BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 59, No. 1, 2011
DOI: 10.2478/v10175-011-0010-5

Stokes flow of an incompressible micropolar fluid past a porous spheroidal shell

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Abstract. Consider a pair of confocal prolate spheroids S_0 and S_1 where S_0 is within S_1. Let the spheroid S_0 be a solid and the annular region between S_0 and S_1 be porous. The present investigation deals with a flow of an incompressible micropolar fluid past S_1 with a uniform stream at infinity along the common axis of symmetry of the spheroids. The flow outside the spheroid S_1 is assumed to follow the linearized version of Eringen's micropolar fluid flow equations and the flow within the porous region is assumed to be governed by the classical Darcy's law. The fluid flow variables within the porous and free regions are determined in terms of Legendre functions, prolate spheroidal radial and angular wave functions and a formula for the drag on the spheroid is obtained. Numerical work is undertaken to study the variation of the drag with respect to the geometric parameter, material parameter and the permeability parameter of the porous region. An interesting feature of the investigation deals with the presentation of the streamline pattern.

Key words: Stokes flow, incompressible micropolar fluid, porous spheroidal shell.

1. Introduction

The later half of the twentieth century has witnessed considerable research dealing with flows through porous media in view of its enormous applicability. This inter disciplinary field, in a broad sense, involves fluid and thermal sciences, geothermal, petroleum and combustion engineering, intricate mathematics and use of a wide range of computational techniques. This study has applications in flows through porous beds, sedimentation of fine particulate suspensions, modeling of micro molecular coils in solvents, floc settling processes and catalytic reactions where porous pellets are used. Studies dealing with enhanced oil reservoir recovery, combustion in an inert porous matrix, under ground spreading of chemical waste and chemical catalytic reactors are also instances where the above research has significant applications.

A class of problems that attracted the attention of a good number of researchers in fluid dynamics involves the study of the flow of a viscous liquid past porous bodies. In view of the simplicity of the geometry, lot of work has been carried out on the flow of a Newtonian liquid past a porous sphere or a porous spherical shell. The usual macroscopic continuum approach to the above 'simple' problem, as a satisfactory approximation in many real processes, is to neglect inertial and volume forces as well as thermal influences, and to treat it as a multi field boundary value problem governed by the steady state Stokes equations in the free flow region and the Darcy or Brinkman equation in the region occupied by the porous sphere or spherical shell [1].

In nature as well as in diverse chemical processes the particles that occur are porous in character. In view of this, for the past few decades, several contributions have been made mainly dealing with viscous fluid flows past axisymmetric porous bodies. While many of the contributions deal with sphere geometry only, there are some exceptions which deal with a porous spheroid or a porous approximate sphere. As early as in 1962, Leonov discussed the slow stationary flow of a viscous fluid about a porous sphere presenting the typical form of the streamline pattern [2]. Subsequently Joseph and Tao studied the effect of the permeability on the slow motion of a porous sphere in a viscous liquid by employing Darcy's law in the porous region and no slip condition at the surface of the sphere [3]. Sutherland and Tan addressed themselves to the study of sedimentation of a porous sphere assuming the continuity of tangential velocity component at the surface of the sphere [4]. The effect of permeability on the drag experienced by a porous sphere in a uniform stream was studied by Singh and Gupta by the method of matched asymptotic expansions in terms of a Reynolds number [5]. Jones, using Darcy's equation for the porous region, solved the problem of creeping flow around a porous spherical shell with rigid concentric spherical core [6]. Gupta studied the slow flow of a viscous fluid past a porous spherical surface in a uniform stream [7]. The studies of Nir on linear shear flow past a porous particle [8] and of Higdon and Kojima on the calculation of Stokes flow past porous particles [9] must also be recorded in this context. Recently, Srinivasacharya has studied the viscous fluid flow past a porous approximate sphere and approximate spherical shell [10, 11].

In this paper the authors propose to study the flow of an incompressible micropolar fluid past a porous spheroidal shell. The theory of micropolar fluids is too well known to be introduced. More than four decades have passed after the introduction of the theory of micropolar fluids by Eringen [12].

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This is a well founded and significant generalization of the classical Navier Stokes model covering both in theory and applications many more phenomenon than the classical one can, as observed by Lukaszewiez [13]. The micropolar fluid flow equations are presentable in terms of the velocity vector $\vec{q}$, and the microrotation $\vec{\nu}$ associated with each particle in the fluid medium. The vector $\vec{\nu}$ represents the rotation in an average sense of the particles centered in a small volume element about the centroid of the element. Physically, a micropolar fluid model can represent fluids whose molecules can rotate independently of the fluid stream flow and its local vorticity. The deformation of the fluid molecules is not taken into consideration. The occurrence of the microrotation vector which differs from the stream flow vorticity vector and also from the angular velocity vector, results in the formation of non-symmetric stresses and couple stresses which consequently result in energy dissipation [14]. The micropolar fluid flow equations constitute a coupled system of vector differential equations involving $\vec{q}$ and $\vec{\nu}$. Further, as are the classical Navier Stokes equations, the micropolar fluid flow equations are non linear in character. Hence, as such, in general, a micropolar fluid flow problem, unless it is extremely simple, cannot be solved exactly. This has led researchers to make some simplifying assumptions to obtain simplified versions of the real problems which are mathematically tractable. One such assumption is the Stokesian assumption (as in the case of viscous fluids) which states that, whenever there is a flow past a body or a flow is generated due to the rotation or oscillations of a body in an infinite expanse of a fluid, nearer to the body the viscous effects predominate the inertial effects when the flow is slow and the fluid is highly viscous. This has paved the way for finding the solutions of complicated problems involving highly non linear differential equations in the micropolar fluid flows as well. Lakshmana Rao, Bhujanga Rao [15], Lakshmana Rao, Iyengar [16], and Iyengar, Srinivasacharya [17] studied flow past axisymmetric bodies dealing with micropolar fluids. Significant contributions are made by Ramkissoon and Majumdar [18] and Ramkissoon [19] as well. All these problems deal with sphere or an approximate sphere or spheroid which are impervious in nature. However a problem dealing with the creeping flow past a porous sphere was studied recently by Srinivasacharya and Rajyalakshmi [20]. Very recently, the present authors studied the slow flow of an incompressible micropolar fluid past a porous spheroid [21].

In this paper we study the flow of an incompressible micropolar fluid past a porous prolate spheroidal shell with a solid core region kept in an infinite expanse of the fluid with uniform streaming at infinity along the direction of the axis of the spheroid. We assume that the flow outside the spheroidal shell is governed by the micropolar fluid flow equations under the Stokesian approximation and that in the porous region in between the outer spheroid and the solid spheroid by the classical Darcy's equation. We determine the velocity field $\vec{q}$, microrotation field $\vec{\nu}$ and the pressure distribution p outside the porous shell and also the velocity components and the pressure distribution within the porous region. The expressions for the velocity and microrotation components are obtained in terms of Legendre functions, Associated Legendre functions, radial prolate spheroidal wave functions and angular prolate spheroidal wave functions [22]. The expressions for the velocity and pressure in the porous region are in terms of Legendre functions and their derivatives. The stresses acting on the outer surface are estimated and the drag experienced by the spheroidal shell is obtained. The variation of the drag on the shell is studied numerically with respect to the geometric parameter, micropolarity parameter and the permeability constant. The results are displayed through graphs. An interesting feature of the present investigation is the presentation of the stream line pattern for various values of the parameters under consideration.

2. Basic equations

The field equations governing an incompressible micropolar fluid flow [1] are

$$\frac{\partial \rho}{\partial t} + div(\rho\, \vec{q}) = 0 \tag{1}$$

$$\begin{aligned} \rho \frac{d\vec{q}}{dt} &= \rho\, \vec{f} - grad\; p + k\; curl\; \vec{\nu} - \\ &- (\mu + k)\; curl\; curl\; \vec{q} + (\lambda_1 + 2\mu + k) grad\; div\; \vec{q}, \end{aligned} \tag{2}$$

$$\begin{aligned} \rho\, j \frac{d\vec{\nu}}{dt} &= \rho\, \vec{l} - 2k\vec{\nu} + k\; curl\; \vec{q} - \gamma\; curl\; curl\; \vec{\nu} + \\ &+ (\alpha + \beta + \gamma) grad\; div\; \vec{\nu} \end{aligned} \tag{3}$$

in which $\vec{q}$, $\vec{\nu}$ are velocity and microrotation vectors, $\overline{f}$, $\overline{l}$ are body force per unit mass, body couple per unit mass respectively and p is the fluid pressure at any point. ρ and j are density of the fluid and gyration parameters respectively and are assumed to be constants. The material constants (λ_1, μ, k) are viscosity coefficients and (α, β, γ) gyroviscosity coefficients. These constants confirm to the inequalities

$$\begin{aligned} k \geq 0; \quad 2\mu + k \geq 0; \quad 3\lambda_1 + 2\mu + k \geq 0; \\ \gamma \geq 0; \quad |\beta| \leq \gamma; \quad 3\alpha + \beta + \gamma \geq 0. \end{aligned} \tag{4}$$

The stress tensor t_{ij} and the couple stress tensor m_{ij} are given by

$$\begin{aligned} t_{ij} &= (-p + \lambda_1 + div\; \overline{q})\, \delta_{ij} + \\ &+ \; (2\mu + k)\, e_{ij} + k\varepsilon_{ijm}\, (w_m - v_m)\,, \end{aligned} \tag{5}$$

$$m_{ij} = \alpha\, (div\; \overline{\nu})\, \delta_{ij} + \beta v_{ij} + \gamma v_{j,i} \tag{6}$$

in which the symbols δ_{ij}, e_{ij}, $2w_m$ and v_m respectively denote Kronecker symbol, components of rate of strain, vorticity vector and microrotation vector. Comma denotes covariant differentiation.

3. Mathematical formulation of the problem

Consider two confocal prolate spheroids S_0 and S_1 with foci P, Q where PQ = 2c units. Let O be the mid point of PQ. Introduce the cylindrical polar coordinate system (r, θ, z) with respect to O as origin and OQ extended on either side as Z axis as in Fig. 1.

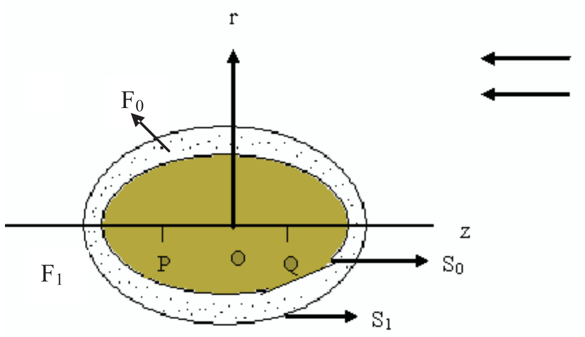

Fig. 1. Schematic diagram of flow past the body

Let us consider the uniform slow stationary flow of an incompressible micropolar fluid past the spheroid S_1 with velocity U in the direction of the z-axis far away from the body. Let us denote the region outside the spheroid S_1 by F_1. Let the region (F_0) between S_0 and S_1 be porous. The spheroid S_0 is a solid.

We examine the flow generated with the assumption that the flow in the porous region is characterized by the classical Darcy's law. Since the flow is slow, we assume that the flow is axisymmetric and is the same in any meridian plane and thus the flow variables are independent of the azimuth angle.

We shall introduce the prolate spheroidal coordinates (ξ, η, ϕ) with $(\overline{e}_\xi, \overline{e}_\eta, \overline{e}_\phi)$ as base vectors and $(\mathrm{h}_1, \mathrm{h}_2, \mathrm{h}_3)$ as the corresponding scale factors through the definition

$$z + ir = \mathrm{c}\ \cosh(\xi + \mathrm{i}\eta). \tag{7}$$

We assume that the flow is Stokesian as in the classical investigation of the problem by Payne and Pell in the case of classical viscous fluid [23] and Lakshmana Rao and Iyengar in the case of micropolar fluid [16]). This enables us to drop the inertial terms in the momentum equation and bilinear terms in the balance of first stress moments.

Let $(\overline{q}^{(1)}, \overline{\nu}^{(1)}, p^{(1)})$ denote the velocity, micro rotation and pressure in the region F_1 and let $(\overline{q}^{(0)}, p^{(0)})$ be the velocity and pressure in the porous region F_0.

In view of the symmetry of the flow, we take

$$\overline{q}^{(1)} = \mathrm{u}^{(1)}(\xi, \eta)\overline{e}_\xi + \mathrm{v}^{(1)}(\xi, \eta)\overline{e}_\eta, \tag{8}$$

$$\overline{\nu}^{(1)} = \mathrm{C}^{(1)}(\xi, \eta)\overline{e}_\phi, \tag{9}$$

$$p^{(1)} = \mathrm{p}^{(1)}(\xi, \eta). \tag{10}$$

Here $\mathrm{u}^{(1)}$, $\mathrm{v}^{(1)}$ are the velocity components in F_1 region and $\mathrm{C}^{(1)}$ is the micro rotation component therein.

Ignoring the body force and body couple $\overline{f}$ and $\overline{l}$ respectively in the field equations, the basic equations governing the Stokesian flow in region F_1 can be written in the form

$$div(\vec{q}^{\,(1)}) = 0, \tag{11}$$

$$-grad p^{(1)} + k curl \vec{\nu}^{\,(1)} - (\mu + k) curl\ curl \vec{q}^{\,(1)} = 0, \tag{12}$$

$$\begin{aligned} -2k\vec{\nu}^{\,(1)} + k\ curl \vec{q}^{\,(1)} - \gamma curl\ curl \vec{\nu}^{\,(1)} + \\ + (\alpha + \beta + \gamma) grad\ div \vec{\nu}^{\,(1)} = 0. \end{aligned} \tag{13}$$

In view of the continuity equation, we introduce the stream function $\psi^{(1)}$ through

$$h_2 h_3 u^{(1)} = -\frac{\partial \psi^{(1)}}{\partial \eta}; \qquad h_1 h_3 v^{(1)} = \frac{\partial \psi^{(1)}}{\partial \xi}. \tag{14}$$

Using (8) and (14)

$$curl \vec{q}^{\,(1)} = \left(\frac{1}{h_3} E^2 \psi^{(1)}\right) \vec{e}_\phi \tag{15}$$

in which the Stokes stream function operator E^2 is given by

$$E^2 = \frac{h_3}{h_1 h_2} \left(\frac{\partial}{\partial \xi} \left(\frac{h_2}{h_1 h_3} \frac{\partial}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left(\frac{h_1}{h_2 h_3} \frac{\partial}{\partial \eta} \right) \right). \tag{16}$$

Evaluating the expressions for $curl\ curl \vec{q}^{\,(1)}$, $div \vec{\nu}^{\,(1)}$ (which is equal to zero), $curl \vec{\nu}^{\,(1)}$, $curl\ curl \vec{\nu}^{\,(1)}$, the basic equations describing the flow in region F_1 are

$$-\frac{1}{h_1}\frac{\partial p^{(1)}}{\partial \xi} + \frac{k}{h_2 h_3}\frac{\partial}{\partial \eta}(h_3 C^{(1)}) - \frac{\mu + k}{h_2 h_3}\frac{\partial}{\partial \eta}(E^2 \psi^{(1)}) = 0, \tag{17}$$

$$-\frac{1}{h_2}\frac{\partial p^{(1)}}{\partial \eta} - \frac{k}{h_1 h_3}\frac{\partial}{\partial \xi}(h_3 C^{(1)}) + \frac{\mu + k}{h_1 h_3}\frac{\partial}{\partial \xi}(E^2 \psi^{(1)}) = 0, \tag{18}$$

$$-2kC^{(1)} + \frac{k}{h_3} E^2 \psi^{(1)} + \gamma \left(\nabla^2 - \frac{1}{h_3^2} \right) C^{(1)} = 0, \tag{19}$$

where ∇^2 is the Laplacian operator given by

$$\nabla^2 = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial \xi} \left(\frac{h_2 h_3}{h_1} \frac{\partial}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left(\frac{h_3 h_1}{h_2} \frac{\partial}{\partial \eta} \right) \right\}. \tag{20}$$

Using the identity

$$h_3 \left(\nabla^2 - \frac{1}{h_3^2} \right) C^{(1)} = E^2 (h_3 C^{(1)}), \tag{21}$$

the Eq. (19) can be recast in the form

$$2k\ h_3 C^{(1)} = k E^2 \psi^{(1)} + \gamma E^2 (h_3 C^{(1)}). \tag{22}$$

Eliminating $p^{(1)}$ from (17) and (18), we have

$$(\mu + k) E^4 \psi^{(1)} - k E^2 (h_3\ C^{(1)}) = 0. \tag{23}$$

From (22) and (23), we get

$$2h_3\ C^{(1)} = E^2 \psi^{(1)} + \frac{\gamma\ (\mu + k)}{k^2} E^4 \psi^{(1)}. \tag{24}$$

Operating E^2 on Eq. (24) and using Eq. (23), we obtain

$$\left(E^6 - \frac{\lambda^2}{c^2} E^4 \right) \psi^{(1)} = 0 \tag{25}$$

which can be written as

$$E^4 \left(E^2 - \frac{\lambda^2}{c^2} \right) \psi^{(1)} = 0, \tag{26}$$

where

$$\frac{\lambda^2}{c^2} = \frac{k(2\mu + k)}{\gamma(\mu + k)}. \tag{27}$$

Here, in Eq. (23), the operator E^4 stands for $\mathrm{E}^2(\mathrm{E}^2)$.

Thus the flow variables in the region F_1 are completely determinable from the system of partial differential Eqs. (26) and (24) using the appropriate boundary and regularity conditions.

As mentioned earlier, the flow in the porous region F_0 is assumed to be Darcian. In view of this, the equations governing the flow in the region F_0 are given by

$$div(\vec{q}^{\,(0)}) = 0, \tag{28}$$

$$\overline{q}^{(0)} = -k^{(1)} grad p^{(0)}, \tag{29}$$

where the velocity $\vec{q}^{\,(0)}$ has components $u^{(0)}$ and $v^{(0)}$. The Eqs. (28) and (29) imply that the pressure $p^{(0)}$ is a harmonic function given by the equation

$$\nabla^2 p^{(0)} = 0. \tag{30}$$

Boundary conditions:

At this stage, a comment regarding the boundary conditions is in order. As observed by Bhatt and Sacheti [24], the existing work concerning the flow past porous bodies in the literature can be divided into three categories:

(i) Using Darcy's law for the flow in the porous region and the Navier Stokes equations for the flow in the free fluid region, with continuity of normal velocity and pressure at the outer surface of the porous body/shell and no slip of the tangential velocity component of the free fluid.

(ii) The same equations as above with continuity of normal velocity and pressure but the slip boundary condition for the tangential component of free fluid velocity at the outer surface of the porous body/shell.

(iii) Using the Brinkman model for the flow inside the porous region and the Navier Stokes equations for the free fluid region together with continuity of velocity, pressure and stresses at the interface (outer surface of the porous body/shell).

As the present problem is being tried with a porous spheroidal shell with solid core and the geometry involved is complicated, as a first trial we have opted for the approach given in (i).

The determination of the relevant flow field variables $\psi^{(i)}$, $C^{(i)}$ and $p^{(i)}$ is subjected to the following boundary and regularity conditions.

(i) Continuity of the normal velocity component on the interfaces[1]:

$$u^{(1)} = u^{(0)} \quad \text{on} \quad S_1 \tag{31}$$

(ii) Vanishing of the tangential velocity components on the interfaces:

$$v^{(1)} = 0 \quad \text{on} \quad S_1 \tag{32}$$

(iii) Vanishing of microrotation on S_1:

$$C^{(1)}(s,t) = 0 \quad \text{on} \quad S_1 \tag{33}$$

(iv) No slip condition on S_0:

$$v^{(0)} = 0 \quad \text{on} \quad S_0 \tag{34}$$

(v) Continuity of pressure on the interfaces:

$$p^{(1)} = p^{(0)} \quad \text{on} \quad S_1. \tag{35}$$

In addition to the above boundary conditions, it is natural to have regularity of the flow field variables on the axis of symmetry. Further as the flow is a uniform stream at infinity we have,

$$\psi = -\frac{1}{2}Ur^2 \quad \text{far away from the body.} \tag{36}$$

4. Solution for the flow in the region S_1

Since, we are dealing with a prolate spheroidal coordinate system, we have

$$h_1 = h_2 = c\sqrt{(s^2-t^2)}, \quad h_3 = c\sqrt{(s^2-1)(1-t^2)}, \tag{37}$$

$$E^2 = \frac{1}{c^2(s^2-t^2)}\left((s^2-1)\frac{\partial^2}{\partial s^2} + (1-t^2)\frac{\partial}{\partial t^2}\right), \tag{38}$$

$$\nabla^2 = \frac{1}{c^2(s^2-t^2)}\left((s^2-1)\frac{\partial^2}{\partial s^2} + (1-t^2)\frac{\partial}{\partial t^2} + \right. \\ \left. + \, 2s\frac{\partial}{\partial s} - 2t\frac{\partial}{\partial t}\right), \tag{39}$$

where

$$s = \cosh\xi; \quad t = \cos\eta. \tag{40}$$

We assume that the boundary of the spheroid is given by $s = s_1$.

The solution of Eq. (26) can be obtained by superposing the solutions of the equations

$$E^4\psi = 0 \tag{41}$$

and

$$\left(E^2 - \frac{\lambda^2}{c^2}\right)\psi = 0. \tag{42}$$

Solution of Eq. (41). The solution of (41) can be written in the form

$$\psi = \psi_0 + \psi_1, \tag{43}$$

where

$$\psi_0 = -\frac{1}{2}Uc^2(s^2-1)(1-t^2) \tag{44}$$

and

$$\psi_1 = c^2(s^2-1)(1-t^2)\sum_{n=0}^{\infty} G_{n+1}(s)P'_{n+1}(t), \tag{45}$$

where $P'_{n+1}(t)$ is the derivative of $P_{n+1}(t)$ with respect to t and the functions $G_{n+1}(s)$ are to be determined later. The function ψ_0 in (44) represents the stream function due to a uniform stream of magnitude U parallel to the axis of symmetry far away from the spheroid. We notice that $E^2\psi_0 = 0$ and hence $E^4\psi_0 = 0$. In view of this, ψ_1 must satisfy

$$E^4\psi_1 = 0. \tag{46}$$

It can be verified that the expression

$$f = c^2(s^2-1)(1-t^2)\sum_{n=0}^{\infty} A_{n+1}Q'_{n+1}(s)\,P'_{n+1}(t), \tag{47}$$

[1] The authors thank the revered referee for the highly illuminating comment on the boundary condition (31). A more appropriate approach is certainly the one mentioned in [24] as (iii). However, at present as a first iteration to the problem, the present approach is adopted. The problem is being considered with approach (iii) by the authors and will be separately communicated.

where $Q'_{n+1}(s)$ is the derivative of Legendre function of second kind $Q_{n+1}(s)$ with respect to s, satisfies $E^2 f = 0$. In view of this, we shall impose the restriction on the functions $G_{n+1}(s)$ through

$$E^2\psi_1 = c^2(s^2-1)(1-t^2)\sum_{n=0}^{\infty} A_{n+1}\, Q'_{n+1}(s)\, P'_{n+1}(t) \quad (48)$$

so that $E^4\psi_1 = 0$.

Now operating E^2 on the Eq. (43) and equating the result with the right hand side of (46), we get

$$\begin{gathered}\sum_{n=0}^{\infty}\big[\{(s^2-1)G_{n+1}(s)\}''- \\ -(n+1)(n+2)G_{n+1}(s)\big]P'_{n+1}(t) = \\ = \sum_{n=0}^{\infty} A_{n+1}c^2(s^2-t^2)Q'_{n+1}(s)P'_{n+1}(t).\end{gathered} \quad (49)$$

Following [Lakshmana Rao & Iyengar [16]], we note that $G_{n+1}(s)$ is governed by the differential equation

$$\begin{gathered}(s^2-1)\,G''_{n+1}(s) + 4s\,G'_{n+1}(s)- \\ -n(n+3)G_{n+1}(s) = g_{n+1}(s),\end{gathered} \quad (50)$$

where

$$\begin{aligned} g_{n+1}(s) = c^2 &\left[\frac{(n+1)(n+2)}{(2n+3)(2n+5)}A_{n+1}- \right.\\ &\left. - \frac{(n+3)(n+4)}{(2n+5)(2n+7)}A_{n+3}\right]Q'_{n+3}(s)- \\ - c^2 &\left[\frac{(n-1)(n)}{(2n-1)(2n+1)}A_{n-1}- \right.\\ &\left. - \frac{(n+1)(n+2)}{(2n+1)(2n+3)}A_{n+1}\right]Q'_{n-1}(s).\end{aligned} \quad (51)$$

The Eqs. (50) and (51) are valid for $n = 0,1,2,3\ldots$ with an understanding that the term involving $Q'_{-1}(s)$ is

$$-\frac{s}{s^2-1} \quad \text{and} \quad A_{-1} = 0. \quad (52)$$

Using the method of variation of parameters we note that

$$\begin{gathered} G_{n+1}(s) = \alpha_{n+1}P'_{n+1}(s) + B_{n+1}Q'_{n+1}(s) - \\ -\frac{P'_{n+1}(s)}{(n+1)(n+2)}\int_{s_1}^{s}(s^2-1)\,Q'_{n+1}(s)\,g_{n+1}(s)\,ds+ \\ +\frac{Q'_{n+1}(s)}{(n+1)(n+2)}\int_{s_1}^{s}(s^2-1)\,P'_{n+1}(s)\,g_{n+1}(s)\,ds \\ \text{for} \quad n = 0,1,2\ldots, \end{gathered} \quad (53)$$

where $s = s_1$ represents the value specifying the spheroid past which the flow is being studied. Thus the flow region F_1 is given by $s > s_1$. As $s \to \infty$, $\psi^{(1)}$ must tend to 0. In view of this we have to take $\alpha_{n+1} = 0$. Hence the appropriate expression for $G_{n+1}(s)$ is given by

$$\begin{gathered} G_{n+1}(s) = B_{n+1}Q'_{n+1}(s)- \\ -\frac{P'_{n+1}(s)}{(n+1)(n+2)}\int_{s_1}^{s}(s^2-1)Q'_{n+1}(s)g_{n+1}(s)ds+ \\ +\frac{Q'_{n+1}(s)}{(n+1)(n+2)}\int_{s_1}^{s}(s^2-1)P'_{n+1}(s)g_{n+1}(s)ds \\ \text{for} \quad n = 0,1,2\ldots. \end{gathered} \quad (54)$$

Thus, the functions $G_{n+1}(s)$ defined in Eq. (45) are completely determined.

As $g_{n+1}(s)$ involves one set $\{A_{n+1}\}$ of arbitrary constants, the functions $G_{n+1}(s)$ involve two sets of arbitrary constants $\{A_{n+1}\}$ and $\{B_{n+1}\}$. Using this in Eq. (45), we get ψ_1.

Solution of Eq. (42). To solve the Eq. (42) (viz.) $\left(E^2 - \frac{\lambda^2}{c^2}\right)\psi = 0$ we take the solution in the form

$$\psi = c\sqrt{(s^2-1)(1-t^2)}R(s)S(t). \quad (55)$$

Substituting this in the Eq. (42), we notice that $R(s)$ and $S(t)$ respectively satisfy the differential equations

$$(s^2-1)R''(s) + 2s\,R'(s) - \left(\Lambda + \lambda^2 s^2 + \frac{1}{s^2-1}\right)R(s) = 0 \quad (56)$$

and

$$(1-t^2)\,S''(t) - 2t\,S'(t) + \left(\Lambda + \lambda^2 t^2 - \frac{1}{1-t^2}\right)S(t) = 0, \quad (57)$$

where Λ is a separation constant [22]. These are spheroidal wave differential equations of radial and angular type respectively. To ensure regularity of solution at infinity and in the flow region we have to choose the solutions of Eqs. (56) and (57) in the form

$$\begin{gathered} R^{(3)}_{1n}(i\lambda, s) = \left[i^{n+2}\sum_{r=0,1}^{\infty}(r+1)(r+2)d^{1n}_r(i\lambda)\right]^{-1} \\ \left(\frac{s^2-1}{s^3}\right)^{1/2}\left(\frac{2}{\pi\lambda}\right)^{1/2}\sum_{r=0,1}^{\infty}(r+1) \\ (r+2)d^{1n}_r(i\lambda)K_{r+3/2}(\lambda s) \end{gathered} \quad (58)$$

and

$$S^{(1)}_{1n}(i\lambda, t) = \sum_{r=0,1}^{\infty} d^{1n}_r(i\lambda)P^{(1)}_{r+1}(t) \quad (59)$$

where

$$P^{(1)}_{r+1}(t) = \sqrt{1-t^2}\frac{d}{dt}P_{r+1}(t) \quad (60)$$

denotes the associated Legendre function of the first kind.

The coefficients $d^{1n}_r(i\lambda)$ in the above expansions are constants depending on the parameter $i\lambda$ and the suffix r has the value $1,3,5\ldots$ or $0,2,4,6\ldots$ depending upon the odd or even values of $n{+}1$. We have therefore the solution

$$\psi_2 = c\sqrt{(s^2-1)(1-t^2)}\sum_{n=1}^{\infty} C_n R_{1n}^{(3)}(i\lambda, s) S_{1n}^{(1)}(i\lambda, t), \quad (61)$$

where C_n's are constants.

Hence, the stream function for the region F_1 is given by

$$\begin{aligned}\psi^{(1)}(s,t) &= -\frac{1}{2}Uc^2(s^2-1)(1-t^2)+ \\ &+ c^2(s^2-1)(1-t^2)\sum_{n=0}^{\infty} G_{n+1}(s)P'_{n+1}(t)+ \\ &+ c\sqrt{(s^2-1)(1-t^2)}\sum_{n=1}^{\infty} C_n R_{1n}^{(3)}(i\lambda, s) S_{1n}^{(1)}(i\lambda, t).\end{aligned} \quad (62)$$

We can see that

$$\begin{aligned}E^2\psi^{(1)} &= c^2(s^2-1)(1-t^2)\sum_{n=0}^{\infty} A_{n+1}Q'_{n+1}(s)P'_{n+1}(t)+ \\ &+ \frac{\lambda^2}{c}\sqrt{(s^2-1)(1-t^2)}\sum_{n=1}^{\infty} C_n R_{1n}^{(3)}(i\lambda, s) S_{1n}^{(1)}(i\lambda, t)\end{aligned} \quad (63)$$

and

$$E^4\psi^{(1)} = \frac{\lambda^4}{c^3}\sqrt{(s^2-1)(1-t^2)}\sum_{n=1}^{\infty} C_n R_{1n}^{(3)}(i\lambda, s) S_{1n}^{(1)}(i\lambda, t). \quad (64)$$

Using Eqs. (63) and (64) in Eqs. (37), we have

$$\begin{aligned}C^{(1)}(s,t) &= \frac{c}{2}\sqrt{(s^2-1)(1-t^2)}\sum_{n=0}^{\infty} A_{n+1}Q'_{n+1}(s)P'_{n+1}(t)+ \\ &+ \frac{\lambda^2}{c^2}\frac{\mu+k}{k}\sum_{n=1}^{\infty} C_n R_{1n}^{(3)}(i\lambda, s) S_{1n}^{(1)}(i\lambda, t).\end{aligned} \quad (65)$$

Pressure distribution in F_1. Equations (17) and (18), using Eq. (37) lead to

$$\frac{\partial p^{(1)}}{\partial s} = \frac{(2\mu+k)}{2c(s^2-1)}\frac{\partial}{\partial t}(E^2\psi^{(1)}) - \frac{\gamma(\mu+k)}{2kc(s^2-1)}\frac{\partial}{\partial t}(E^4\psi^{(1)}) \quad (66)$$

and

$$\frac{\partial p^{(1)}}{\partial t} = -\frac{(2\mu+k)}{2c(1-t^2)}\frac{\partial}{\partial s}(E^2\psi^{(1)}) + \frac{\gamma(\mu+k)}{2kc(1-t^2)}\frac{\partial}{\partial s}(E^4\psi^{(1)}). \quad (67)$$

Using the expressions in Eqs. (63) and (64) in (66) and (67), and integrating the resulting equations, we get

$$\begin{aligned}p^{(1)}(s,t) &= -\frac{(2\mu+k)c}{2} \\ &\sum_{n=0}^{\infty} A_{n+1}Q_{n+1}(s)(n+1)(n+2)P_{n+1}(t).\end{aligned} \quad (68)$$

Thus $\psi^{(1)}(s,t)$, $C^{(1)}(s,t)$ and $p^{(1)}(s,t)$ given in Eqs. (62), (65) and (66) give respectively the stream function, micro rotation and pressure distribution for the region F_1. These involve the three sets of constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ as can be seen from Eqs. (61), (64) and (65).

5. Solution for the flow in the region F_0

We have seen earlier that the flow in the porous region F_0 is governed by the Eqs. (28) and (29) which lead to the Eq. (30). The Eq. (30) implies that the pressure distribution $p^{(0)}(s,t)$ in F_0 is harmonic and hence it is given by

$$p^{(0)}(s,t) = \sum_{n=0}^{\infty}(\alpha_n P_n(s) + \beta_n Q_n(s))\,P_n(t), \quad (69)$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ constitute two sets of arbitrary constants to be determined. The velocity components $u^{(0)}(s,t)$ and $v^{(0)}(s,t)$ can be determined from Eqs. (29) and (69).

In view of the continuity equation in the region F_0, we introduce the stream function $\psi^{(0)}$ through

$$h_2h_3u^{(0)} = -\frac{\partial\psi^{(0)}}{\partial\eta}; \qquad h_1h_3v^{(0)} = \frac{\partial\psi^{(0)}}{\partial\xi} \quad (70)$$

as in Eq. (14).Using (69) and (29), the stream function $\psi^{(0)}$ takes the form

$$\begin{aligned}\psi^{(0)}(s,t) &= -k^{(1)}c(s^2-1)\sum_{n=0}^{\infty} \\ &\left(\alpha_{2n+1}P'_{2n+1}(s) + \beta_{2n+1}Q'_{2n+1}(s)\right)\int_{-1}^{t} P_{2n+1}(t)dt.\end{aligned} \quad (71)$$

Thus, in all, we have five sets of unknown constants $\{A_n\}$, $\{B_n\}$, $\{C_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ and these can be determined by using the boundary conditions given by the Eqs. (31)–(34) and (35).

6. Velocity and microrotation components in the regions F_0, F_1

The expressions for the velocity components $u^{(1)}(s,t)$ and $v^{(1)}(s,t)$as

$$\begin{aligned}u^{(1)}(s,t) &= \frac{1}{c^2\sqrt{(s^2-t^2)(s^2-1)}}\frac{\partial\psi^{(1)}}{\partial t}, \\ v^{(1)}(s,t) &= \frac{1}{c^2\sqrt{(s^2-t^2)(1-t^2)}}\frac{\partial\psi^{(1)}}{\partial s}.\end{aligned} \quad (72)$$

Further

$$\begin{aligned}u^{(0)}(s,t) &= -\frac{k^{(1)}\sqrt{s^2-1}}{c\sqrt{(s^2-t^2)}}\frac{\partial p^{(0)}}{\partial s}, \\ v^{(0)}(s,t) &= \frac{k^{(1)}\sqrt{1-t^2}}{c\sqrt{(s^2-t^2)}}\frac{\partial p^{(0)}}{\partial t}.\end{aligned} \quad (73)$$

These can be obtained by using the expressions for $\psi^{(1)}$ given in Eqs. (62) and $p^{(0)}$ given in Eq. (69). Thus the expressions for the velocity components $u^{(1)}$, $v^{(1)}$; $u^{(0)}$, $v^{(0)}$; the micro rotation component $C^{(1)}$ can all be written explicitly. Using these expressions and those of $p^{(0)}$ and $p^{(1)}$ in the boundary conditions given by Eqs. (31)–(34) and (35), we can write the equations that lead to the determination of the arbitrary constants.

7. Determination of arbitrary constants

In view of the continuity of the normal velocity components on the interface $s = s_1$ given by Eq. (31), we have

$$\begin{aligned} & Uc^2(s_1^2-1) - c^2(s_1^2-1)\cdot \\ & \cdot \sum_{n=0}^{\infty} G_{n+1}(s_1)(n+1)(n+2)P_{n+1}(t) - \\ & - c\sqrt{s_1^2-1}\sum_{n=1}^{\infty} C_n R_{1n}^{(3)}(i\lambda, s_1)\frac{d}{dt}\cdot \\ & \cdot, \left(\sqrt{1-t^2}S_{1n}^{(1)}(i\lambda, t)\right) = \\ & = -k^{(1)}c(s_1^2-1)\sum_{n=0}^{\infty}(\alpha_{n+1}P'_{n+1}(s_1) \\ & + \beta_{n+1}Q'_{n+1}(s_1))\,P_{n+1}(t). \end{aligned} \tag{74}$$

As the tangential velocity components are to vanish on the boundaries, the Eq. (32) leads to

$$\begin{aligned} & -Uc^2 s_1(1-t^2)P_1'(t) + \\ & + c^2\sum_{n=0}^{\infty}\frac{d}{ds}\left((s^2-1)G_{n+1}(s)\right)_{s=s_1}(1-t^2)P'_{n+1}(t) + \\ & c\sum_{n=1}^{\infty} C_n\frac{d}{ds}\left[\sqrt{s^2-1}R_{1n}^{(3)}(i\lambda, s)\right]_{ons=s_1} \\ & \sum_{r=0,1}^{\infty} d_r^{1n}(i\lambda)(1-t^2)P'_{r+1}(t) = 0. \end{aligned} \tag{75}$$

The condition (33) on micro rotation gives rise to the equation

$$\begin{aligned} & \frac{c}{2}\sqrt{(s_1^2-1)}\sum_{n=0}^{\infty} A_{n+1}Q'_{n+1}(s_1)\sqrt{1-t^2}P'_{n+1}(t) + \\ & + \frac{(\mu+k)}{k}\frac{\lambda^2}{c^2}\sum_{n=1}^{\infty} C_n R_{1m}^{(3)}(i\lambda, s_1)\cdot \\ & \cdot\sum_{r=0,1}^{\infty} d_r^{1n}(i\lambda)\sqrt{1-t^2}P'_{r+1}(t) = 0. \end{aligned} \tag{76}$$

The no slip condition on S_0 given by (34) leads to

$$\begin{aligned} & -\frac{k^{(1)}\sqrt{(1-t^2)}}{c\sqrt{s^2-t^2}}\sum_{n=0}^{\infty}(\alpha_{n+1}P_{n+1}(s_0) + \beta_{n+1}Q_{n+1}(s_0))\cdot \\ & \cdot P'_{n+1}(t) = 0. \end{aligned} \tag{77}$$

The continuity of pressure on the interfaces given by Eq. (35) yield

$$\begin{aligned} & -\frac{(2\mu+k)c}{2}\sum_{n=0}^{\infty} A_{n+1}Q_{n+1}(s_1)(n+1)(n+2)P_{n+1}(t) = \\ & = \sum_{n=0}^{\infty}(\alpha_{n+1}P_{n+1}(s_1) + \beta_{n+1}Q_{n+1}(s_1))P_{n+1}(t). \end{aligned} \tag{78}$$

Using the orthogonality property of Legendre functions and the associated Legendre functions, the Eqs. (74) to (78) give rise to the following equations adopting some simple algebraic manipulation:

$$\begin{aligned} & Uc^2(s_1^2-1)\delta_{n0} - c^2(s_1^2-1)B_{n+1}Q'_{n+1}(s_1)(n+1)(n+2) - \\ & - c(n+1)(n+2)\sum_{m=1}^{\infty} C_m\sqrt{s_1^2-1}R_{1n}^{(3)}(i\lambda, s_1)d_n^{1m}(i\lambda) = \\ & = -k^{(1)}c(s_1^2-1)\left(\alpha_{n+1}P'_{n+1}(s_1) + \beta_{n+1}Q'_{n+1}(s_1)\right), \end{aligned} \tag{79}$$

$$\begin{aligned} & -Uc^2 s_1\delta_{0n} + c^2 B_{n+1}(n+1)(n+2)Q_{n+1}(s_1) + \\ & c\sum_{m=1}^{\infty} C_m\frac{d}{ds}\left[\sqrt{s^2-1}R_{1n}^{(3)}(i\lambda, s)\right]_{ons=s_1} d_n^{1m}(i\lambda) = 0, \end{aligned} \tag{80}$$

$$\begin{aligned} & \frac{c}{2}\sqrt{(s_1^2-1)}A_{n+1}Q'_{n+1}(s_1) + \frac{(\mu+k)}{k}\frac{\lambda^2}{c^2} \\ & \sum_{m=1}^{\infty} C_m R_{1m}^{(3)}(i\lambda, s_1)d_n^{1m}(i\lambda) = 0, \end{aligned} \tag{81}$$

$$\begin{aligned} & -\frac{(2\mu+k)c}{2}A_{n+1}Q_{n+1}(s_1)(n+1)(n+2) = \\ & = \alpha_{n+1}P_{n+1}(s_1) + \beta_{n+1}Q_{n+1}(s_1), \end{aligned} \tag{82}$$

$$\alpha_{n+1}P_{n+1}(s_0) + \beta_{n+1}Q_{n+1}(s_0) = 0. \tag{83}$$

The coefficients $d_r^{1n}(i\lambda)$ in the equations above are constants depending on the parameter $i\lambda$ and the suffix r has the value $1, 3, 5\ldots$ or $0, 2, 4\ldots$ depending upon the odd or even nature of $n+1$ [22]. From Eqs. (79) and (80), the coefficient B_{n+1} can be eliminated and using (81), (82) and (83), we get a non homogeneous linear system of algebraic equations for the determination of constants $\{C_n\}$. This system is seen to be

$$\sum_{m=1}^{\infty} D_{nm}C_m = -U\,c\delta_{0n}, \quad n = 0, 1, 2\ldots \tag{84}$$

where

$$D_{nm} = d_{2n}^{1m}(i\lambda)\begin{bmatrix} \left\{\sqrt{s_1^2-1}\frac{d}{ds}R_{1m}^{(3)}(i\lambda, s_1) + \frac{s_1}{\sqrt{s_1^2-1}}R_{1m}^{(3)}(i\lambda, s_1)\right\}(s_1^2-1)Q'_{2n+1}(s_1) \\ -2(n+1)(2n+1)\sqrt{s_1^2-1}Q_{2n+1}(s_1)R_{1m}^{(3)}(i\lambda, s_1) \\ -(2\mu+k)(n+1)(2n+1)\frac{\mu+k}{k}\frac{\lambda^2}{c^2}2k^{(1)}(s_1^2-1)\frac{Q_{2n+1}(s_1)}{Q'_{2n+1}(s_1)}\frac{Q_{2n+1}(s_1)}{\sqrt{s_1^2-1}}R_{1m}^{(3)}(i\lambda, s_1) \\ \frac{Q_{2n+1}(s_0)P'_{2n+1}(s_1) - P_{2n+1}(s_0)Q'_{2n+1}(s_1)}{P_{2n+1}(s_0)Q_{2n+1}(s_1) - P_{2n+1}(s_1)Q_{2n+1}(s_0)} \end{bmatrix}. \tag{85}$$

The above linear system splits into two complementary sub systems where n is even and n is odd. The subsystem when n is odd reduces to the homogeneous set of equations

$$\sum_{m=1}^{\infty} D_{2n+1,2m} C_{2m} = 0 \tag{86}$$

and we therefore have $C_2 = C_4 = C_6 \ldots = 0$. Hence A_n, B_n are all zero when n is even. The analytical determination of the odd suffixed constants is not possible. In view of this, we propose to determine them numerically. Here we truncate the system (85) to fifth order and numerically evaluate the coefficients C_1, C_3, C_5, C_7 and C_9. This is the maximum extent to which the order of truncation can be extended since the coefficients of spheroidal wave functions needed for a higher order truncation are not explicitly available in the standard literature [22].

After determining these, it is possible to evaluate numerically the other constants. The details of the manipulations are omitted in view of the lengthiness of the expressions and the final system only is reported here.

8. Determination of drag

To evaluate the drag on the body, we need the stress components and the couple stress components. The stress tensor is given by Eq. (5) and we need to evaluate the rate of strain components e_{ij} and the spin component ω_ϕ.

The velocity vector $\vec{q}$ can be written in the form

$$\vec{q} = u\,\vec{\mathrm{e}}_\xi + v\,\vec{\mathrm{e}}_\eta, \tag{87}$$

where

$$\begin{aligned} u &= \frac{1}{c^2\sqrt{(s^2-t^2)(s^2-1)}}\frac{\partial\psi}{\partial t}, \\ v &= \frac{1}{c^2\sqrt{(s^2-t^2)(1-t^2)}}\frac{\partial\psi}{\partial s}. \end{aligned} \tag{88}$$

The rate of strain components are given by

$$\begin{aligned} e_{\xi\xi} &= \frac{1}{c^3(s^2-t^2)}\left(\psi_{st} + \frac{t}{s^2-t^2}\psi_s - \right. \\ &\qquad \left. - \frac{s(2s^2-1-t^2)}{(s^2-t^2)(s^2-1)}\psi_t\right), \\ e_{\xi\eta} = e_{n\xi} &= \frac{(s^2-1)\psi_{ss} - (1-t^2)\psi_{tt}}{2c^3(s^2-t^2)\sqrt{(s^2-1)(1-t^2)}} - \\ &- \frac{s\sqrt{s^2-1}}{c^3(s^2-t^2)^2\sqrt{1-t^2}}\psi_s - \frac{t\sqrt{1-t^2}}{c^3(s^2-t^2)^2\sqrt{s^2-1}}\psi_t, \\ e_{\mu\eta} &= \frac{1}{c^3(s^2-t^2)}\left(-\psi_{st} + \frac{s}{s^2-t^2}\psi_t + \right. \\ &\qquad \left. + \frac{s(2t^2-1-s^2)}{(s^2-t^2)(1-t^2)}\psi_s\right), \\ e_{\xi\xi} &= \frac{1}{c^3(s^2-t^2)}\left(\frac{s}{s^2-1}\psi_t + \frac{t}{(1-t^2)}\psi_{st}\right), \\ e_{\xi\phi} &= e_{\phi\xi} = e_{\eta\phi} = e_{\phi\eta} = 0 \end{aligned} \tag{89}$$

The spin = $\frac{1}{2} curl \vec{\mathrm{q}}$ has only one non zero component ω_ϕ in the direction of the vector $\vec{\mathrm{e}}_\phi$ and this is given by

$$\omega_\phi = \frac{1}{2c\sqrt{(s^2-1)(1-t^2)}}E^2\psi. \tag{90}$$

The surface stress t_{ij} for the micropolar fluid is given by Eq. (5) and we find that the only non vanishing components of t_{ij} are $t_{\xi\xi}$, $t_{\eta\eta}$, $t_{\phi\phi}$, $t_{\xi\eta}$ and $t_{\eta\xi}$. These are given by

$$\begin{aligned} t_{\eta\eta} &= -p + (2\mu+k)e_{\eta\eta}, \\ t_{\phi\phi} &= -p + (2\mu+k)e_{\phi\phi}, \\ t_{\xi\eta} &= (2\mu+k)e_{\xi\eta}, \\ t_{\eta\xi} &= (2\mu+k)e_{\eta\xi}. \end{aligned} \tag{91}$$

The stress vector $\vec{t}$ on the boundary of the body is given by

$$\vec{t} = t_{\xi\xi}\,\vec{\mathrm{e}}_\xi + t_{\xi\eta}\,\vec{\mathrm{e}}_\eta. \tag{92}$$

We find that

$$\begin{aligned} (t_{\xi\xi})_{s=s_1} &= -p^{(1)}(s_1,t) + \frac{k^{(1)}s_1(2s_1^2-1-t^2)(2\mu+k)}{c^2(s_1^2-t^2)^2}\cdot \\ &\cdot \sum_{n=0}^{\infty}\left(\alpha_{n+1}P'_{n+1}(s_1) + \beta_{n+1}Q'_{n+1}(s_1)\right)P_{n+1}(t) \end{aligned} \tag{93}$$

and

$$\begin{aligned} (t_{\xi\eta})_{\mathrm{s=s_1}} &= \frac{(2\mu+k)c}{2}\sqrt{(\mathrm{s}_1^2-1)(1-t^2)} \\ &\sum_{n=0}^{\infty} A_{n+1}Q'_{n+1}(s_1)P'_{n+1}(t) + \\ &+ \frac{(2\mu+k)}{2}\frac{\lambda^2}{c^2}\sum_{n=0}^{\infty} C_n R_{1n}^{(3)}(i\lambda,s_1)S_{1n}^{(1)}(i\lambda,t) + \\ &\frac{(2\mu+k)}{c^2(s_1^2-t^2)}\sqrt{(\mathrm{s}_1^2-1)(1-t^2)} \\ &\sum_{n=0}^{\infty}(\alpha_{n+1}P'_{n+1}(s_1) + \beta_{n+1}Q'_{n+1}(s_1))P'_{n+1}(t) - \\ &\frac{(2\mu+k)s_1k^{(1)}}{c^2(s_1^2-t^2)^2}\sqrt{(\mathrm{s}_1^2-1)(1-t^2)} \\ &\sum_{n=0}^{\infty}(\alpha_{n+1}P'_{n+1}(s_1) + \beta_{n+1}Q'_{n+1}(s_1))P'_{n+1}(t) + \\ &\frac{k}{2c\sqrt{(s_1^2-1)(1-t^2)}} \\ &\left(\begin{array}{l} c^2(s_1^2-1)(1-t^2)\sum\limits_{n=0}^{\infty} A_{n+1}Q'_{n+1}(s_1)P'_{n+1}(t) \\ +\dfrac{\lambda^2}{c}\sqrt{(s_1^2-1)(1-t^2)}\sum\limits_{n=0}^{\infty} C_n R_{1n}^{(3)}(i\lambda,s)S_{1n}^{(1)}(i\lambda,t) \end{array}\right). \end{aligned} \tag{94}$$

The stress vector has the component

$$\begin{aligned} &(stress)_{axial} = \\ &= \frac{1}{\sqrt{s_1^2-t^2}}\left(t\sqrt{s^2-1}\mathrm{t}_{\xi\xi} - s\sqrt{1-t^2}\mathrm{t}_{\xi\eta}\right)_{s=s_1} \end{aligned} \tag{95}$$

in the direction of the axis of symmetry and

$$(stress)_{radial} = \\ = \frac{1}{\sqrt{s_1^2 - t^2}} \left(s\sqrt{1-t^2}\ \mathrm{t}_{\xi\xi} + t\sqrt{s^2-1}\ \mathrm{t}_{\xi\eta} \right)_{s=s_1} \tag{96}$$

in the radial direction of the meridian plane. The resultants of these two vector components over the entire surface of the body are obtained by integration and it is seen that the radial component integrates to zero. Thus the resultant of the stress vector on the body is the force in the direction of the axis of symmetry and this gives the drag on the body. The drag D can be written in the form

$$D{=}2\pi\ c^2\sqrt{s_1^2{-}1}\int_{-1}^{1}\left(t\sqrt{s^2{-}1}\ \mathrm{t}_{\xi\xi}{-}s\sqrt{1{-}t^2}\ \mathrm{t}_{\xi\eta}\right)_{s=s_1} dt \tag{97}$$

and this simplifies to

$$2\pi\ c^2\sqrt{s_1^2-1}\left(\begin{array}{l} \int_{-1}^{1} t\sqrt{s_1^2-1}\, p^{(1)}(s_1,t)dt+ \\ \dfrac{2(2\mu+k)s_1(s_1^2-1)^{3/2}k^{(1)}}{c^2} \\ \sum_{n=0}^{\infty}\left(\alpha_{n+1}P'_{n+1}(s)+\beta_{n+1}Q'_{n+1}(s_1)\right) \\ \int_{-1}^{1}\frac{tP_{n+1}(t)}{(s_1^2-t^2)^2}dt - \frac{(2\mu+k)}{c^2}s_1\sqrt{s_1^2-1}\ \mathrm{k}^{(1)} \\ \sum_{n=0}^{\infty}\left(\alpha_{n+1}P'_{n+1}(s)+\beta_{n+1}Q'_{n+1}(s_1)\right) \\ \int_{-1}^{1}\frac{(1-t^2)P'_{n+1}(t)}{(s_1^2-t^2)}dt- \\ (\mu+k)cs_1\sqrt{s_1^2-1}\sum_{n=0}^{\infty}A_{n+1}Q'_{n+1}(s_1) \\ \int_{-1}^{1}(1-t^2)P'_{n+1}(t)dt- \\ (\mu+k)s_1\frac{\lambda^2}{c^2}\sum_{n=0}^{\infty}C_nR_{1n}^{(3)}(i\lambda,s_1) \\ \int_{-1}^{1}\sqrt{(1-t^2)}S_{1n}^{(1)}(i\lambda,t)dt \end{array}\right). \tag{98}$$

Using the relations

$$\int_{-1}^{1}\frac{(1-t^2)P'_n(t)}{(s_1^2-t^2)}dt = -\frac{2}{s_1}(s_1^2-1)Q'_n(s_1) \tag{99}$$

and

$$\int_{-1}^{1}\frac{tP_n(t)}{(s_1^2-t^2)^2}dt = -\frac{1}{s_1}Q'_n(s_1), \tag{100}$$

drawn from "The Theory of Spherical and Ellipsoidal Harmonics" due to Hobson [25], the drag simplifies to

$$D = 2\pi\ \mathrm{c}\sqrt{s_1^2-1}\left(\begin{array}{l} cA_1^{2/3}\sqrt{s_1^2-1}((2\mu+k)Q_1(s_1)- \\ -2(\mu+k)s_1Q'_1(s_1)) \\ -\dfrac{4}{3}(\mu+k)\dfrac{\lambda^2}{c^2}s_1\sum_{n=0}^{\infty}/ \\ C_nR_{1n}^{(3)}(i\lambda,s_1)d_0^{1n}(i\lambda) \end{array}\right). \tag{101}$$

Using the Eq. (81), we may eliminate the series involving the constants C_n in the above expression for the drag and after further simplification we see that the drag due to the surface stress is given by the simple formula

$$D = \frac{4}{3}(2\mu+k)\pi\ c^3A_1. \tag{102}$$

Introducing the non dimensionalization scheme given by

$$A_{n+1} = \frac{U}{c^2}\widetilde{A}_{n+1}, \\ C_n = Uc\widetilde{C}_n \tag{103} \\ D = 4\pi(2\mu+k)Uc^3\widetilde{D}$$

it is seen, after dropping the tildes, that the nondimensional drag D is given by

$$D = \frac{1}{3}A_1, \tag{104}$$

where

$$A_1 = -\frac{2(\mu+k)}{k}\lambda^2\frac{\sum_{m=1}^{\infty}C_mR_{1m}^{(3)}(i\lambda,s_1)d_0^{1m}(i\lambda)}{\sqrt{s_1^2-1}Q'_1(s_1)}. \tag{105}$$

This depends upon the eccentricity of the spheroid, the material constant λ, the micropolarity parameter $pl = \dfrac{k}{\mu+k}$ and the non dimensional permeability parameter *kp* defined through $kp = k^{(1)}\dfrac{\mu+k}{c^2}$.

9. Numerical discussion

We have computed the non dimensional drag D for various values of the parameters λ, pl, kp, s_1 and a fixed value of s_0. This requires solving of the infinite non homogeneous system of equations given in Eq. (84). As explained already, we truncated this to a 5×5 system. This involves the determination of constants of various radial prolate spheroidal wave functions $R_{1m}^{(3)}(i\lambda, s)$, the needed Legendre functions and their derivatives. All the programs necessary were written in 'C' and the constants C_1, $C_3 \ldots C_9$ have been evaluated. We have fixed s_0 as 1.2 and varied s_1 through 1.5, 1.8 and 2.0. The drag is calculated for $\lambda = 1.0, 1.2, 1.5$ and 1.8; pl = 0.2, 0.4, 0.6, 0.8 and kp = 0.001 and 0.005.

The variation of drag is presented through Figs. 2 to 7. For a given s_0, s_1 and any prescribed pl, the drag is seen to increase as the parameter λ increases (see Figs. 2 and 3). For a given s_0, given λ, given permeability parameter kp, for any pl, the drag is increasing with s_1. This is natural because as s_1 increases, the size of the spheroid increases and a larger body experiences a greater drag. Figures 6 and 7, for a given s_0 and s_1 and for different values of λ depict the variation of drag with reference to the parameter pl. As pl increases, the drag is seen to decrease. An increase in pl indicates greater microrotation leading to dissipation of energy which results in the reduction in the drag.

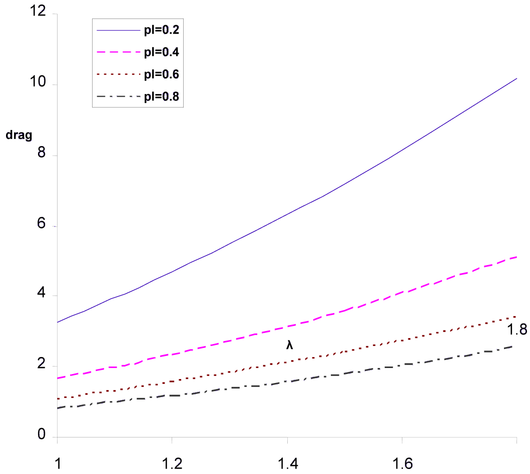

Fig. 2. Variation of drag with respect to λ for different values of the polarity parameter pl, when $s_0 = 1.2$, $s_1 = 1.5$ and permeability parameter $kp = 0.001$

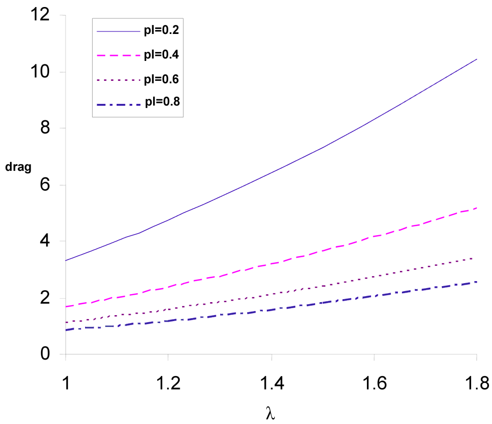

Fig. 3. Variation of drag with respect to $lambda$ for different values of the polarity parameter pl, when $s_0 = 1.2$, $s_1 = 1.5$ and permeability parameter $kp = 0.005$

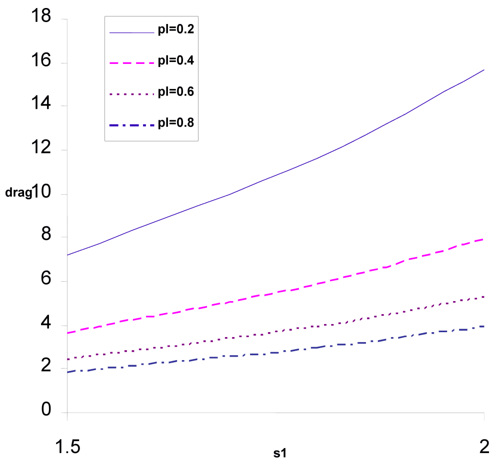

Fig. 4. Variation of drag with s_1 for different values of the polarity parameter pl, when $s_0 = 1.2$, $s_1 = 1.5$ and permeability parameter $kp = 0.001$

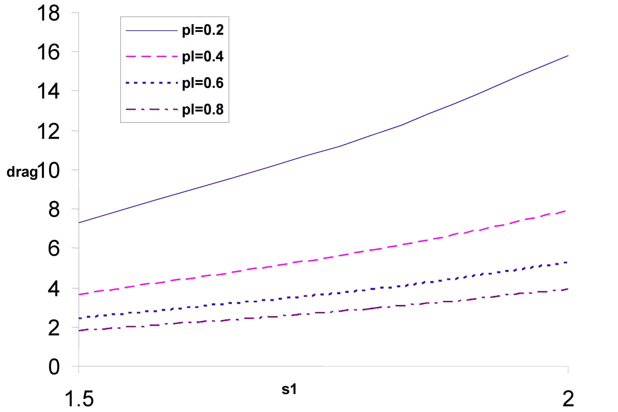

Fig. 5. Variation of drag with s_1 for different values of the polarity parameter pl, when $s_0 = 1.2$, $s_1 = 1.5$ and permeability parameter $kp = 0.005$

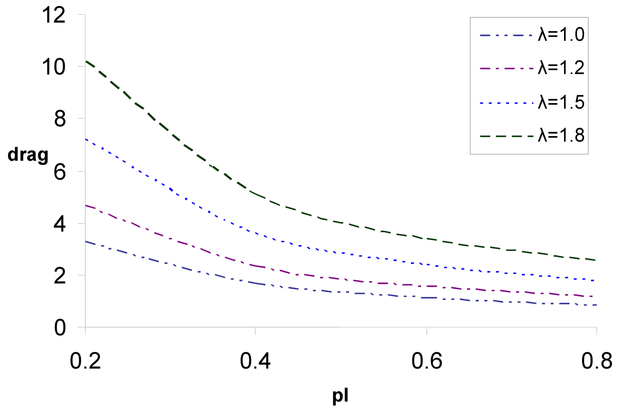

Fig. 6. Variation of drag with respect to the polarity pl for different values of λ when $s_0 = 1.2$, $s_1 = 1.5$ and permeability parameter $kp = 0.001$

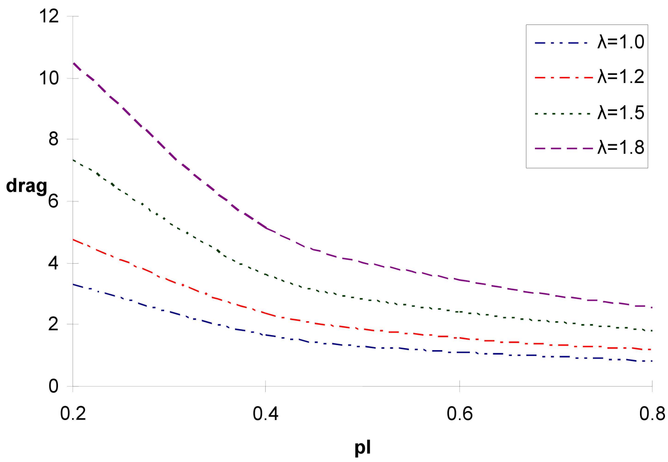

Fig. 7. Variation of drag with respect to the polarity pl for different values of λ when $s_0 = 1.2$, $s_1 = 1.5$ and permeability parameter $kp = 0.005$

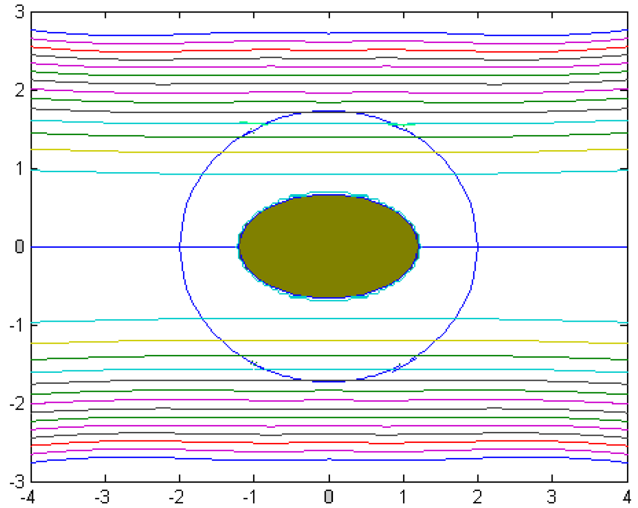

Fig. 8. Streamlines for kp = 0.01, pl = 0.4 and λ = 2.5

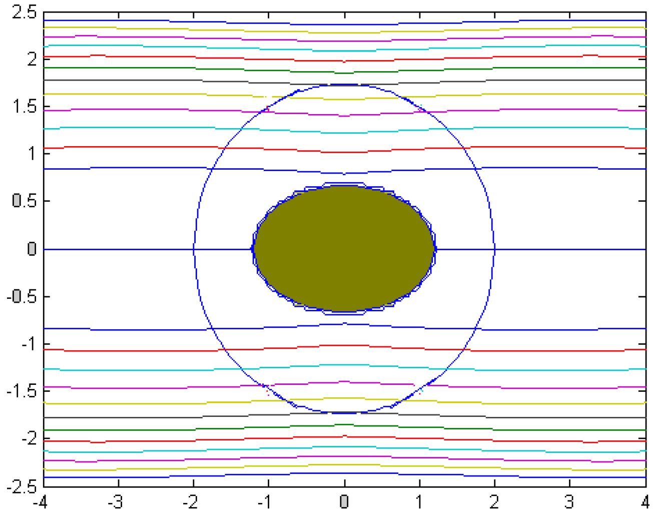

Fig. 9. Streamlines for kp = 0.01, pl = 0.4 and λ = 2.0

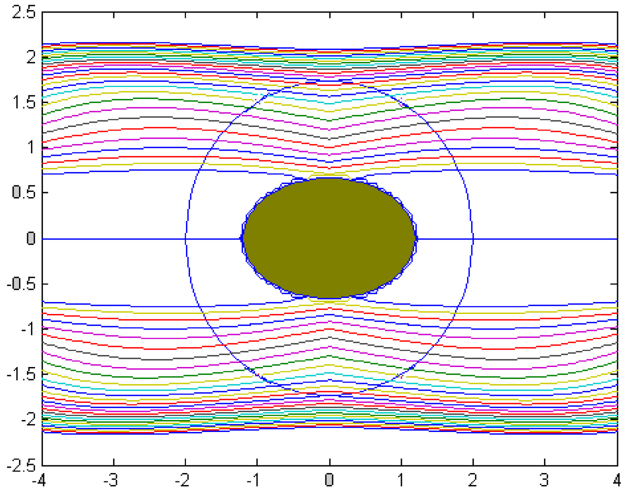

Fig. 10. Streamlines for kp = 0.01, pl = 0.6 and λ = 2.0

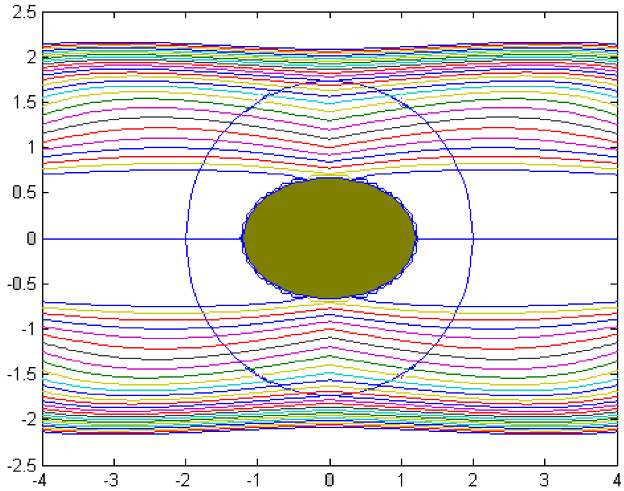

Fig. 11. Streamlines for kp = 0.01, pl = 0.6 and λ = 2.5

Figures 8 to 11 present the streamline pattern. For small values of kp and pl, the flow is seen to be less disturbed (see Figs. 8, 9). For larger values of kp, the fluid nearer the boundary s_1 is sucked into the porous region as seen in Fig. 10. This is analogous to that observed by Raja Sekhar and Osamu Sano in their study of viscous flow past a circular/ spherical void in porous media [26]. For still higher values of λ, the flow is further disturbed and divided steamline pattern is observed (see Fig. 11). This is analogous to the streamline pattern found by the present authors in connection with their investigation with respect to a porous spheroid [21]. The analysis for an oblate spheroid can also be similarly carried out and salient features of the investigations can be communicated in a separate paper.

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