $\int_0^t Ce^{(t-\tau)A}Bu_o(\tau)d\tau = 0$, we obtain $Ce^{(t-\tau)A}Bu_o(\tau) = 0$ for $0 \leq \tau \leq t$. Differentiating $n - 1$ times both sides of the last equality with respect to t we obtain the following sequence of equalities

$$\begin{gathered} Ce^{(t-\tau)A}Bu_o(\tau) = 0, \\ CAe^{(t-\tau)A}Bu_o(\tau) = 0, \\ \vdots \\ CA^{n-1}e^{(t-\tau)A}Bu_o(\tau) = 0. \end{gathered} \tag{14}$$

The assumption concerning observability yields now $e^{(t-\tau)A}Bu_o(\tau) = 0$ and consequently, $Bu_o(\tau) = 0$ for all $0 \leq \tau \leq t$. Since the first nonzero Markov parameter $CA^{\nu}B$ is nonsingular, the matrix B has full column rank. As a consequence, we obtain $u_o(\tau) = 0$ for all $0 \leq \tau \leq t$. Finally, since t can be fixed arbitrarily, we can write $u_o(t) = 0$ for all $t \geq 0$.

In this way we have shown that $(x_o,\ u_o(t))$ is the trivial output-zeroing input and the theorem is proved.

Remark 5. Theorems 4 and 5 remain valid for non-square positive systems (1) when the assumption of uniform rank is replaced by the assumption that the first nonzero Markov parameter has full column rank. The proofs follow the same lines.

3.2. Finite zeros in positive systems of uniform rank.

Theorem 6. Suppose that in the positive proper system (1) of uniform rank the observability matrix has the full column rank n. Then the system has no finite zeros.

Proof. We proceed the proof via *reductio ad absurdum*. To this end, suppose that a number λ is a finite zero of the system. Then, by virtue of Theorem 2 (or Theorem 3), λ is real and $0 \neq x^o \in R_+^n$, $g \in R_+^m$. From the second equality in (5) we have $Cx^o + Dg = 0$ and, by virtue of Theorem 1, we obtain $Cx^o = 0$ and $Dg = 0$. Since, by assumption, D is nonsingular, we obtain $g = 0$. Now, the condition (5) yields the equalities $\lambda x^o - Ax^o = 0$, $Cx^o = 0$. Pre-multiplying subsequently the first equality by C, CA, ..., CA^{n-2} and taking into account the second equality, we get the following sequence of equalities $Cx^o = 0$, $CAx^o = 0, \ldots, CA^{n-1}x^o = 0$. Since the observability matrix has full column rank, we obtain $x^o = 0$ which contradicts the assumption $x^o \neq 0$ (comp. Definition 2).

Theorem 7. Suppose that in a positive strictly proper system (1) of uniform rank the observability matrix has the full column rank n. Then the system has no finite zeros.

Proof. Suppose that Theorem 7 is not valid and let a number λ be a finite zero of the system. Then, by virtue of Theorem 3, λ is real and $0 \neq x^o \in R_+^n$ and $g \in R_+^m$. Of course, λ generates the nontrivial output-zeroing input $(x_o,\ u_o(t))$ where $x_o = x^o \neq 0$ and $u_o(t) = e^{\lambda t} g$. However, this contradicts Theorem 5.

Corollary 1. Consider a SISO (single input, single output, i.e., $m = r = 1$) strictly proper or proper positive system (1) which is observable as a standard system. Then the positive system has no finite zeros.

Remark 6. As is known [9], a standard strictly proper system (1) of uniform rank with $CA^{\nu}B$ as the first nonzero Markov parameter has $n - m(\nu + 1)$ invariant (Smith) zeros. In particular, a SISO strictly proper standard system has $n - (\nu + 1)$ invariant (Smith) zeros. For a proper standard system (1) of uniform rank the number of invariant (Smith) zeros equals n (the same holds for a SISO standard proper system) [9].

Remark 7. Note that Theorems 6 and 7 remain valid for nonsquare positive systems (1) if the assumption of uniform rank is replaced by the assumption that the first nonzero Markov parameter has full column rank. The proofs follow the same lines.

4. Examples

Example 1. Consider a positive SISO system (1) with the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad c = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

The assumptions of Corollary 1 are fulfilled and the system has no finite zeros. On the other hand, the transfer function of this system equals $g(s) = \dfrac{s^2 + s + 1}{s^3}$ and the system, treated as a standard one, has two invariant (Smith) zeros.

Example 2. Consider a positive system (1) with the matrices

$$A = \begin{bmatrix} -1/3 & 0 & 2 \\ 0 & -1/3 & 1 \\ 0 & 0 & -1/3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Equations (5) take the form

$$\left(\lambda + \frac{1}{3}\right) x_1^o - 2x_3^o - g_1 - g_2 = 0,$$

$$\left(\lambda + \frac{1}{3}\right) x_2^o - x_3^o - g_1 = 0,$$

$$\left(\lambda + \frac{1}{3}\right) x_3^o = 0,$$

$$x_2^o + 2x_3^o + g_1 = 0,$$

where

$$x^o = \begin{bmatrix} x_1^o \\ x_2^o \\ x_3^o \end{bmatrix}, \qquad g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}.$$

It is easy to verify that for any given $\lambda > -1/3$ and $x_1^o > 0$ the triple

$$\lambda, \qquad x^o = \begin{bmatrix} x_1^o \\ 0 \\ 0 \end{bmatrix}, \qquad g = \begin{bmatrix} 0 \\ \left(\lambda + \frac{1}{3}\right) x_1^o \end{bmatrix}$$

satisfies Definition 2 (see also Theorem 3). This means that any real number greater than $-1/3$ is a finite zero of the system. Moreover, for any given $x_1^o > 0$, the triple

$$\lambda = -\frac{1}{3}, \qquad x^o = \begin{bmatrix} x_1^o \\ 0 \\ 0 \end{bmatrix}, \qquad g = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

also satisfies Definition 2 (or Theorem 3) (i.e., $\lambda = -1/3$ is also a finite zero of the system).

On the other hand, the considered system, treated as a standard one, is degenerate [9, 10] (i.e., each complex number is its invariant zero). Moreover, it has exactly two Smith zeros ($\lambda = -1/3$ and $\lambda = -4/3$) and $\lambda = -1/3$ is simultaneously the output decoupling zero.

Example 3. Consider a positive system (1) with the matrices

$$A = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solving (5) it is easy to verify that for each $\lambda > -1$ and $x_2^o > 0$ the triple

$$\lambda, \qquad x^o = \begin{bmatrix} 0 \\ x_2^o \\ 0 \\ 0 \end{bmatrix}, \qquad g = \begin{bmatrix} 0 \\ (\lambda + 1)x_2^o \end{bmatrix}$$

satisfies Definition 2 (see also Theorem 3). For a given $x_2^o > 0$ the triple

$$\lambda = -1, \qquad x^o = \begin{bmatrix} 0 \\ x_2^o \\ 0 \\ 0 \end{bmatrix}, \qquad g = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

also satisfies Definition 2 (or Theorem 3). Hence each real number $\lambda \geq -1$ constitutes a finite zero of the positive system.

The considered system, treated as a standard one, is degenerate (i.e., each complex number is its invariant zero). Moreover, $\lambda = -1$ is an output-decoupling zero of the system.

5. Conclusions

The notion of finite zeros for positive continuous-time linear systems has been introduced (Definition 2). This notion uses the assumption that finite zeros generate admissible (i.e., nonnegative) output-zeroing inputs and the corresponding solutions. As a consequence, finite zeros of a positive continuous-time system (if they exist) are real numbers, while the corresponding state-zero and input-zero directions remain in R_+^n and R_+^m respectively (Theorems 2 and 3).

It has been shown that positive continuous-time strictly proper or proper systems of uniform rank satisfying observability condition as standard systems do not possess nontrivial output-zeroing inputs (Theorems 4 and 5) nor finite zeros (Theorems 6 and 7). Theorems 4–7 remain valid for nonsquare systems with the first nonzero Markov parameter of full column rank (Remarks 5 and 7). The considerations have been illustrated by simple numerical examples.

The obtained results clearly show that positivity constraints (Definition 1 and Theorem 1) imposed on continuous-time linear systems result in limitations concerning internal dynamics and location of zeros (comp. [6, 13, 14]).

An open problem is the state space (dynamical) characterization of poles in positive discrete-time and continuous-time linear systems.

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