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Computation of positive stable realizations for linear continuous-time systems

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Abstract. Conditions for the existence of positive stable realizations with system Metzler matrices for proper transfer function are established. It is shown that there exists a proper stable realization of transfer function of second order if and only if the transfer function has real negative poles. Sufficient conditions for the existence of positive stable realizations of transfer function of third order are established. A method based on the decomposition of transfer functions into the first, second and third orders transfer functions for computation of positive stable realizations is proposed. A method for computation of positive stable realizations of transfer functions with real negative poles and zeros is given.

Key words: positive, stable, realization, system Metzler matrix, procedure, linear continuous-time.

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs [1, 2]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

An overview on the positive realization problem is given in [1–3]. The realization problem for positive continuous-time and discrete-time linear systems has been considered in [4–7] and the positive minimal realization problem for singular discrete-time systems with delays in [8]. The realization problem for fractional linear systems has been analyzed in [9, 10] and for positive 2D hybrid systems in [11]. A method based on the similarity transformation of the standard realizations to the desired form has been proposed in [10].

In this paper sufficient conditions will be established for the existence of positive stable realizations with system Metzler matrices and procedures for computation of the realizations of proper transfer functions will be proposed.

The paper is organized as follows. In Sec. 2 some definitions and theorems concerning positive continuous-time linear systems are recalled and the problem formulation is given. Problem solution is presented in Secs. 3, 4 and 5. In Sec. 3 first systems of the second and third orders are analyzed. Next the general case is solved. The systems with real poles by the use of Gilbert method are analyzed. Systems with real poles and zeros are considered in Sec. 5. Concluding remarks and open problems are given in Sec. 6.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n=\Re_+^{n\times 1}$, $\Re^{n\times m}[s]$ – the set of $n\times m$ polynomial matrices in s with real coefficients, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. Preliminaries and the problem formulation

Consider the continuous-time linear system

$$\dot{x}(t)=Ax(t)+Bu(t), \tag{1a}$$

$$y(t)=Cx(t)+Du(t), \tag{1b}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

Definition 1. [1, 2] The system (1) is called (internally) positive if $x(t)\in\Re_+^n$, $y(t)\in\Re_+^p$, $t\geq 0$ for any initial conditions $x(0)=x_0\in\Re_+^n$ and all inputs $u(t)\in\Re_+^m$, $t\geq 0$.

Theorem 1. [1, 2] The system (2.1) is positive if and only if

$$\begin{gathered} A\in M_n, \quad B\in\Re_+^{n\times m}, \\ C\in\Re_+^{p\times n}, \quad D\in\Re_+^{p\times m}. \end{gathered} \tag{2}$$

The transfer matrix of the system (1) is given by

$$T(s)=C[I_ns-A]^{-1}B+D. \tag{3}$$

The transfer matrix is called proper if

$$\lim_{s\to\infty}T(s)=K\in\Re^{p\times m} \tag{4}$$

and it is called strictly proper if $K=0$.

Definition 2. Matrices (2) are called a positive realization of transfer matrix $T(s)$ if they satisfy the equality (3).

The realization is called (asymptotically) stable if the matrix A is a (asymptotically) stable Metzler matrix (Hurwitz Metzler matrix).

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Theorem 2. [2] The positive realization (2) is stable if and only if all coefficients of the polynomial

$$p_A(s) = \det[I_n s - A] = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 \quad (5)$$

are positive, i.e. $a_i > 0$ for $i = 0, 1, ..., n-1$.

The problem under considerations can be stated as follows.

Given a rational matrix $T(s) \in \Re^{p\times m}(s)$, find its positive stable realization

$$A \in M_{nS}, \quad B \in \Re_+^{n\times m}, \quad C \in \Re_+^{p\times n}, \quad D \in \Re_+^{p\times m}, \quad (6)$$

where M_{nS} is the set of $n \times n$ (asymptotically) stable Metzler matrices.

3. Problem solution

3.1. Particular case. First we shall consider the positive system (1) with the transfer function

$$T(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0}. \quad (7)$$

The positive system with (7) is (asymptotically) stable if and only if $a_i > 0$ for $i = 0, 1$ [2].

Knowing the transfer function (7) we can find the matrix D by the use of the formula [2]

$$D = \lim_{s\to\infty} T(s) = b_2 \quad (8)$$

and the strictly proper transfer function

$$T_{sp}(s) = T(s) - b_2 = \frac{\overline{b}_1 s + \overline{b}_0}{s^2 + a_1 s + a_0}, \quad (9)$$

where $\overline{b}_i = b_i - a_i b_2$, $i = 0, 1$.

It is assumed that the transfer function (7) (and also (9)) satisfies the condition

$$a_1^2 - 4a_0 \geq 0 \quad (10)$$

and it has two real negative poles

$$\begin{aligned} s_1 = -\alpha = \frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2}, \\ s_2 = -\beta = \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2}. \end{aligned} \quad (11)$$

Theorem 3. There exists a positive stable realization of the form

$$A = \begin{bmatrix} -\alpha & 1 \\ 0 & -\beta \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (12)$$
$$C = \begin{bmatrix} b_2(a_1\alpha - a_0) - b_1\alpha + b_0 & b_1 - b_2 a_1 \end{bmatrix}$$

of the transfer function (7) if and only if the following conditions are satisfied:

1. the condition (10) is met,
2. $b_2(a_1\alpha - a_0) - b_1\alpha + b_0 \geq 0$ and $b_1 - b_2 a_1 \geq 0$, (13)
3. $b_2 \geq 0$ (14)

Proof. The transfer function (9) has two real negative poles if and only if the condition (10) is met. The matrices (12) are the positive stable realization of the strictly proper transfer function (9) since

$$\begin{aligned} &C[I_2 s - A]^{-1}B = \begin{bmatrix} b_2(a_1\alpha - a_0) - b_1\alpha + b_0 & b_1 - b_2 a_1 \end{bmatrix} \cdot \\ &\cdot \begin{bmatrix} s+\alpha & -1 \\ 0 & s+\beta \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \\ &= \begin{bmatrix} b_2(a_1\alpha - a_0) - b_1\alpha + b_0 & b_1 - b_2 a_1 \end{bmatrix} \frac{1}{(s+\alpha)(s+\beta)} \cdot \\ &\cdot \begin{bmatrix} s+\beta & 1 \\ 0 & s+\alpha \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \\ &= \frac{\begin{bmatrix} b_2(a_1\alpha - a_0) - b_1\alpha + b_0 & b_1 - b_2 a_1 \end{bmatrix}}{(s+\alpha)(s+\beta)} \begin{bmatrix} 1 \\ s+\alpha \end{bmatrix} = \\ &= \frac{\overline{b}_1 s + \overline{b}_0}{s^2 + a_1 s + a_0} = T_{sp}(s) \end{aligned} \quad (15)$$

Note that $C \in \Re_+^{1\times 2}$ and $D \in \Re_+^{1\times 1}$ if and only if the conditions (13) and (14) are satisfied.

Remark 1. The matrix

$$A = \begin{bmatrix} -a_1 & a_2 \\ a_3 & -a_4 \end{bmatrix} a_{i>0, i=1,...,4} \quad (16)$$

is a Metzler matrix if and only if it has two real eigenvalues since its characteristic polynomial

$$\begin{aligned} \det[I_2 s - A] = \begin{vmatrix} s + a_1 & -a_2 \\ -a_3 & s + a_4 \end{vmatrix} = \\ = s^2 + (a_1 + a_4)s + a_1 a_4 - a_2 a_3 \end{aligned} \quad (17)$$

satisfies the condition

$$(a_1 + a_4)^2 - 4(a_1 a_4 - a_2 a_3) = (a_1 - a_4)^2 + 4a_2 a_3 \geq 0. \quad (18)$$

Example 1. Find the positive stable realization of the transfer function

$$T(s) = \frac{2s^2 + 7s + 7}{s^2 + 3s + 2}. \quad (19)$$

The conditions of Theorem 3 are satisfied since

$$a_1^2 - 4a_0 = 9 - 8 > 0, \quad b_2 = 2 > 0,$$
$$s_1 = -\alpha = -1, \quad s_2 = -\beta = -2$$

and

$$b_2(a_1\alpha - a_0) - b_1\alpha + b_0 = 2, \quad b_1 - b_2 a_1 = 1.$$

Using (8) and (9) we obtain

$$D = \lim_{s\to\infty} T(s) = 2 \quad (20)$$

and

$$T_{sp}(s) = T(s) - D = \frac{s+3}{s^2 + 3s + 2}. \quad (21)$$

The positive stable realization of (21) has the form

$$A = \begin{bmatrix} -\alpha & 1 \\ 0 & -\beta \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} b_2(a_1\alpha - a_0) - b_1\alpha + b_0 & b_1 - b_2 a_1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix}. \quad (22)$$

The desired positive stable realization of (19) is given by (22) and (20).

Now let us consider the positive system with the transfer function

$$T(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}. \tag{23}$$

In this case the matrix D has the form

$$D = \lim_{s \to \infty} T(s) = b_3 \tag{24}$$

and

$$T_{sp}(s) = T(s) - D = \frac{\overline{b}_2 s^2 + \overline{b}_1 s + \overline{b}_0}{s^3 + a_2 s^2 + a_1 s + a_0}, \tag{25}$$

where $\overline{b}_i = b_i - a_i b_3$, $i = 0, 1, 2$.

A realization of (25) has the form

$$\overline{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{26}$$
$$\overline{C} = [\ \overline{b}_0 \quad \overline{b}_1 \quad \overline{b}_2\ ].$$

Note that the realization (26) for stable system is not positive since the stability of (25) implies $a_i > 0$, $i = 0, 1, 2$ and in this case $\overline{A}$ is not a Metzler matrix.

We are looking for a nonsingular matrix $P \in \Re^{3\times 3}$ such that [10]

$$A = P\overline{A}P^{-1} \in M_{3S}, \quad B = P\overline{B} \in \Re^{3}_{+}, \quad C = \overline{C}P^{-1} \in \Re^{1\times 3}_{+}. \tag{27}$$

It is well-known [10, 15] that

$$\det[I_3 s - A] = \det[I_3 s - \overline{A}] \tag{28}$$

for any nonsingular matrix $P \in \Re^{3\times 3}$. It is easy to check that if

$$P = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \alpha^2 & 2\alpha & 1 \end{bmatrix}, \quad (\alpha \text{ – arbitrary}). \tag{29}$$

then

$$A = P\overline{A}P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \alpha^2 & 2\alpha & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -\alpha & 1 & 0 \\ \alpha^2 & -2\alpha & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -\alpha & 1 & 0 \\ 0 & -\alpha & 1 \\ \alpha^3 - a_2\alpha^2 + a_1\alpha - a_0 & -3\alpha^2 + 2a_2\alpha - a_1 & 2\alpha - a_2 \end{bmatrix}, \tag{30a}$$

$$B = P\overline{B} = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \alpha^2 & 2\alpha & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{30b}$$

$$C = \overline{C}P^{-1} = [\ \overline{b}_0 \quad \overline{b}_1 \quad \overline{b}_2\ ] \begin{bmatrix} 1 & 0 & 0 \\ -\alpha & 1 & 0 \\ \alpha^2 & -2\alpha & 1 \end{bmatrix} = [\ \overline{b}_0 - \overline{b}_1\alpha + \overline{b}_2\alpha^2 \quad \overline{b}_1 - 2\alpha\overline{b}_2 \quad \overline{b}_2\ ]. \tag{30c}$$

We choose α so that

$$p_1(\alpha) = \alpha^3 - a_2\alpha^2 + a_1\alpha - a_0 \geq 0 \tag{31a}$$

and the matrix

$$A_2 = \begin{bmatrix} -\alpha & 1 \\ -3\alpha^2 + 2a_2\alpha - a_1 & 2\alpha - a_2 \end{bmatrix} \tag{31b}$$

has two real negative eigenvalues or equivalently its characteristic polynomial

$$\det[I_2 s - A_2] = s^2 + (a_2 - \alpha)s + a_1 + \alpha^2 - a_2\alpha \tag{31c}$$

has two real negative zeros

$$s_{1,2} = \frac{\alpha - a_2 \pm \sqrt{(a_2 - \alpha)^2 - 4(a_1 + \alpha^2 - a_2\alpha)}}{2}. \tag{31d}$$

This implies $p_2(\alpha) = (a_2 - \alpha)^2 - 4(a_1 + \alpha^2 - a_2\alpha) = -3\alpha^2 + 2a_2\alpha - 4a_1 + a_2^2 \geq 0$ and there exists $\alpha > 0$ satisfying $p_2(\alpha) = 0$ if and only if

$$\alpha_{1,2} = \frac{2a_2 \pm 4\sqrt{a_2^2 - 3a_1}}{6} \tag{32}$$

is a real number, i.e.

$$a_2^2 \geq 3a_1. \tag{33}$$

Note that the polynomial $p_1(\alpha)$ reaches its extremum for α satisfying

$$\frac{dp_1(\alpha)}{d\alpha} = 3\alpha^2 - 2a_2\alpha + a_1 = 0. \tag{34}$$

In this case the matrix (31b) takes the form

$$A_2 = \begin{bmatrix} -\alpha & 1 \\ 0 & 2\alpha - a_2 \end{bmatrix}. \tag{35}$$

From (34) we have

$$a_1 = -3\alpha^2 + 2a_2\alpha \tag{36}$$

and substituting (36) into (31a) we obtain

$$p_1(\alpha) = a_2\alpha^2 - 2\alpha^3 - a_0 \geq 0 \tag{37}$$

and

$$A = \begin{bmatrix} -\alpha & 1 & 0 \\ 0 & -\alpha & 1 \\ -2\alpha^3 + a_2\alpha^2 - a_0 & 0 & 2\alpha - a_2 \end{bmatrix}. \tag{38}$$

From (30c) it follows that $C \in \Re^{1\times 3}_{+}$ if

$$\begin{aligned} \overline{b}_0 - \overline{b}_1\alpha + \overline{b}_2\alpha^2 &= b_0 - b_1\alpha + b_2\alpha^2 + \\ &+ (-a_2\alpha^2 + a_1\alpha - a_0)b_3 \geq 0, \\ \overline{b}_1 - 2\alpha\overline{b}_2 &= b_1 - 2\alpha b_2 + (2a_2\alpha - a_1)b_3 \geq 0, \\ \overline{b}_2 &= b_2 - a_2 b_3 \geq 0. \end{aligned} \tag{39}$$

Therefore, the following theorem has been proved.

Theorem 4. There exists a positive stable realization (38), (30b) and (30c) of the transfer function (23) if the following conditions are satisfied:

1. $\lim_{s\to\infty} T(s) = b_3 \in \Re_+$,
2. the condition (33) is met and α can be chosen so that (37) holds,
3. the conditions (39) are satisfied.

Example 2. Find a positive stable realization of the strictly proper transfer function

$$T(s) = \frac{s^2+5s+8}{s^3+7s^2+16s+10}. \tag{40}$$

The transfer function (40) has one real pole $s_1 = -1$ and two complex conjugate poles $s_2 = -3+j$, $s_3 = -3-j$. In this case the condition (33) is met since $a_2^2 - 3a_1 = 7^2 - 48 = 1$. We choose $\alpha = 2$ for which the conditions (37) and (39) are satisfied since

$$\begin{gathered} p_1(\alpha) = 7\cdot(2)^2 - 2\cdot 8 - 10 = 2, \\ \overline{b}_0 - \overline{b}_1\alpha + \overline{b}_2\alpha^2 = 8 - 10 + 4 = 2, \\ \overline{b}_1 - 2\alpha\overline{b}_2 = 5 - 4 = 1, \\ \overline{b}_2 = 1 \end{gathered}$$

and the matrix C has the form

$$C = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}. \tag{41}$$

The matrices A and B are

$$\begin{aligned} A &= \begin{bmatrix} -\alpha & 1 & 0 \\ 0 & -\alpha & 1 \\ -2\alpha^3 + a_2\alpha^2 - a_0 & 0 & 2\alpha - a_2 \end{bmatrix} = \\ &= \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 2 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \end{aligned} \tag{42}$$

The desired positive stable realization of (40) is given by (42), (41) and $D = [0]$.

3.2. General case. Consider the positive continuous-time linear system (1) with the transfer function

$$T(s) = \frac{b_n s^n + b_{n-1}s^{n-1} + ... + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0}. \tag{43}$$

The positive system with (43) is (asymptotically) stable if and only if $a_i > 0$ for $i = 0, 1, ..., n-1$ [1, 2].

Knowing the transfer function (43) we can find the matrix D by the use of the formula

$$D = \lim_{s\to\infty} T(s) = b_n \tag{44}$$

and the strictly proper transfer function

$$\begin{aligned} T_{sp}(s) &= T(s) - D = C[I_n s - A]^{-1}B = \\ &= \frac{b_n s^n + b_{n-1}s^{n-1} + ... + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0} - b_n = \\ &= \frac{\overline{b}_{n-1}s^{n-1} + ... + \overline{b}_1 s + \overline{b}_0}{s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0}, \end{aligned} \tag{45}$$

where $\overline{b}_i = b_i - a_i b_n$, $i = 0, 1, ..., n-1$.

The transfer function (45) can be written as follows

$$T_{sp}(s) = \sum_{i=1}^{k} T_i(s), \tag{46a}$$

where $T_i(s)$ may have one of the following forms

$$T_i(s) = \begin{cases} \dfrac{T_i}{s+s_i}, \\[2ex] \dfrac{\overline{b}_1 s + \overline{b}_0}{s^2 + a_1 s + a_0}, \\[2ex] \dfrac{\overline{b}_2 s^2 + \overline{b}_1 s + \overline{b}_0}{s^3 + a_2 s^2 + a_1 s + a_0}. \end{cases} \tag{46b}$$

The polynomial $s^2 + a_1 s + a_0$ has two real negative zeros and the polynomial $s^3 + a_2 s^2 + a_1 s + a_0$ has one real negative zero and two complex conjugate stable zeros.

The coefficients of $T_i(s)$ can be found by the comparison of the coefficients of (46a) and (45) at the same powers of s.

Theorem 5. Let $A_i, B_i, C_i \quad i = 1, ..., k$ be positive stable realizations of the transfer functions (46b) of (46a), then the desirable realization of (45) is given by

$$\begin{gathered} A = \text{blockdiag}\begin{bmatrix} A_1 & ... & A_k \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ \vdots \\ B_k \end{bmatrix}, \\ C = \begin{bmatrix} C_1 & ... & C_k \end{bmatrix}. \end{gathered} \tag{47}$$

Proof. Using (47) and (45) we obtain

$$\begin{aligned} C[Is - A]^{-1}B &= \begin{bmatrix} C_1 & ... & C_k \end{bmatrix} \cdot \\ &\cdot \begin{bmatrix} Is - A_1 & 0 & ... & 0 \\ 0 & Is - A_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & Is - A_k \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ \vdots \\ B_k \end{bmatrix} = \\ &= \begin{bmatrix} C_1 & ... & C_k \end{bmatrix} \cdot \\ &\cdot \begin{bmatrix} [Is - A_1]^{-1} & 0 & ... & 0 \\ 0 & [Is - A_2]^{-1} & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & [Is - A_k]^{-1} \end{bmatrix} \cdot \\ &\cdot \begin{bmatrix} B_1 \\ \vdots \\ B_k \end{bmatrix} = \sum_{i=1}^{k} C_i[Is - A_i]^{-1}B_i = \sum_{i=1}^{k} T_i(s) = T_{sp}(s). \end{aligned}$$

The details of the procedure will be demonstrated on the following two examples.

Example 3. Find a positive stable realization of the strictly proper transfer function

$$T_{sp}(s) = \frac{3s^3 + 21s^2 + 50s + 36}{s^4 + 9s^3 + 30s^2 + 42s + 20} = \frac{3s^3 + 21s^2 + 50s + 36}{(s+1)(s+2)(s+3+j)(s+3-j)}. \tag{48}$$

We decomposed (48) in the following two parts

$$T_{sp}(s) = T_1(s) + T_2(s), \tag{49}$$

where

$$T_1(s) = \frac{2}{s+2}, \quad T_2(s) = \frac{s^2 + 5s + 8}{s^3 + 7s^2 + 16s + 10}. \tag{50}$$

Note that (48) can be also decomposed in two following parts

$$\overline{T}_1(s) = \frac{2.8s + 3.6}{(s+1)(s+2)}, \quad \overline{T}_2(s) = \frac{0.2s}{(s+3+j)(s+3-j)} \tag{51}$$

but the transfer function $\overline{T}_2(s)$ has not a positive stable realization because the condition (10) is not satisfied.

A positive stable realization of $T_1(s)$ has the form

$$A_1 = [-2], \qquad B_1 = [1], \qquad C_1 = [2] \tag{52}$$

and a stable realization of $T_2(s)$ is given by (42) and (41). Using (47) we obtain the desired positive stable realization of (48) in the form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 2 & 0 & -3 \end{bmatrix},$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{53}$$

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 & 1 \end{bmatrix}.$$

Example 4. Find a positive stable realization of the strictly proper transfer function

$$T_{sp}(s) = \frac{2s^4 + 18s^3 + 62s^2 + 92s + 46}{s^5 + 10s^4 + 39s^3 + 72s^2 + 62s + 20} = \frac{2s^4 + 18s^3 + 62s^2 + 92s + 46}{(s+1)^2(s+2)(s+3+j)(s+3-j)}. \tag{54}$$

We decomposed (54) in two parts

$$T_{sp}(s) = T_1(s) + T_2(s), \tag{55}$$

where

$$T_1(s) = \frac{s+3}{(s+1)(s+2)} = \frac{s+3}{s^2 + 3s + 2}, \tag{56a}$$

$$T_2(s) = \frac{s^2 + 5s + 8}{(s+1)(s+3+j)(s+3-j)} = \frac{s^2 + 5s + 8}{s^3 + 7s^2 + 16s + 10}. \tag{56b}$$

A positive stable realization of (56a) is given by (22) and of (56b) by (42) and (41).

Using (47) we obtain the desired positive stable realization of (54) in the form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 2 & 0 & -3 \end{bmatrix},$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{57}$$

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 & 1 & 1 \end{bmatrix}.$$

The considerations can be extended to multi-input multi-output linear systems [10].

4. System with real poles

In this section using Gilbert method [14] a procedure for finding positive stable realizations with system Metzler matrices will be presented for transfer matrices with real negative poles.

Consider a linear continuous-time system with m-inputs, p-outputs and the strictly proper transfer matrix

$$T_{sp}(s) = \frac{N(s)}{d(s)} \in \Re^{p \times m}(s), \tag{58}$$

where $N(s) \in \Re^{p \times m}[s]$ and is the least common denominator of all entries of the matrix

$$d(s) = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 \tag{59}$$

is the least common denominator of all entries of the matrix.

It is assumed that the equation $d(s) = 0$ has only distinct real negative roots $s_1, s_2, ..., s_n$ ($s_i \neq s_j$ for $i \neq j$), i.e. $d(s) = (s - s_1)(s - s_2)...(s - s_n)$. In this case the transfer matrix (58) can be written in the form

$$T_{sp}(s) = \sum_{i=1}^{n} \frac{T_i}{s - s_i}, \tag{60}$$

where

$$T_i = \lim_{s \to s_i} (s - s_i) T_{sp}(s) = \frac{N(s_i)}{\prod\limits_{j=1, j \neq i}^{n} (s_i - s_j)}, \quad i = 1, ..., n. \tag{61}$$

Let

$$\text{rank } T_i = r_i \leq \min(p, m). \tag{62}$$

It is easy to show [14] that

$$T_i = C_i B_i, \quad \text{rank } C_i = \text{rank } B_i = r_i, \quad i = 1, ..., n, \tag{63a}$$

where

$$C_i = [\ C_{i,1} \quad C_{i,2} \quad \dots \quad C_{i,r_i}\ ] \in \Re^{p\times r_i},$$

$$B_i = \begin{bmatrix} B_{i,1} \\ B_{i,2} \\ \vdots \\ B_{i,r_i} \end{bmatrix} \in \Re^{r_i\times m}. \tag{63b}$$

We shall show that the matrices are the desired positive stable realization with system Metzler matrix

$$A = \text{blockdiag}[\ I_{r_1}s_1 \quad \dots \quad I_{r_n}s_n\ ],$$

$$B = \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix}, \qquad C = [\ C_1 \quad \dots \quad C_n\ ]. \tag{64}$$

Using (64), (63) and (60) we obtain

$$T_{sp}(s) = C[Is-A]^{-1}B = [\ C_1 \quad \dots \quad C_n\ ]\cdot$$
$$\cdot\left(\text{blockdiag}[\ I_{r_1}(s-s_1)^{-1} \quad \dots \quad I_{r_n}(s-s_n)^{-1}\ ]\right)\cdot$$
$$\cdot\begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} = \sum_{i=1}^{n}\frac{C_iB_i}{s-s_i} = \sum_{i=1}^{n}\frac{T_i}{s-s_i}. \tag{65}$$

From (64) it follows that:

1. if $s_1, s_2, ..., s_n$ are real negative then the matrix A is stable and is a Metzler matrix,
2. if
$$T_i \in \Re_+^{p\times m} \quad \text{for} \quad i = 1, ..., n. \tag{66}$$
Then we can find
$$C_i \in \Re_+^{p\times r_i} \quad \text{and} \quad B_i \in \Re_+^{r_i\times m} \quad \text{for} \quad i = 1, ..., n \tag{67}$$
and $B \in \Re_+^{\overline{n}\times m}$, $C \in \Re_+^{p\times\overline{n}}$, $\overline{n} = \sum_{i=1}^{n} r_i$.

If $T(\infty) \in \Re_+^{p\times m}$ then from

$$D = \lim_{s\to\infty} T(s) \tag{68}$$

we have $D \in \Re_+^{p\times m}$. Therefore, the following theorem has been proved.

Theorem 6. There exists a positive stable realization (64) and $D \in \Re_+^{p\times m}$ of the proper transfer matrix $T(s) \in \Re^{p\times m}(s)$ if the following conditions are satisfied:

1. The poles of $T(s)$ are distinct real and negative $s_i \neq s_j$ for $i \neq j$, $s_i < 0$, $i = 1, ..., n$.
2. $T_i \in \Re_+^{p\times m}$ for $i = 1, ..., n$,
3. $T(\infty) \in \Re_+^{p\times m}$.

If the conditions of Theorem 6 are satisfied the following procedure can be used to find the desired positive stable realization with system Metzler matrix.

Procedure 1.

Step 1. Using (68) find the matrix D and the strictly proper transfer matrix $T_{sp}(s) = T(s) - D$ and write it in the form (58).

Step 2. Find the real zeros $s_1, s_2, ..., s_n$ of the polynomial (59).

Step 3. Using (61) find the matrices $T_1, ..., T_n$ and their decomposition (63).

Step 4. Using (64) find the matrices A, B, C.

Example 5. Using Procedure 1 find a positive stable realization with system Metzler matrix of the transfer matrix

$$T(s) = \begin{bmatrix} \dfrac{s+3}{s+1} & \dfrac{2s+5}{s+2} \\ \dfrac{1}{s+2} & \dfrac{s+4}{s+3} \end{bmatrix}. \tag{69}$$

Step 1. The matrix D with nonnegative entries has the form

$$D = \lim_{s\to\infty} T(s) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \tag{70}$$

and the strictly proper transfer matrix is given by

$$T_{sp}(s) = T(s) - D = \begin{bmatrix} \dfrac{2}{s+1} & \dfrac{1}{s+2} \\ \dfrac{1}{s+2} & \dfrac{1}{s+3} \end{bmatrix} \tag{71}$$

Step 2. The transfer matrix (71) can be written in the form

$$T_{sp}(s) = \frac{1}{(s+1)(s+2)(s+3)}\cdot$$
$$\cdot\begin{bmatrix} 2(s+2)(s+3) & (s+1)(s+3) \\ (s+1)(s+3) & (s+2)(s+1) \end{bmatrix} = \frac{N(s)}{d(s)}. \tag{72}$$

In this case $d(s) = (s+1)(s+2)(s+3)$, $s_1 = -1$, $s_2 = -2$, $s_3 = -3$ and the condition 1) of Theorem 6 is met.

Step 3. Using (51) and (53) we obtain

$$T_1 = \frac{1}{(s+2)(s+3)}\cdot$$
$$\cdot\left.\begin{bmatrix} 2(s+2)(s+3) & (s+1)(s+3) \\ (s+1)(s+3) & (s+2)(s+1) \end{bmatrix}\right|_{s=-1} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix},$$
$$r_1 = \text{rank}T_1 = 1, \qquad T_1 = C_1B_1,$$
$$B_1 = [\ 1 \quad 0\ ], \qquad C_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \tag{73a}$$

$$T_2 = \frac{1}{(s+1)(s+3)} \cdot$$
$$\cdot \left[\begin{array}{cc} 2(s+2)(s+3) & (s+1)(s+3) \\ (s+1)(s+3) & (s+2)(s+1) \end{array} \right] \Bigg|_{s=-2} = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right],$$
$$r_2 = \mathrm{rank} T_2 = 2, \qquad T_2 = C_2 B_2,$$
$$B_2 = [\ B_{21} \quad B_{22}\ ] = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right],$$
$$C_2 = [\ C_{21} \quad C_{22}\ ] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \tag{73b}$$

$$T_3 = \frac{1}{(s+1)(s+2)} \cdot$$
$$\cdot \left[\begin{array}{cc} 2(s+2)(s+3) & (s+1)(s+3) \\ (s+1)(s+3) & (s+2)(s+1) \end{array} \right] \Bigg|_{s=-3} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right],$$
$$r_3 = \mathrm{rank} T_3 = 1, \qquad T_3 = C_3 B_3,$$
$$B_3 = [\ 0 \quad 1\ ], \qquad C_3 = \left[\begin{array}{c} 0 \\ 1 \end{array} \right]. \tag{73c}$$

From (73) it follows that the conditions 2) of Theorem 6 are satisfied.

Step 4. Using (64) and (73) we obtain

$$A = \left[\begin{array}{ccc} I_{r_1} s_1 & 0 & 0 \\ 0 & I_{r_2} s_2 & 0 \\ 0 & 0 & I_{r_1} s_3 \end{array} \right] = \left[\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{array} \right],$$
$$B = \left[\begin{array}{c} B_1 \\ B_2 \\ B_3 \end{array} \right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{array} \right],$$
$$C = [\ C_1 \quad C_2 \quad C_3\ ] = \left[\begin{array}{cccc} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right]. \tag{74}$$

The desired positive stable realization of (69) is given by (70) and (74).

This approach can be extended for transfer matrices with multiple real negative poles [17].

5. Systems with real poles and zeros

Consider the stable strictly proper irreducible transfer function

$$\begin{aligned} T_{sp}(s) &= \frac{\overline{b}_{n-1}s^{n-1} + ... + \overline{b}_1 s + \overline{b}_0}{s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0} = \\ &= \frac{\overline{b}_{n-1}(s-z_1)...(s-z_{n-1})}{(s-s_1)(s-s_2)...(s-s_n)}, \end{aligned} \tag{75}$$

where $s_1, ..., s_n$ are the real negative poles and $z_1, ..., z_{n-1}$ are real negative zeros of the transfer function.

Theorem 7. There exists a positive stable realization of (75) if

$$s_k < z_k < s_{k+1} \quad \text{for} \quad k = 1, ..., n-1. \tag{76}$$

Proof. From (61) we have

$$T_i = \frac{(s_i - z_1)(s_i - z_2)...(s_i - z_{n-1})}{(s_i - s_1)...(s_i - s_{i-1})(s_i - s_{i+1})...(s_i - s_n)} > 0 \quad \text{for} \quad k = 1, ..., n \tag{77}$$

if the condition (76) is satisfied. By Theorem 6 the matrices

$$A = \mathrm{diag}[\ s_1 \quad ... \quad s_n\ ], \qquad B = \left[\begin{array}{c} b_1 \\ \vdots \\ b_n \end{array} \right], \tag{78}$$
$$C = [\ c_1 \quad ... \quad c_n\ ], \qquad T_i = b_i c_i, \qquad i = 1, ..., n$$

are a positive stable realization of the transfer function (75).

Example 6. Find a positive realization of the strictly proper transfer function

$$T_{sp}(s) = \frac{s+2}{s^2 + 4s + 3}. \tag{79}$$

In this case $s_1 = -1$, $s_2 = -3$, $z_1 = -2$ and the condition (76) is satisfied. Using (61) (or (73)) we obtain

$$T_1 = \frac{s+2}{s+3}\Big|_{s=-1} = \frac{1}{2}, \qquad T_2 = \frac{s+2}{s+1}\Big|_{s=-3} = \frac{1}{2}$$

and

$$T_1 = b_1 c_1 = \frac{1}{2}, \qquad b_1 = 1, \qquad c_1 = \frac{1}{2},$$
$$T_2 = b_2 c_2 = \frac{1}{2}, \qquad b_2 = 1, \qquad c_2 = \frac{1}{2}.$$

The desired positive realization has the form

$$A = \left[\begin{array}{cc} s_1 & 0 \\ 0 & s_2 \end{array} \right] = \left[\begin{array}{cc} -1 & 0 \\ 0 & -3 \end{array} \right],$$
$$B = \left[\begin{array}{c} b_1 \\ b_2 \end{array} \right] = \left[\begin{array}{c} 1 \\ 1 \end{array} \right], \tag{80}$$
$$C = [\ c_1 \quad c_2\ ] = \left[\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \end{array} \right].$$

Now let us consider the strictly proper transfer matrix (58) rewritten in the form

$$T_{sp}(s) = \frac{1}{(s-s_1)...(s-s_n)} \cdot$$
$$\cdot \left[\begin{array}{ccc} (s-z_{11}^1)...(s-z_{11}^{n_{11}}) & ... & (s-z_{1,m}^1)...(s-z_{1,m}^{n_{1,m}}) \\ \vdots & ... & \vdots \\ (s-z_{p,1}^1)...(s-z_{p,1}^{n_{p,1}}) & ... & (s-z_{p,m}^1)...(s-z_{p,m}^{n_{p,m}}) \end{array} \right], \tag{81}$$

where $s_1,...,s_n$ are real negative poles and z_{ij}^k, $i=1,...,p$; $j=1,...,m$; $k=1,...,n_{ij}$ are real negative zeros.

Theorem 8. There exists a positive stable realization of (81) if

$$\begin{gathered} s_k \le z_{i,j}^k \le s_{k+1} \\ \text{for} \quad i=1,...,p; \quad j=1,...,m; \quad k=1,...,n_{ij}. \end{gathered} \tag{82}$$

Proof is similar to the proof of Theorem 7.

If the condition (82) is satisfied then a positive stable realization (64) of (81) can be found by the use of Procedure 1.

Example 7. Using Procedure 1 find a positive realization of the strictly proper transfer matrix

$$\begin{gathered} T_{sp}(s) = \frac{1}{(s+1)(s+3)(s+5)} \cdot \\ \cdot \begin{bmatrix} (s+2)(s+4) & (s+1)(s+4) \\ (s+2)(s+5) & (s+2)(s+4) \end{bmatrix}. \end{gathered} \tag{83}$$

In this case we have $s_1=-1$, $s_2=-3$, $s_3=-5$, $z_{11}^1=-2$, $z_{11}^2=-4$, $z_{12}^1=-1$, $z_{12}^2=-4$, $z_{21}^1=-2$, $z_{21}^2=-5$, $z_{22}^1=-2$, $z_{12}^2=-4$ and the conditions (76) are satisfied. Therefore, by Theorem 8 there exists a positive stable realization of the transfer matrix (83). Using (61) and (83) we obtain

$$\begin{gathered} T_1 = \frac{1}{(s+3)(s+5)} \cdot \\ \cdot \left[\begin{bmatrix} (s+2)(s+4) & (s+1)(s+4) \\ (s+2)(s+5) & (s+2)(s+4) \end{bmatrix} \right]_{s=-1} = \begin{bmatrix} \frac{3}{8} & 0 \\ \frac{1}{2} & \frac{3}{8} \end{bmatrix}, \\ \text{rank}T_1 = 2, \qquad T_1 = C_1B_1, \\ C_1 = \begin{bmatrix} \frac{3}{8} & 0 \\ \frac{1}{2} & \frac{3}{8} \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \end{gathered} \tag{84a}$$

$$\begin{gathered} T_2 = \frac{1}{(s+1)(s+5)} \cdot \\ \cdot \left[\begin{bmatrix} (s+2)(s+4) & (s+1)(s+4) \\ (s+2)(s+5) & (s+2)(s+4) \end{bmatrix} \right]_{s=-3} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}, \\ \text{rank}T_2 = 2, \qquad T_2 = C_2B_2, \\ C_2 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \end{gathered} \tag{84b}$$

$$\begin{gathered} T_3 = \frac{1}{(s+1)(s+3)} \cdot \\ \cdot \left[\begin{bmatrix} (s+2)(s+4) & (s+1)(s+4) \\ (s+2)(s+5) & (s+2)(s+4) \end{bmatrix} \right]_{s=-5} = \begin{bmatrix} \frac{3}{8} & \frac{1}{2} \\ 0 & \frac{3}{8} \end{bmatrix}, \\ \text{rank}T_3 = 2, \qquad T_3 = C_3B_3, \\ C_3 = \begin{bmatrix} \frac{3}{8} & \frac{1}{2} \\ 0 & \frac{3}{8} \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{gathered} \tag{84c}$$

The desired positive stable realization has the form

$$\begin{gathered} A = \begin{bmatrix} I_{r_1}s_1 & 0 & 0 \\ 0 & I_{r_2}s_2 & 0 \\ 0 & 0 & I_{r_3}s_3 \end{bmatrix} = \\ = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix}, \\ B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ C = [\, C_1 \quad C_2 \quad C_3 \,] = \frac{1}{8} \begin{bmatrix} 3 & 0 & 2 & 4 & 3 & 4 \\ 4 & 3 & 4 & 2 & 0 & 3 \end{bmatrix}. \end{gathered} \tag{85}$$

6. Concluding remarks

Conditions for the existence of positive stable realizations with system Metzler matrices of proper transfer function have been established. It has been shown that a stable transfer function of second order has a positive stable realization if and only if the poles of the transfer function are real negative (Theorem 3). Sufficient conditions for the existence of positive stable realizations of a stable proper transfer function of third order have been also established (Theorem 4). In general case a method based on the decomposition of the transfer function into first, second and third order transfer functions has been proposed (Theorem 5). Using Gilbert method a procedure has been proposed for computation of positive stable realizations for transfer functions with real negative poles (Theorem 6) and for transfer functions with real poles and zeros (Theorem 7). The considerations have been illustrated by numerical examples. The following are open problems:

1. find necessary and sufficient conditions for the existence of positive stable realizations with system Metzler matrices of proper transfer matrices,

2. give a method for finding positive stable realizations with system Metzler matrices which is not based on the similarity transformation of proper transfer matrices.

An extension of the presented procedure for fractional linear systems [12, 13, 16] is also an open problem.

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