BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 59, No. 2, 2011
DOI: 10.2478/v10175-011-0023-0

Stability of continuous-discrete linear systems described by the general model

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Abstract. New necessary and sufficient conditions for asymptotic stability of positive continuous-discrete linear systems described by the general 2D model are established. A procedure for checking the asymptotic stability is proposed. The effectiveness of the procedure is demonstrated on examples.

Key words: stability, continuous-discrete linear systems, general model.

1. Introduction

In positive systems inputs, state variables and outputs take only nonnegative values. A variety of models having positive systems behavior can be found in engineering, management science, economics, social sciences, biology and medicine etc. An overview of state of the art in positive systems is given in the monographs [1, 2]. The positive continuous-discrete 2D linear systems have been introduced in [3], positive hybrid linear systems in [4] and the positive fractional 2D hybrid systems in [5]. Different methods of solvability of 2D hybrid linear systems have been discussed in [6] and the solution to singular 2D hybrids linear systems has been derived in [7]. The realization problem for positive 2D hybrid systems has been addressed in [8]. Some problems of dynamics and control of 2D hybrid systems have been considered in [9, 10]. The problems of stability and robust stability of 2D continuous-discrete linear systems have been investigated in [11–17]. The stability of positive continuous-time linear systems with delays has been addressed in [18]. Recently the stability and robust stability of Fornasini-Marchesini type model and of Roesser type model of scalar continuous-discrete linear systems have been analyzed by Busłowicz in [12–14].

In this note new necessary and sufficient conditions for asymptotic stability of positive continuous-discrete linear systems described by the general 2D model and a procedure for checking the stability will be presented.

The following notation will be used: $\Re$ – the set of real numbers, Z_+ – the set of nonnegative integers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, I_n – the $n\times n$ identity matrix.

2. Preliminaries

Consider the continuous-discrete linear 2D system [2, 3]

$$\begin{aligned}\dot{x}(t,i+1) &= A_0x(t,i) + A_1\dot{x}(t,i)+\\ &+ A_2x(t,i+1) + Bu(t,i),\end{aligned} \tag{1}$$

$$t\in\Re_+,\qquad i\in Z_+,$$

where

$$\dot{x}(t,i) = \frac{\partial x(t,i)}{\partial t},$$

$$x(t,i)\in\Re^n,\qquad u(t,i)\in\Re^m,$$

$$A_0,A_1,A_2\in\Re^{n\times n},\qquad B\in\Re^{n\times m}.$$

Definition 1. The continuous-discrete linear 2D system (1) is called (internally) positive if $x(t,i)\in\Re_+^n$, $t\in\Re_+$, $i\in Z_+$ for any input $u(t,i)\in\Re_+^m$ and all initial conditions

$$\begin{aligned}&x(0,i)\in\Re_+^n,\qquad i\in Z_+,\\ &x(t,0)\in\Re_+^n,\qquad \dot{x}(t,0)\in\Re_+^n,\qquad t\in\Re_+.\end{aligned} \tag{2}$$

Theorem 1. [2, 3] The continuous-discrete linear 2D system (1) is positive if and only if

$$\begin{aligned}&A_2\in M_n,\qquad A_0,A_1\in\Re_+^{n\times n},\\ &A_0+A_1A_2\in\Re_+^{n\times n}\quad\text{and}\quad B\in\Re_+^{n\times m},\end{aligned} \tag{3}$$

where M_n is the set of $n\times n$ Metzler matrices (with nonnegative off-diagonal entries).

Definition 2. The continuous-discrete linear 2D system (1) is called asymptotically stable if

$$\lim_{t,i\to\infty} x(t,i) = 0 \tag{4}$$

for bounded initial conditions and for $u(t,i)=0$.

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The matrix $A \in \Re^{n\times n}$ is called asymptotically stable (Hurwitz) if all its eigenvalues lie in the open left half of the complex plane.

Definition 3. The point x_e is called equilibrium point of he asymptotically stable system (1) if for $Bu = 1_n = [1 \ \dots \ 1]^T \in \Re^n_+$ if

$$0 = A_0 x_e + A_2 x_e + 1_n. \tag{5}$$

Asymptotic stability implies $\det[A_0 + A_2] \neq 0$ and from (5) we have

$$x_e = -[A_0 + A_2]^{-1} 1_n. \tag{6}$$

Remark 1. From (1) for $B = 0$ it follows that the positive system is asymptotically stable only if the matrix $A_1 - I_n$ is Hurwitz Metzler matrix [1, 2].

In what follows it is assumed that the matrix $A_1 - I_n$ is a Hurwitz Metzler matrix.

Theorem 2. The linear continuous-discrete positive 2D system (1) is asymptotically stable if and only if all coefficients of the polynomial

$$\begin{aligned}\det[I_n s(z+1) - A_0 - A_1 s - A_2(z+1)] = \\ = s^n z^n + \overline{a}_{n,n-1} s^n z^{n-1} + \\ + \overline{a}_{n-1,n} s^{n-1} z^n + \ldots + \overline{a}_{10} s + \overline{a}_{01} z + \overline{a}_{00}\end{aligned} \tag{7}$$

are positive, i.e.

$$\overline{a}_{k,l} > 0 \qquad \text{for} \quad k,l = 0,1,\ldots,n (\overline{a}_{n,n} = 1). \tag{8}$$

3. Main result

Theorem 3. Let the matrix $A_1 - I_n$ be a Hurwitz Metzler matrix. The positive continuous-discrete linear 2D system (1) is asymptotically stable if and only if there exists a strictly positive vector $\lambda \in \Re^n_+$ (all components of the vectors are positive) such that

$$(A_0 + A_2)\lambda < 0. \tag{9}$$

Proof. Integrating the Eq. (1) with $B = 0$ in the interval $(0, +\infty)$ for $i \to +\infty$ we obtain

$$\begin{aligned} x(+\infty, +\infty) - x(0, +\infty) = \\ = A_0 \int_0^{+\infty} x(\tau, +\infty) d\tau + A_1 x(+\infty, +\infty) - \\ - A_1 x(0, +\infty) + A_2 \int_0^{+\infty} x(\tau, +\infty) d\tau. \end{aligned} \tag{10}$$

If the system is asymptotically stable then by (4) from (10) we obtain

$$(A_1 - I_n) x(0, +\infty) = (A_0 + A_2) \int_0^{+\infty} x(\tau, +\infty) d\tau. \tag{11}$$

If the matrix $A_1 - I_n$ is Hurwitz Metzler matrix then for every $x(0, +\infty) > 0$ such that $(A_1 - I_n)x(0, +\infty)$ is a strictly negative vector, $\lambda = \int_0^{+\infty} x(\tau, +\infty) d\tau$ is a strictly positive vector and (9) holds.

Now we can show that if there exists a strictly positive vector λ such that (9) holds then the positive system (1) is asymptotically stable. It is well-known that the positive system (1) with $B = 0$ is asymptotically stable if and only if the corresponding transpose positive system

$$\begin{aligned}\dot{x}(t, i+1) = A_0^T x(t,i) + A_1^T \dot{x}(t,i) + A_2^T x(t, i+1), \\ t \in \Re_+, \qquad i \in Z_+\end{aligned} \tag{12}$$

is asymptotically stable. As a candidate for a Lapunov function for the positive system (12) we chose

$$V(t, x(i)) = x^T(t,i)\lambda, \qquad \lambda > 0 \tag{13}$$

which is positive for every nonzero $x(t,i) \in \Re^n_+$. Using (13) and (12) we obtain

$$\begin{aligned}\Delta \dot{V}(t, x(i)) = \dot{V}(t, x(i+1)) - \dot{V}(t, x(i)) = \\ = \dot{x}^T(t, i+1)\lambda - \dot{x}^T(t,i)\lambda \\ = \dot{x}^T(t,i)[A_1 - I_n]\lambda + x^T(t,i)A_0\lambda + x^T(t,i+1)A_2\lambda \\ \leq \begin{cases} x^T(t,i)(A_0 + A_2)\lambda & \text{for} \quad x(t,i) \geq x(t,i+1) \\ x^T(t,i+1)(A_0 + A_2)\lambda & \text{for} \quad x(t,i) < x(t,i+1) \end{cases}\end{aligned} \tag{14}$$

since by assumption $[A_1 - I_n]\lambda < 0$. If (9) holds then from (14) we have $\Delta \dot{V}(t, x(i)) < 0$ and the positive system is asymptotically stable.

Remark 2. As the strictly positive vector λ we may choose the equilibrium point (6) since for $\lambda = x_e$ we have

$$(A_0 + A_2)\lambda = -(A_0 + A_2)(A_0 + A_2)^{-1} 1_n = -1_n. \tag{15}$$

Theorem 4. The positive system (1) is asymptotically stable if and only if both matrices

$$A_1 - I_n, \qquad A_0 + A_2 \tag{16}$$

are Hurwitz Metzler matrices.

Proof. From Remark 1 it follows that the positive system (1) is asymptotically stable only if the matrix $A_1 - I_n$ is Hurwitz Metzler matrix. By Theorem 3 the positive system is asymptotically stable if and only if there exists a strictly positive vector λ such that (9) holds but this is equivalent that the matrix $A_0 + A_2$ is Hurwitz Metzler matrix.

To test of the matrices (16) are Hurwitz Metzler matrices the following theorem is recommended [2, 19].

Theorem 5. The matrix $A \in \Re^{n\times n}$ is a Hurwitz Metzler matrix if and only if one of the following equivalent conditions is satisfied:

i) all coefficients $a_0, \ldots, a_{n-1}$ of the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 \tag{17}$$

are positive, i.e. $a_i > 0$, $i = 0, 1, \ldots, n-1$,

ii) the diagonal entries of the matrices

$$A_{n-k}^{(k)} \quad \text{for} \quad k = 1, \dots, n-1 \tag{18}$$

are negative, where

$$A_n^{(0)} = A = \begin{bmatrix} a_{11}^{(0)} & \dots & a_{1,n}^{(0)} \\ \vdots & \dots & \vdots \\ a_{n,1}^{(0)} & \dots & a_{n,n}^{(0)} \end{bmatrix} = \begin{bmatrix} A_{n-1}^{(0)} & b_{n-1}^{(0)} \\ c_{n-1}^{(0)} & a_{n,n}^{(0)} \end{bmatrix},$$

$$A_{n-1}^{(0)} = \begin{bmatrix} a_{11}^{(0)} & \dots & a_{1,n-1}^{(0)} \\ \vdots & \dots & \vdots \\ a_{n-1,1}^{(0)} & \dots & a_{n-1,n-1}^{(0)} \end{bmatrix},$$

$$b_{n-1}^{(0)} = \begin{bmatrix} a_{1,n}^{(0)} \\ \vdots \\ a_{n-1,n}^{(0)} \end{bmatrix},$$

$$c_{n-1}^{(0)} = [\ a_{n,1}^{(0)} \ \ \dots \ \ a_{n,n-1}^{(0)}\ ],$$

$$A_{n-k}^{(k)} = A_{n-k}^{(n-1)} - \frac{b_{n-k}^{(k-1)} c_{n-k}^{(k-1)}}{a_{n-k+1,n-k+1}^{(k-1)}} =$$

$$= \begin{bmatrix} a_{11}^{(k)} & \dots & a_{1,n-k}^{(k)} \\ \vdots & \dots & \vdots \\ a_{n-k,1}^{(k)} & \dots & a_{n-k,n-k}^{(k)} \end{bmatrix} =$$

$$= \begin{bmatrix} A_{n-k-1}^{(k)} & b_{n-k-1}^{(k)} \\ c_{n-k-1}^{(k)} & a_{n-k,n-k}^{(k)} \end{bmatrix},$$

$$b_{n-k-1}^{(k)} = \begin{bmatrix} a_{1,n-k}^{(k)} \\ \vdots \\ a_{n-k-1,n-k}^{(k)} \end{bmatrix},$$

$$c_{n-k-1}^{(k)} = [\ a_{n-k,1}^{(k)} \ \ \dots \ \ a_{n-k,n-k-1}^{(k)}\ ] \tag{19}$$

for $k = 0, 1, \dots, n-1$.

To check the stability of the positive system (1) the following procedure can be used.

Procedure 1.

Step 1. Check if at least one diagonal entry of the matrix $A_1 \in \Re_+^{n\times n}$ is equal or greater then 1. If this holds then positive system ( ) is unstable [2].

Step 2. Using Theorem 5 check if the matrix $A_1 - I_n$ is Hurwitz Metzler matrix. If not the positive system (1) is unstable.

Step 3. Using Theorem 5 check if the matrix $A_0 + A_2$ is Hurwitz Metzler matrix. If yes the positive system (1) is asymptotically stable.

Example 1. Consider the positive system (1) with the matrices

$$A_0 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.5 & 0.1 \\ 0.2 & -0.6 \end{bmatrix}. \tag{20}$$

By Theorem 1 the system is positive since $A_2 \in M_n$, $A_0, A_1 \in \Re_+^{n\times n}$ and $A_0 + A_1 A_2 = \begin{bmatrix} 0.04 & 0.02 \\ 0.11 & 0.13 \end{bmatrix} \in \Re_+^{2\times 2}$.

Using Procedure 1 we obtain the following

Step 1. All diagonal entries of the matrix A_1 are less then 1.

Step 2. The matrix $A_1 - I_n$ is Hurwitz since the coefficient of the polynomial

$$\det[I_2 s - A_1 + I_n] = \begin{vmatrix} s+0.6 & -0.2 \\ -0.1 & s+0.7 \end{vmatrix} = s^2 + 1.3s + 0.4$$

are positive.

Step 3. The matrix

$$A = A_0 + A_2 = \begin{bmatrix} -0.3 & 0.2 \\ 0.3 & -0.3 \end{bmatrix}$$

is also Hurwitz since (using condition ii) of Theorem 5)

$$A_1^{(??)} = -0.3 + \frac{0.2 * 0.3}{0.3} = -0.1 < 0.$$

By Theorem 4 the positive system (1) with (20) is asymptotically stable.

The polynomial (7) for positive system has the form

$$\det[I_2 s(z+1) - A_0 - A_1 s - A_2(z+1)] =$$

$$= \begin{vmatrix} s(z+1) - 0.2 - 0.4s + 0.5(z+1) & -0.1 - 0.2s - 0.1(z+1) \\ -0.1 - 0.1s - 0.2(z+1) & s(z+1) - 0.3 - 0.3s + 0.6(z+1) \end{vmatrix} =$$

$$= s^2z^2 + 1.3s^2z + 1.1sz^2 + 1.26sz + $$
$$+ 0.28z^2 + 0.26z + 0.4s^2 + 0.31s + 0.03.$$

All coefficients of the polynomial are positive. Therefore, by Theorem 4 the positive system is also asymptotically stable.

It is well-known [2] that substituting $A_0 = 0$, $B = 0$ in (1) we obtain the autonomous second Fornasini-Marchesini continuous-discrete linear 2D system

$$\dot{x}(t, i+1) = A_1 \dot{x}(t, i) + A_2 x(t, i+1), \quad t \in \Re_+, \quad i \in Z_+. \tag{21}$$

The autonomous Roesser type continuous-discrete model has the form [2]

$$\begin{bmatrix} \dot{x}^h(t,i) \\ x^v(t,i+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h(t,i) \\ x^v(t,i) \end{bmatrix}, \tag{22}$$

$$t \in \Re_+, \qquad i \in Z_+,$$

where $\dot{x}(t,i) = \dfrac{\partial x(t,i)}{\partial t}$, $x^h(t,i) \in \Re^{n_1}$ and $x^v(t,i) \in \Re^{n_2}$ are the horizontal and vertical vectors and $A_{kl} \in \Re^{n_k \times n_l}$, $k,l = 1,2$. The model (22) is positive if and only if [2] A_{11} is a Metzler matrix and $A_{12} \in \Re_+^{n_1 \times n_2}$, $A_{21} \in \Re_+^{n_2 \times n_1}$, $A_{22} \in \Re_+^{n_2 \times n_2}$. The positive model (22) is a particular case of the model (21) for [2]

$$A_1 = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}, \qquad A_2 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}. \tag{23}$$

By Theorem 2 the positive Roesser type continuous-discrete model (22) is asymptotically stable if and only if the coefficients of the polynomial

$$\det\begin{bmatrix} I_{n_1}s(z+1) - A_{11}(z+1) & -A_{12}(z+1) \\ -A_{21}s & I_{n_2}s(z+1) - A_{22}s \end{bmatrix} = $$
$$= s^{n_1}z^{n_2} + \widehat{a}_{n_1,n_2-1}s^{n_1}z^{n_2-1} +$$
$$+ \widehat{a}_{n_1-1,n_2}s^{n_1-1}z^{n_2} + \ldots + \widehat{a}_{11}sz + \widehat{a}_{10}s + \widehat{a}_{01}z + \widehat{a}_{00} \tag{24}$$

are positive.

Proof. To transform the model (22) to the model (21) we perform the following two operations:

1) In the equation

$$\dot{x}^h(t,i) = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \begin{bmatrix} x^h(t,i) \\ x^v(t,i) \end{bmatrix}$$

we substitute i by $i+1$.

2) We differentiate with respect to t the equation

$$x^v(t,i+1) = \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h(t,i) \\ x^v(t,i) \end{bmatrix}.$$

Note that to operation 1) corresponds the multiplication of the Z transform by z and to the operation 2) the multiplication of the Laplace transform by s. These operations do not change the asymptotic stability of the positive system (model). To shift the unit circle of the complex plane in the left half of the complex plane we substitute z by $z+1$.

Taking into account

$$\begin{bmatrix} I_{n_1}sz - A_{11}z & -A_{12}z \\ -A_{21}s & I_{n_2}s(z+1) - A_{22}s \end{bmatrix} =$$
$$= \begin{bmatrix} I_{n_1}z & 0 \\ 0 & I_{n_2}s \end{bmatrix} \begin{bmatrix} I_{n_1}s - A_{11} & -A_{12} \\ -A_{21} & I_{n_2}(z+1) - A_{22} \end{bmatrix}$$

and Theorem 5 we conclude that the positive Roesser type model (22) is asymptotically stable if and only if all coefficients of the polynomial (24) are positive.

Example 2. Consider the positive scalar model (22) with [14]

$$A_1 = \begin{bmatrix} 0 & 0 \\ a_{21} & a_{22} \end{bmatrix}, \qquad A_2 = \begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix}, \tag{25}$$
$$a_{11} < 0, \qquad a_{12} \geq 0, \qquad a_{21} \geq 0, \qquad a_{22} \geq 0.$$

The polynomial (24) for (25) has the form

$$\det\begin{bmatrix} s(z+1) - a_{11}(z+1) & -a_{12}(z+1) \\ -a_{21}s & s(z+1) - a_{22}s \end{bmatrix} =$$
$$= s^2z^2 + (2 - a_{22})s^2z - a_{11}sz^2 + (1 - a_{22})s^2 + \tag{26}$$
$$+(-2a_{11} + a_{11}a_{22} - a_{12}a_{21})sz +$$
$$+ (a_{11}a_{22} - a_{12}a_{21} - a_{11})s$$

and its coefficients are positive if and only if $a_{11} < 0$, $0 \leq a_{22} < 1$ and $a_{11}a_{22} - a_{12}a_{21} > a_{11}$. This result is consent with the one obtained in [14] by different method.

4. Concluding remarks

New necessary and sufficient conditions for the asymptotic stability of continuous-discrete linear systems described by the general model have been established (Theorem 3 and 4). A procedure for checking the stability has been proposed and its effectiveness has been demonstrated on examples. The considerations can be also extended for fractional positive 2D continuous-discrete linear systems and linear continuous-discrete 2D systems with delays.

Acknowledgements. This work was supported by the Ministry of Science and Higher Education in Poland under work S/WE/1/11.

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