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Finite zeros of positive linear discrete time systems

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Abstract. The notion of finite zeros of discrete-time positive linear systems is introduced. It is shown that such zeros are real nonnegative numbers. It is also shown that a square positive strictly proper or proper system of uniform rank with the observability matrix of full column rank has no finite zeros. The problem of zeroing the system output for positive systems is defined. It is shown that a square positive strictly proper or proper system of uniform rank with the observability matrix of full column rank has no nontrivial output-zeroing inputs. The obtained results remain valid for non-square positive systems with the first nonzero Markov parameter of full column rank.

Key words: finite zeros, output-zeroing problem, positive discrete-time linear systems.

1. Introduction

In positive systems inputs, state variables and outputs take merely non-negative values. Examples of positive systems can be found in industrial processes involving chemical reactors, distillation columns, storage systems, water and atmospheric pollution models. A number of mathematical models with positive linear behaviour can be found in management science, economics, biology and medicine, social sciences, etc. A number of topics in the area of positive systems is extensively discussed in the literature, in particular: state space properties (e.g., stability, reachability, observability), behavioral approach, positive realization problems, positive systems and related disciplines, positive 2-D systems. An overview of the state of the art in positive linear systems can be found in [1–3]. Some new results concerning stability and stabilizability of positive fractional order systems can be found in [4, 5].

Unfortunately, the notions of zeros and poles of positive linear systems are not widely discussed in the literature. The notions of decoupling zeros of positive discrete-time systems are introduced in [6] and the relationship between decoupling zeros of standard and positive discrete-time systems are analyzed. The presented approach is based on the notions of reachability and observability for positive discrete-time systems and on a canonical decomposition of the pairs of matrices (A, B) and (A, C) of a linear discrete-time positive system.

In the present paper we introduce the notion of finite zeros for discrete-time linear positive systems. This notion is based on state-zero and input-zero directions [7, 8] and uses the additional assumption concerning positivity of inputs and solutions generated by such zeros. In this way, the finite zeros of positive systems constitute a counterpart of the notion of invariant zeros for standard discrete-time systems [8].

The paper is organized as follows. In Sec. 2 the basic definitions and theorems concerning positive systems are recalled and definitions of the output-zeroing problem and finite zeros for positive systems are introduced. The main results of the paper are presented in Sec. 3. Section 4 contains simple numerical examples and concluding remarks are given in Sec. 5.

2. Preliminary results

The set of all $n \times m$ complex (real) matrices is denoted by $C^{n\times m}$ ($R^{n\times m}$) respectively and by definition $C^{n\times 1} := C^n$ ($R^{n\times 1} := R^n$). The set of all $n \times m$ real matrices with nonnegative entries is denoted by $R_+^{n\times m}$ and $R_+^{n\times 1} := R_+^n$. The set of all nonnegative integers is denoted by Z_+.

Consider a linear discrete time system of the form

$$\begin{array}{l} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{array}, \quad k \in Z_+ \tag{1}$$

where $x(k) \in R^n$, $u(k) \in R^m$, $y(k) \in R^r$ are the state, input and output vectors and $A \in R^{n\times n}$, $0 \neq B \in R^{n\times m}$, $0 \neq C \in R^{r\times n}$, $D \in R^{r\times m}$. System (1) is called proper if $D \neq 0$; otherwise the system is called strictly proper. The matrices D, CB, CAB, ..., CA^lB, ... are called the *Markov parameters* of (1). By

$$\begin{bmatrix} C \\ CA \\ \cdot \\ CA^{n-1} \end{bmatrix}$$

we denote the observability matrix for (1).

Definition 1. [1, 2] The system (1) is called (internally) positive if and only if $x(k) \in R_+^n$ and $y(k) \in R_+^r$, $k \in Z_+$, for every initial condition $x_o \in R_+^n$ and any input sequence $u(k) \in R_+^m$, $k \in Z_+$.

Theorem 1. [1, 2] The system (1) is (internally) positive if and only if $A \in R_+^{n\times n}$, $B \in R_+^{n\times m}$, $C \in R_+^{r\times n}$, $D \in R_+^{r\times m}$.

By analogy to the standard case, for positive discrete-time systems we can consider the problem of zeroing the system output (comp. [8, p.21], [9]). To this end we will use the

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following formulation of the *output-zeroing problem* for the positive system (1). Find all pairs (x_o, $u_o(k)$) consisting of an admissible initial state $x_o \in R^n_+$ and an admissible input $u_o(k) \in R^m_+$, $k \in Z_+$, such that the corresponding output is identically zero, i.e., $y(k) = 0$ for all $k \in Z_+$. Any nontrivial pair of this kind (i.e., such that $x_o \neq 0$ or $u_o(k)$ is not identically zero) will be called the *output-zeroing input*. Of course, by virtue of Theorem 1, for the corresponding solution

$$x_o(k) = \begin{cases} x_o & \text{for } k = 0, \\ A^k x_o + \sum_{l=0}^{k-1} A^{k-1-l} B u_o(l) & \text{for } k = 1, 2, ... \end{cases} \tag{2}$$

of the state equation the condition $x_o(k) \in R^n_+$ for all $k \in Z_+$ will be satisfied. In each output-zeroing input (x_o, $u_o(k)$), $u_o(k)$ should be understood as an open-loop real-valued control signal which, when applied to the positive system (1) exactly at the initial state $x(0) = x_o$ yields the solution $x_o(k)$ of the form (2) and the system response $y(k) = 0$ for all $k \in Z_+$. Naturally, the trivial pair ($x_o = 0$, $u_o(k) \equiv 0$) also yields $y(k) \equiv 0$; this pair will be called the *trivial output-zeroing input*.

Remark 1. In this remark we recall some basic facts concerning zeros of the standard system (1). For such system, the most commonly used notion of zeros are the Smith zeros ([7], [8, p.2]). These zeros are defined on the basis of the Smith canonical form of the system (Rosenbrock) matrix

$$P(z) = \begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix}. \tag{3}$$

With the aid of elementary row and column operations (3) is transformed to the Smith diagonal form. The product of diagonal polynomials is called the zero polynomial and its roots are the Smith zeros of (1). The Smith zeros can be equivalently defined as those points of the complex plane for which the pencil (3) loses its normal (determinantal) rank.

In [10] it is shown that a more general concept of zeros of (1) than the Smith zeros can be derived from the generalized eigenvalue problem for the matrix (3) when the latter is written as

$$P(z) = zN - M, \tag{4}$$

where

$$N = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} A & B \\ -C & -D \end{bmatrix}.$$

In the generalized eigenvalue problem for (4) we are looking for all complex numbers $\lambda \in C$ such that $\lambda N w = M w$ for some vector $0 \neq w \in C^{n+m}$. Each $\lambda \in C$ with the above property is called the generalized eigenvalue of the pair (N, M) and the corresponding to it vector w is called the generalized eigenvector. It is clear that λ is a generalized eigenvalue of (N, M) if and only if rank of $\lambda N - M$ is smaller than $n+m$. The set of all generalized eigenvalues of (N, M) we denote as $\sigma(N, M) := \{\lambda \in C : \, rank(\lambda N - M) < n+m\}$. It is important to note that the generalized eigenvalues of (N, M) are not only those complex numbers λ for which rank of $\lambda N - M$ is smaller than normal rank of $zN - M$. In general case, $\sigma(N, M)$ may be empty, finite or equal to the whole complex plane. The last case takes place if for example $n + r < n + m$ or, more generally, if rank of $zN - M$ is smaller than $\min\{n+r, \, n+m\}$. Immediately from the definition of Smith zeros, it follows that each Smith zero of (1) is also a generalized eigenvalue of (N, M). In fact, if λ is a Smith zero of (1), then

$$rank(\lambda N - M) < normal\, rank(zN - M) \leq \\ \leq \min\{n + r, \, n + m\} \leq n + m$$

and consequently, $\lambda \in \sigma(N, M)$.

The definition of generalized eigenvalues for the pair (N, M) can be expressed in the form:
a number $\lambda \in C$ is a generalized eigenvalue of P(z) (4), i.e., $\lambda \in \sigma(N, M)$, if and only if there exists a nonzero vector $\begin{bmatrix} x^o \\ g \end{bmatrix} \in C^{n+m}$ such that $P(\lambda) \begin{bmatrix} x^o \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, i.e.,

$$\lambda x^o - A x^o = B g, \qquad C x^o + D g = 0. \tag{5}$$

As a zero of the standard system (1) we take any generalized eigenvalue $\lambda \in \sigma(N, M)$ which satisfies the following two conditions:

a) λ generates a nontrivial solution of the state equation in (1) (i.e., a solution which is not the identically zero solution),

b) this solution yields the identically zero system response, i.e., $y(k) = 0$ for all $k \in Z_+$.

It is easy to note that if $\lambda \in \sigma(N, M)$, then $\lambda^k x^o$ is a solution of the state equation in (1) which corresponds to the initial condition x^o and to the input $\lambda^k g$. The term $\lambda^k x^o$ is called the solution of the state equation in (1) generated by λ. Of course, λ, x^o and g may in general case be complex. Note that if λ, x^o, g satisfy (5), then the triple $\overline{\lambda}$, $\overline{x}^o$, $\overline{g}$ consisted of complex conjugates also satisfies (5). On the other hand, we are interested only in real initial conditions, inputs and solutions. The latter we obtain (see [8, p. 25]) by taking $\mathrm{Re}\, x^o$ as the initial condition and $\mathrm{Re}(\lambda^k g)$, $\mathrm{Re}(\lambda^k x^o)$ as the corresponding input and solution or $\mathrm{Im}\, x^o$ as the initial condition and $\mathrm{Im}(\lambda^k g)$, $\mathrm{Im}(\lambda^k x^o)$ as the corresponding input and solution (Re and Im denote real and imaginary part of a complex value).

As is shown in [10], the set of zeros of the standard system (1) consists of all those generalized eigenvalues of (N, M) for which there exists a generalized eigenvector $\begin{bmatrix} x^o \\ g \end{bmatrix}$ with the property $x^o \neq 0$. The elements of this set we call, in order to distinguish from the Smith zeros, the invariant zeros of (1). In this way we obtain the following definition ([8, 10]):
a number $\lambda \in C$ is an invariant zero of the standard system (1) if and only if there exist vectors $0 \neq x^o \in C^n$ and $g \in C^m$ such that the triple λ, x^o, g satisfies (5).

As is known [8], for the standard system (1) the set of its invariant zeros is an extension of the set of Smith zeros (i.e., each Smith zero is also an invariant zero).

Let $\lambda \in C$ be an invariant zero of the standard system (1), i.e., let a triple λ, $x^o \neq 0$, g satisfy (5). Denote

$\lambda = \operatorname{Re}\lambda + \mathrm{j}\operatorname{Im}\lambda$, $x^o = \operatorname{Re}x^o + \mathrm{j}\operatorname{Im}x^o$, $g = \operatorname{Re}g + \mathrm{j}\operatorname{Im}g$. Then the equalities (5) take the form

$$\begin{aligned}\operatorname{Re}\lambda\operatorname{Re}x^o - \operatorname{Im}\lambda\operatorname{Im}x^o - A\operatorname{Re}x^o = B\operatorname{Re}g,\\ \operatorname{Im}\lambda\operatorname{Re}x^o + \operatorname{Re}\lambda\operatorname{Im}x^o - A\operatorname{Im}x^o = B\operatorname{Im}g\end{aligned} \tag{6}$$

and

$$\begin{aligned}C\operatorname{Re}x^o + D\operatorname{Re}g = 0,\\ C\operatorname{Im}x^o + D\operatorname{Im}g = 0,\end{aligned} \tag{7}$$

while the real valued initial conditions (x_o), inputs ($u_o(k)$) and solutions ($x_o(k)$) generated by $\lambda = |\lambda|\,e^{j\varphi}$ are of the form (comp. [8, pp. 25–28]):

$$\begin{aligned}&x_o = \operatorname{Re}x^o,\\ &u_o(k) = \operatorname{Re}(\lambda^k g) =\\ &= \begin{cases}\operatorname{Re}g & \text{for } k=0\\ |\lambda|^k(\operatorname{Re}g\cos k\varphi - \operatorname{Im}g\sin k\varphi) & \text{for } k=1,2,...\end{cases}\\ &x_o(k) = \operatorname{Re}(\lambda^k x^o) =\\ &= \begin{cases}\operatorname{Re}x^o & \text{for } k=0\\ |\lambda|^k(\operatorname{Re}x^o\cos k\varphi - \operatorname{Im}x^o\sin k\varphi) & \text{for } k=1,2,...\end{cases}\end{aligned} \tag{8}$$

and

$$\begin{aligned}&x_o = \operatorname{Im}x^o,\\ &u_o(k) = \operatorname{Im}(\lambda^k g) =\\ &= \begin{cases}\operatorname{Im}g & \text{for } k=0\\ |\lambda|^k(\operatorname{Re}g\sin k\varphi + \operatorname{Im}g\cos k\varphi) & \text{for } k=1,2,...\end{cases}\\ &x_o(k) = \operatorname{Im}(\lambda^k x^o) =\\ &= \begin{cases}\operatorname{Im}x^o & \text{for } k=0\\ |\lambda|^k(\operatorname{Re}x^o\sin k\varphi + \operatorname{Im}x^o\cos k\varphi) & \text{for } k=1,2,...\end{cases}\end{aligned} \tag{9}$$

Remark 2. Note that if $\lambda \in C$ such that $\operatorname{Im}\lambda \neq 0$ is an invariant zero of the standard system (1), i.e., a triple λ, $x^o \neq 0$, g satisfies (5), then $\operatorname{Im}g \neq 0$ or $\operatorname{Im}x^o \neq 0$. In fact, suppose that $\operatorname{Im}g = 0$ and $\operatorname{Im}x^o = 0$. Then, from the second equality in (6), it follows that $\operatorname{Im}\lambda\operatorname{Re}x^o = 0$, and consequently, $\operatorname{Re}x^o = 0$. Hence, $x^o = 0$, contrary to the assumption.

For positive system (1) we adopt the following definition of zeros.

Definition 2. A number $\lambda \in C$ is a finite zero of a strictly proper or proper positive discrete time system (1) if and only if there exist vectors $0 \neq x^o \in C^n$ and $g \in C^m$ such that the triple λ, $x^o \neq 0$, g satisfies (5) and λ generates an admissible (i.e., nonnegative) real valued input and an admissible (i.e., nonnegative) real valued solution of the state equation.

3. Main results

Theorem 2. If λ is a finite zero of a strictly proper or proper positive discrete-time system (1) and λ, $x^o \neq 0$, g satisfy (5), then λ is real and nonnegative, i.e., $\lambda \geq 0$; moreover, $x^o \in R^n_+$ and $g \in R^m_+$.

Proof. For the proof of the first assertion of the theorem it is enough to show that if $\lambda \in C$ satisfies (5) and λ is such that $\operatorname{Im}\lambda \neq 0$ or λ is a negative real number, then the conditions of Definition 2 are not fulfilled, i.e., λ does not generate admissible inputs and solutions.

Suppose first that a triple λ, $x^o \neq 0$, g satisfies (5) and $\operatorname{Im}\lambda \neq 0$. We consider the following two disjoint cases: $g = 0$ and $g \neq 0$.

In the first case, i.e., $g = 0$, we have, by virtue of Remark 2, $\operatorname{Im}x^o \neq 0$. Let the j-th component $(\operatorname{Im}x^o)_j$ of the vector $\operatorname{Im}x^o$ be nonzero. Then the j-th component $(x_o(k))_j$ of the solution $x_o(k)$ in (8) takes, for $k = 1, 2, ...$, the form

$$(x_o(k))_j = c\,|\lambda|^k\cos(\alpha + k\varphi),$$

where

$$c = \sqrt{(\operatorname{Re}x^o)_j^2 + (\operatorname{Im}x^o)_j^2}$$

and

$$\sin\alpha = \frac{(\operatorname{Im}x^o)_j}{c}.$$

It means that this component changes sign and consequently, $x_o(k)$ can not remain in R^n_+. The same reasoning applies to $x_o(k)$ in (9), where the j-th component takes, for $k = 1, 2, ...$, the form $(x_o(k))_j = c\,|\lambda|^k\sin(\alpha + k\varphi)$.

In the second case, i.e., $g \neq 0$, we have $\operatorname{Re}g \neq 0$ or $\operatorname{Im}g \neq 0$. Suppose that for the j-th component of g is $(\operatorname{Re}g)_j \neq 0$ or $(\operatorname{Im}g)_j \neq 0$. Then the j-th component $(u_o(k))_j$ of the input $u_o(k)$ in (8) takes, for $k = 1, 2, ...$, the form

$$(u_o(k))_j = d\,|\lambda|^k\cos(\beta + k\varphi),$$

where

$$d = \sqrt{(\operatorname{Re}g)_j^2 + (\operatorname{Im}g)_j^2}$$

and

$$\sin\beta = \frac{(\operatorname{Im}g)_j}{d}.$$

It means that this component changes sign when k changes and consequently, $u_o(k)$ can not remain in R^m_+. In the same way we analyze the j-th component of $u_o(k)$ in (9). Then, for $k = 1, 2, ...$, we have $(u_o(k))_j = d\,|\lambda|^k\sin(\beta + k\varphi)$ and $u_o(k)$ is not contained in R^m_+.

Suppose now that a triple λ, $x^o \neq 0$, g satisfies (5) and λ is a negative real number. Of course, x^o and g are taken as real vectors. Similarly as above we discuss separately the cases $g = 0$ and $g \neq 0$.

Let $g = 0$ and let $(x^o)_j \neq 0$. Then the j-th component of the solution generated by λ takes the form $(x_o(k))_j = (-1)^k|\lambda|^k(x^o)_j$, for $k = 0, 1, 2, ...$, and $x_o(k)$ is not admissible.

Suppose now that $g \neq 0$ and let $(g)_j \neq 0$. Then the j-th component $(u_o(k))_j$ of the input $u_o(k)$ generated by λ takes the form $(u_o(k))_j = (-1)^k|\lambda|^k(g)_j$ and consequently, $u_o(k)$ is not admissible.

In this way we have shown that if $\lambda \in C$ satisfies Definition 2, then $\operatorname{Im}\lambda = 0$, i.e., λ is a real number. Moreover, if a real number λ satisfies Definition 2, then λ is nonnegative, i.e., $\lambda \geq 0$.

The last assertion of the theorem follows directly from the above and Definition 2. This completes the proof.

In the remaining part of this section we consider a positive proper or strictly proper system (1) of *uniform rank* which means that the system is square (the number of inputs equals the number of outputs, i.e., $m = r$) and the first nonzero Markov parameter is nonsingular. For strictly proper systems the first nonzero Markov parameter is denoted by $CA^{\nu}B$, where $0 \leq \nu \leq n-1$.

Theorem 3. Suppose that in a positive proper system (1) of uniform rank the observability matrix has the full column rank n. Then the system has no finite zeros.

Proof. We proceed the proof via *reductio ad absurdum*. To this end, suppose that a number λ is a finite zero of the system. Then, by virtue of Theorem 2, $\lambda \geq 0$ and $x^o \in R_+^n$, $g \in R_+^m$. From the second equality in (5) we have $Cx^o + Dg = 0$ and, by virtue of Theorem 1, we obtain $Cx^o = 0$ and $Dg = 0$. Since, by assumption, D is nonsingular, we obtain $g = 0$. Now, the condition (5) yields the equalities $\lambda x^o - Ax^o = 0$, $Cx^o = 0$. Premultiplying subsequently the first equality by C, CA, ..., CA^{n-2} and taking into account the second equality, we get the following sequence of equalities $Cx^o = 0$, $CAx^o = 0, ..., CA^{n-1}x^o = 0$. Since the observability matrix has full column rank, we obtain $x^o = 0$ which contradicts the assumption $x^o \neq 0$ (comp. Definition 2).

Theorem 4. Suppose that in a positive strictly proper system (1) of uniform rank the observability matrix has the full column rank n. Then the system has no finite zeros.

Proof. Let $CA^{\nu}B$, $0 \leq \nu \leq n-1$, be the first nonzero Markov parameter, i.e., $CB = CAB = ... = CA^{\nu-1}B = 0$ and $CA^{\nu}B$ is nonsingular. Suppose that Theorem 4 is not valid and let a number λ be a finite zero of the system. Then, by virtue of Theorem 2, $\lambda \geq 0$, $x^o \in R_+^n$ and $g \in R_+^m$. The condition (5) takes the form $\lambda x^o - Ax^o = Bg$, $Cx^o = 0$. By successive multiplication of the first equality from the left by C, CA, ..., CA^{ν} and by using $CB = CAB = ... = CA^{\nu-1}B = 0$ and $Cx^o = 0$, we obtain the relation $CA^{\nu}Bg = -CA^{\nu+1}x^o$. By virtue of Theorems 1 and 2 this relation can be fulfilled if and only if $CA^{\nu}Bg = 0$ and $CA^{\nu+1}x^o = 0$. In view of nonsingularity of $CA^{\nu}B$, we obtain $g = 0$. The remaining part of the proof follows the same lines as the proof of Theorem 3.

Corollary 1. Consider a SISO (single input, single output, i.e., $m = r = 1$) strictly proper or proper positive system (1) which is observable as a standard system. Then the positive system has no finite zeros.

Remark 3. As is known ([8, pp. 140–142]), a standard strictly proper system (1) of uniform rank with $CA^{\nu}B$ as the first nonzero Markov parameter has $n-m(\nu+1)$ invariant (Smith) zeros. In particular, a SISO strictly proper standard system has $n-(\nu+1)$ invariant (Smith) zeros. For a proper standard system (1) of uniform rank the number of invariant (Smith) zeros equals n (the same holds for a SISO standard proper system) (see [8, p. 142]).

Remark 4. Note that Theorems 3 and 4 remain valid for non-square positive systems (1) if the assumption of uniform rank is replaced by the assumption that the first nonzero Markov parameter has full column rank. The proofs follow the same lines.

Theorem 5. Suppose that in a positive proper system (1) of uniform rank the observability matrix has full column rank. Then the system has only the trivial output-zeroing input.

Proof. Let $(x_o, u_o(k))$ be a nontrivial output-zeroing input and let $x_o(k)$ denote the corresponding solution of the state equation. At this assumption for each $k \in Z_+$ the following equalities hold

$$\begin{aligned} x_o(k+1) &= Ax_o(k) + Bu_o(k), \qquad x_o(0) = x_o, \\ 0 &= Cx_o(k) + Du_o(k). \end{aligned} \tag{10}$$

By virtue of Theorem 1 and the definition of the output-zeroing problem for positive systems, we obtain from the second equality in (10) the relation $Du_o(k) = -Cx_o(k)$ and consequently, $Cx_o(k) = 0$ and $Du_o(k) = 0$ for all $k \in Z_+$. The last equality yields $u_o(k) \equiv 0$ and consequently, $x_o(k) = A^k x_o$. Hence, $Cx_o(k) = CA^k x_o = 0$ for all $k \in Z_+$. In particular, we can write $Cx_o = 0$, $CAx_o = 0, ..., CA^{n-1}x_o = 0$ and, in view of the assumption concerning the observability matrix, we obtain $x_o = 0$. This contradicts the assumption that $(x_o, u_o(k))$ is nontrivial.

Theorem 6. Suppose that in a positive strictly proper system (1) of uniform rank the observability matrix has the full column rank n. Then the system has only the trivial output-zeroing input.

Proof. It is enough to show that each output-zeroing input is trivial. To this end, let $(x_o, u_o(k))$ be a nontrivial output-zeroing input and let $x_o(k)$ denote the corresponding solution of the state equation. At this assumption we have the following equalities

$$\begin{aligned} x_o(k+1) &= Ax_o(k) + Bu_o(k), \qquad x_o(0) = x_o, \\ Cx_o(k) &= 0, \end{aligned} \tag{11}$$

which are valid for each $k \in Z_+$. By successive multiplication of the first equality in (11) from the left by C, CA, ..., CA^{ν} and by using $CB = CAB = ... = CA^{\nu-1}B = 0$ and $Cx_o(k) = 0$, we obtain at the last step the equality

$$CA^{\nu}Bu_o(k) = -CA^{\nu+1}x_o(k) \tag{12}$$

which holds for each $k \in Z_+$. By virtue of Theorem 1 and the definition of output-zeroing inputs for the positive system (1), from (12) it follows that for each $k \in Z_+$ we have $CA^{\nu+1}x_o(k) = 0$ and $CA^{\nu}Bu_o(k) = 0$. In view of non-singularity of $CA^{\nu}B$, this last equality means that $u_o(k) = 0$ for each $k \in Z_+$. In this way, from (2) we obtain $x_o(k) = A^k x_o$ for each $k \in Z_+$ and from (11) we have $CA^k x_o = 0$ for all $k \in Z_+$. In particular, we get $Cx_o = 0$, $CAx_o = 0, ..., CA^{n-1}x_o = 0$ and from the assumption concerning the observability matrix we obtain $x_o = 0$. In this way, we have shown that $(x_o, u_o(k))$ is the trivial output-zeroing input.

Remark 5. Theorems 5 and 6 remain valid for non-square positive systems (1) when the assumption of uniform rank is replaced by the assumption that the first nonzero Markov parameter has full column rank. The proofs follow the same lines.

4. Examples

Example 1. Consider a positive SISO system (1) with the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$c = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

The transfer function of the system equals zero identically (i.e., $g(z) = c(zI - A)^{-1}b \equiv 0$). Solving Eq. (5), it is easy to verify that the triple

$$\lambda = 1/2, \qquad x^o = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \qquad g = 1$$

satisfies (5) and generates the following output-zeroing input for the system

$$x_o = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad u_o(k) = g\,\lambda^k = (1/2)^k.$$

This means that Definition 2 is satisfied, i.e., $\lambda = 1/2$ is a finite zero of the system. The corresponding solution takes the form

$$x_o(k) = x_o\lambda^k = (1/2)^k \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}.$$

Furthermore, any triple of the form

$$1/2 \neq \lambda > 0, \qquad x^o = \begin{bmatrix} x_1^o \\ 0 \\ 0 \end{bmatrix},$$

where

$$x_1^o > 0, \qquad g = \lambda x_1^o$$

also satisfies Definition 2. Hence, any $1/2 \neq \lambda > 0$ is also a finite zero of the system.

Note that the considered positive system treated as a standard system is degenerate, i.e., each complex number is its invariant zero (comp. [8, pp. 50–53]).

Example 2. Consider a positive system (1) with the matrices

$$A = \begin{bmatrix} 1/3 & 0 & 2 \\ 0 & 1/3 & 1 \\ 0 & 0 & 1/3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Equations (5) take the form

$$\left(\lambda - \frac{1}{3}\right) x_1^o - 2x_3^o - g_1 - g_2 = 0,$$

$$\left(\lambda - \frac{1}{3}\right) x_2^o - x_3^o - g_1 = 0,$$

$$\left(\lambda - \frac{1}{3}\right) x_3^o = 0,$$

$$x_2^o + 2x_3^o + g_1 = 0,$$

where

$$x^o = \begin{bmatrix} x_1^o \\ x_2^o \\ x_3^o \end{bmatrix}, \qquad g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}.$$

It is easy to verify that for any given $\lambda > 1/3$ and $x_1^o > 0$ the triple

$$\lambda, \qquad x^o = \begin{bmatrix} x_1^o \\ 0 \\ 0 \end{bmatrix}, \qquad g = \begin{bmatrix} 0 \\ \left(\lambda - \frac{1}{3}\right) x_1^o \end{bmatrix}$$

satisfies Definition 2. This means that any real number greater than 1/3 is a finite zero of the system. Moreover, for any given $x_1^o > 0$, the triple

$$\lambda = \frac{1}{3}, \qquad x^o = \begin{bmatrix} x_1^o \\ 0 \\ 0 \end{bmatrix}, \qquad g = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

also satisfies Definition 2 (i.e., $\lambda = 1/3$ is also a finite zero of the system).

On the other hand, the considered system, treated as a standard one, is degenerate (i.e., each complex number is its invariant zero). Moreover, it has exactly two Smith zeros ($\lambda = 1/3$ and $\lambda = -2/3$) and $\lambda = 1/3$ is simultaneously the output decoupling zero.

Example 3. Consider a SISO positive system (1) with the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$c = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

The assumptions of Corollary 1 are fulfilled and the system has no finite zeros. On the other hand, the transfer function of this system equals $g(z) = \dfrac{z^2 + z + 1}{z^3}$ and the system, treated as a standard one, has two invariant (Smith) zeros.

5. Conclusions

The notion of finite zeros for positive discrete-time linear systems has been introduced (Definition 2). This notion uses the assumption that finite zeros generate admissible (i.e., nonnegative) output-zeroing inputs and the corresponding solutions. As a consequence, finite zeros of a positive discrete-time system (if they exist) are nonnegative real numbers, while the corresponding state-zero and input-zero directions remain in R_+^n and R_+^m respectively (Theorem 2).

It has been shown that positive discrete-time strictly proper or proper systems of uniform rank do not possess finite zeros (Theorems 3, 4 and Corollary 1) nor nontrivial output-zeroing inputs (Theorems 5, 6). Theorems 3-6 remain valid for non-square systems with the first nonzero Markov parameter of full column rank (Remarks 4, 5). The considerations have been illustrated by simple numerical examples.

The obtained results clearly show that positivity constraints (Definition 1 and Theorem 1) imposed on discrete-time linear systems result in limitations concerning internal dynamics and location of zeros (comp. [6, 11, 12]).

An open problem is an extension of the above considerations to positive continuous-time linear systems.

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