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Some applications of fractional order calculus

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Abstract. This paper presents some recent results in the area of application of fractional order system models. After the introduction to the dynamic systems modelling with the fractional order calculus the paper concentrates on the possibilities of using this approach to the modelling of real-world phenomena. Two examples of such systems are considered. First one is the ultracapacitor where fractional order models turn out to be more precise in the wider range of frequencies than other models used so far. Another example is the beam heating problem where again the fractional order model allows to obtain better modelling accuracy. The theroetical models were tested experimentally and the results of these experiments are described in the paper.

Key words: fractional order, state-space system; ultracapacitors modelling; heat transfer models.

1. Introduction

Although the application of fractional order differentials and integrals is a matter of recent decades the theory of fractional calculus has quite a long and prominent history. In fact, one may trace it back to the very origins of differential calculus itself. However, its complexity prevented it from being used in practice until only very recently. In the last decades, the results of work on the theory of chaos revealed some relations with fractional derivatives and integrals. This in turn renewed interest in the field. Some fundamental facts of the fractional calculus theory and its properties may be found in e.g., [1, 2]. As far as the applications of fractional calculus are concerned there is a large volume of research on viscoelasticity/damping, see e.g., [3, 4] and chaos/fractals, see e.g., [5]. Also, other areas of science and technology have started to pay more attention to these concepts and it may be noted that fractional calculus is being adopted in the fields of signal processing, system modelling and identification, and control to name just a few. What is most interesting from our point of view is the application of fractional calculus in the last two areas. Several researchers on automatic control have proposed control algorithms both in frequency [6, 7] and time [8] domains based on the concepts of fractional calculus. This work is still in a fairly early stage and a lot remains to be done.

One of the fundamental problems in control is the stability analysis of the dynamic system. The stability problem for linear, continuous-time, fractional order state-space systems has been considered for some time and some properties and stability results for these systems are presented and discussed e.g. in [9] and [10].

For the discrete-time fractional order systems however the discussion of this problem is much less common. There are very few results dealing with the stability of such systems. It is even more so for the state-space description of these systems. Some basic results of defining the fractional order state-space systems are presented in e.g. [11]. Some remarks on poles and zeros of fractional order systems are given in [12].

Also other system properties for fractional order systems like controlability and observability have been addressed only in recent year (see e.g. [13]).

Two recent good expositions of the fractional order calculus and its applications can be found e.g. in [14] and [15].

The aim of this paper is to summarise some theoretical developments in the area of fractional order systems, and most notably to present some research results of application of these models to several physical phenomena like ultracapacitors, and the beam heating problem.

2. Fractional order differential calculus introduction

Differential calculus is only the generalization of full integer order integral and differential calculus to real or even complex order. In the section below main definitions of fractional order integrals and derivatives are presented.

2.1. Definition of fractional order differ-integral. In this paper the following definition of the fractional order derivative [1, 2] is used.

Definition 1. Riemann-Liouville definition of fractional order differ-integral:

$$ {}^{RL}{}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt} \int_a^t (t-\tau)^{m-\alpha-1} f(\tau) d\tau, $$

where

$$ m-1 < \alpha \leq m \in \mathbb{N} $$

and $\alpha \in \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers) is a fractional order of the differ-integral of the function $f(t)$.

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The Laplace transform of the R-L fractional order differ-integral is given as follows:

$$L[{}^{RL}{}_0D_t^\alpha f(t)] = \begin{cases} s^\alpha F(s) & \text{for} \quad \alpha < 0, \\ s^\alpha F(s) - F'(s) & \text{for} \quad \alpha > 0, \end{cases}$$

where $F'(s) = \sum_{k=0}^{n-1} s_0^k D_t^{\alpha-k-1} f(0)$ are the initial conditions terms, and $n - 1 < \alpha \leq n \in \mathbb{N}$.

Definition 2. Caputo's definition of fractional order differ-integral:

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha - m)} \int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau,$$

where

$$m - 1 < \alpha \leq m \in \mathbb{N} \quad \text{and} \quad \alpha \in \mathbb{R}$$

is a fractional order of the differ-integral of the function $f(t)$.

The Laplace transform of the differ-integral of Definition 2 is given as:

$$L[{}_0^C D_t^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0)$$

where

$$n - 1 < \alpha \leq n \in \mathbb{N}.$$

Rieman-Liouville and Caputo definitions are pretty close. The difference is the order of initial conditions (IC). In Caputo definition these conditions are of integer order which makes them easier to interpret. This is not the case of R-L definition where the IC are of fractional order.

As stated above all of the definitions of fractional order calculus mentioned are in many respects equivalent and for $\alpha > 0$ give results of the fractional order derivative, for $\alpha < 0$ fractional order integral and for $\alpha = 0$ the function itself. This is why these definitions are called differ-integrals definitions.

Different approach to differ-integral of non-integer order was presented by Grünwald-Letnikov:

Definition 3. Grünwald-Letnikov definition of fractional order differ-integral:

$${}^{GL}{}_aD_t^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{\frac{t}{h}} (-1)^j \binom{\alpha}{j} f(t - jh).$$

The variation of this definition is used in the next part of the article.

2.2. Fractional order integrator. Let us assume the following transfer function of a fractional order integrator

$$G(s) = \frac{1}{(Ts)^\alpha}. \tag{1}$$

The spectral transfer function of (1) is

$$G(j\omega) = \frac{1}{(Tj\omega)^\alpha} = \frac{1}{(T\omega)^\alpha \left(\cos \frac{\pi}{2}\alpha + j \sin \frac{\pi}{2}\alpha\right)}. \tag{2}$$

The magnitude of the transfer function is given as follows [16]

$$A(\omega) = \sqrt{\frac{\left(\cos^2 \frac{\pi}{2}\alpha + \sin^2 \frac{\pi}{2}\alpha\right)}{(T\omega)^{2\alpha}}} = \frac{1}{(T\omega)^\alpha}, \tag{3}$$

which yields

$$M(\omega) = 20 \log A(\omega) = -\alpha 20 \log(T) - \alpha 20 \log(\omega). \tag{4}$$

The phase properties are obtained from the following:

$$\varphi(\omega) = \arg\left[\frac{1}{(T\omega)^\alpha} j^{-\alpha}\right] = -\alpha \frac{\pi}{2}.$$

The Bode diagram of the fractional order integrator for different values of α is presented in Fig. 1.

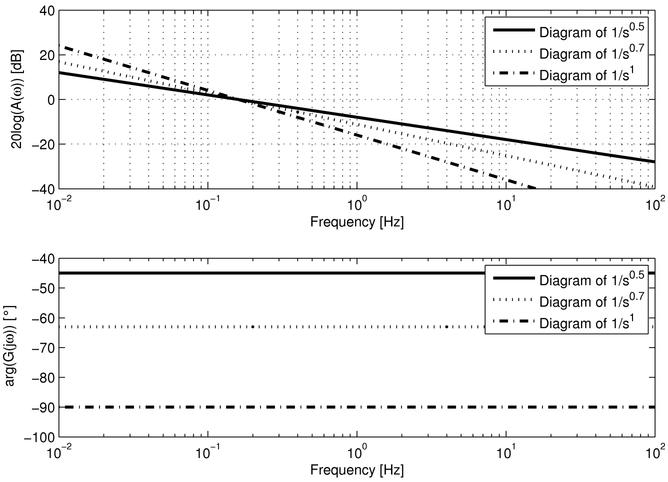

Fig. 1. Bode diagrams of $\frac{1}{s^\alpha}$ systems for $\alpha = 0.5, 0.7, 1$

2.3. Transfer function $\mathbf{e}^{(\mathbf{Ts})^\alpha}$ and its Bode diagram. Let us assume the following transfer function

$$G(s) = e^{(Ts)^\alpha}, \tag{5}$$

for which the spectral transfer function is given by

$$G(j\omega) = e^{(Tj\omega)^\alpha} = e^{(T\omega)^\alpha(\cos \frac{\pi}{2}\alpha + j \sin \frac{\pi}{2}\alpha)}. \tag{6}$$

The magnitude of the transfer function is given as follows

$$A(\omega) = e^{(T\omega)^\alpha \cos \frac{\pi}{2}\alpha}, \tag{7}$$

which gives

$$M(\omega) = 20 \log(e^{(T\omega)^\alpha \cos \frac{\pi}{2}\alpha}). \tag{8}$$

The phase properties are given as follows:

$$\varphi(\omega) = (T\omega)^\alpha \sin \frac{\pi}{2}\alpha.$$

The Bode diagram of this transfer function for different values of T and $\alpha = 0.5$ is presented in Fig. 2.

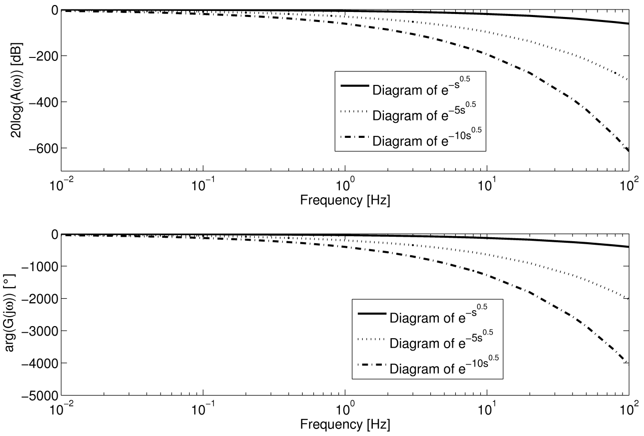

Fig. 2. Bode diagrams of $e^{-(Ts)^{0.5}}$ systems for $T = 1, 5, 10$

The interpretation of the transfer function $e^{-(Ts)^{0.5}}$ is not as easy as the interpretation of the $e^{-(Ts)}$ function. Looking at the frequency response it can be noted that this is not a pure delay system where the phase shift changes exponentially, but there is also a non-zero effect in the magnitude.

3. Fractional order state-space system

The continuous time state-space system is given by the following definition:

Definition 4. Linear continuous fractional order state-space system is given as a following set of equations:

$$ {}^{GL}{}_aD_t^{\alpha}x(t) = Ex(t) + Fu(t), \tag{9}$$

$$ y = Gx(t) + Hu(t), \tag{10}$$

where $E \in \mathbb{R}^{N\times N}$, $F \in \mathbb{R}^{N\times m}$, $G \in \mathbb{R}^{p\times N}$, $H \in \mathbb{R}^{p\times m}$, m is a number of outputs, p is a number of inputs, N is a number of state equations, a $\dfrac{d^{\alpha}x(t)}{dt^{\alpha}}$ is a fractional order derivative of a system order $\alpha \in \mathbb{R}$.

In order to present the discrete fractional order state-space system, let us define the shifted G-L definition of the fractional order differ-integral, which is a variant of classical G-L approach:

Definition 5. The shifted Grünwald-Letnikov definition of fractional order differ-integral is given as follows: [17]

$$ {}^{SGL}{}_aD_t^{\alpha}x(t) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{j=0}^{t/h+1}(-1)^j\binom{\alpha}{j}x(t-(j-1)h),$$

where, $\alpha \in \mathbb{R}$, as previously defined is a fractional degree.

The binomial term in Definition 5 can be obtained from the following relation:

$$\left(\begin{array}{c}\alpha\\ j\end{array}\right) = \begin{cases} 1 & \text{for} \quad j=0 \\ \frac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!} & \text{for} \quad j>0 \end{cases}. \tag{11}$$

In order to introduce discrete fractional order state-space system, let us substitute Definition 5 into 4. This gives

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{j=0}^{t/h+1}(-1)^j\binom{\alpha}{j}x(t-(j-1)h) = \\ = Ex(t) + Fu(t),\end{aligned} \tag{12}$$

$$ y = Gx(t) + Hu(t). \tag{13}$$

Analogically we can define shifted fractional order difference:

Definition 6. Shifted fractional order difference is given as follows:

$$\Delta^{\alpha}x_{k+1} = \sum_{j=0}^{k+1}(-1)^j\binom{\alpha}{j}x_{k-j+1},$$

where, $\alpha \in \mathbb{R}$, is a fractional degree, $\mathbb{R}$, is the set of real numbers and $k \in \mathbb{N}$ ($\mathbb{N}$, is the set of natural numbers) is the number of a sample for which the approximation of the derivative is calculated.

Using the Eq. (12) with some relatively small value of h we can obtain the following structure of the discrete fractional order state-space model. In the general case the values of the discrete system matrices are not the same as in continuous case and have to be found by the discretization process or by identification.

Definition 7. The linear discrete fractional order system in state-space representation is given as follows:

$$\Delta^{\alpha}x_{k+1} = Ax_k + Bu_k, \tag{14}$$

$$x_{k+1} = \Delta^{\alpha}x_{k+1} - \sum_{j=1}^{k+1}(-1)^j\binom{\alpha}{j}x_{k-j+1} \tag{15}$$

$$y_k = Cx_k + Du_k, \tag{16}$$

where $\alpha \in \mathbb{R}$ is a system order.

The value of fractional order difference of state vector for time instant $k+1$ is obtained according to (14), from this value the state vector x_{k+1} is calculated using relation (15). The output equation is given by (16).

For practical realization the number of samples taken into consideration has to be reduced to the predefined number L. In this case the Eq. (15) is rewritten as

$$x_{k+1} = \Delta^{\alpha}x_{k+1} - \sum_{j=1}^{L}(-1)^j\binom{\alpha}{j}x_{k-j+1}, \tag{17}$$

where L is a number of samples taken into account, called memory length and with assumption that $x_k = 0$ for $k < 0$.

The system given by the Definition 7 can be rewritten as an infinite dimensional system in the following way

Definition 8. The infinite dimensional form of the linear discrete fractional order state-space system is defined as follows

$$\begin{bmatrix} x_{k+1} \\ x_k \\ x_{k-1} \\ \vdots \end{bmatrix} = \mathbb{A} \begin{bmatrix} x_k \\ x_{k-1} \\ x_{k-2} \\ \vdots \end{bmatrix} + \mathbb{B}u_k,$$

$$y_k = \mathbb{C} \begin{bmatrix} x_k \\ x_{k-1} \\ x_{k-2} \\ \vdots \end{bmatrix},$$

where

$$\mathbb{A} = \begin{bmatrix} (A + I\alpha) & -(-1)^2 I\binom{\alpha}{2} & -(-1)^3 I\binom{\alpha}{3} & \dots \\ I & 0 & 0 & \dots \\ 0 & I & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

$$\mathbb{B} = \begin{bmatrix} B \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \quad \mathbb{C} = \begin{bmatrix} C & 0 & 0 & \dots \end{bmatrix},$$

where I is the identity matrix.

4. Fractional order difference equation

In order to identify the system parameters the difference equation describing input-output dynamic relation is more convenient than the state-space representation. Let the system equations (14) and (16) be rewritten using $\mathcal{Z}$ transform with zero initial conditions ($x_j = 0$ for $j \leq 0$) as follows:

$$z\Delta^\alpha(z)X(z) = AX(z) + BU(z)$$

$$Y(z) = CX(z).$$

This immediately gives for a Single-Input-Single-Output case the relation

$$\frac{Y(z)}{U(z)} = C(I(z\Delta^\alpha(z)) - A)^{-1}B,$$

where $z\Delta^\alpha(z)$ is a polynomial of z given as follows:

$$z\Delta^\alpha(z) = \sum_{j=0}^{k+1} (-1)^j \binom{\alpha}{j} z^{-j+1}.$$

This leads to the relation

$$G(z) = \frac{Y(z)}{U(z)} = \\ = \frac{(z\Delta^\alpha(z) - z_{b,N-1}) \dots (z\Delta^\alpha(z) - z_{b,1})}{(z\Delta^\alpha(z) - z_{a,N}) \dots (z\Delta^\alpha(z) - z_{a,1})} = \\ = \frac{b_{N-1} z^{N-1} \Delta^{\alpha(N-1)}(z) + \dots + b_0}{z^N \Delta^{\alpha N}(z) + \dots + a_1 z\Delta^\alpha(z) + a_0}, \tag{18}$$

where $z_{a,k}$ for $k = 1 \dots N$ are the eigenvalues of the matrix A and are associated with the system poles, $z_{b,k}$ for $k = 1 \dots N-1$ are associated with the zeros of the system.

Then, the Fractional Difference Equation is as follows:

$$\Delta^{\alpha N} y_k + a_{N-1} \Delta^{\alpha(N-1)} y_{k-1} + \dots + a_0 y_{k-N} = \\ = b_{N-1} \Delta^{\alpha(N-1)} u_{k-1} + \dots + b_0 u_{k-N},$$

where parameters a_k and b_k for $k = 0 \dots N-1$ are the entries of the system matrices, eg. in the canonical form.

5. Fractional order discrete systems identification

The fractional difference equation model of a dynamic system presented in Sec. 4 can form a basis for a control law. However, in order to construct any control law it is essential to know the parameters of the model. Using difference equation defined in previous section it is possible to determine parameters of the estimation process in the following way. This reasoning is in principle a version of the RLS approach to parametric identification.

$$\varphi_k = [\ -\Delta^{\alpha^*_{N-1}} y_{k-1} \quad \dots \quad -y_{k-N} \tag{19}$$

$$\Delta^{\alpha^*_{N-1}} u_{k-1} \quad \dots \quad u_{k-N}\ ] \tag{20}$$

$$\theta^T = \begin{bmatrix} a_{N-1} & \dots & a_0 & b_{N-1} & \dots & b_0 \end{bmatrix} \tag{21}$$

$$Y_k = \begin{bmatrix} \Delta^{\alpha^*_N} y_k s \end{bmatrix} \tag{22}$$

The parameters may be obtained by solving the equation (usually overdetermined)

$$\begin{bmatrix} Y_k \\ Y_{k-1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \varphi_k \\ \varphi_{k-1} \\ \vdots \end{bmatrix} \theta. \tag{23}$$

The use of the approach is demonstrated in the following Section.

5.1. Identification example. The continuous time state-space system (12), (13) is given by the following matrices:

$$E = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \qquad F = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$G = \begin{bmatrix} 2 & 3 \end{bmatrix}, \qquad H = \begin{bmatrix} 0 \end{bmatrix}, \qquad \alpha = 0.5$$

The transfer function of the system has the form:

$$G(s) = \frac{3s^{0.5} + 2}{s^1 + 3s^{0.5} + 2} = \frac{-1}{s^{0.5} + 1} + \frac{4}{s^{0.5} + 2}.$$

The step response of the system is

$$h(t) = -t^{0.5} \mathrm{E}_{0.5,1}(-t^{0.5}) + 4t^{0.5} \mathrm{E}_{0.5,1}(-t^{0.5}),$$

where $\mathrm{E}_{\alpha,\beta}$ is a Mittag-Leffler function.

In parametric identification of this system the following discrete transfer function form is assumed:

$$G(z) = \frac{b_1 z\Delta^{0.5}(z) + b_0}{z^2 \Delta^1(z) + a_1 z\Delta^{0.5}(z) + a_0} \tag{24}$$

which is rewritten in the form

$$\varphi_k = [\ -\Delta^{0.5} y_{k+1}, \quad -y_k, \quad \Delta^{0.5} u_{k+1}, \quad u_k\ ] \tag{25}$$

$$\theta^T = \begin{bmatrix} a_1 & a_0 & b_1 & b_0 \end{bmatrix} \quad (26)$$

$$Y_k = \begin{bmatrix} \Delta^1 y_k \end{bmatrix} \quad (27)$$

Solving the equation (23) we obtain the following parameters:

$$a_1 = -0.5867, \qquad a_0 = -0.1705,$$

$$b_1 = 0.6181, \qquad b_0 = 0.1677.$$

This gives the following discrete time model

$$A = \begin{bmatrix} 0 & 1 \\ -0.1705 & -0.5867 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \alpha = 0.5$$

$$C = \begin{bmatrix} 0.1677 & 0.6181 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \end{bmatrix}.$$

The comparison of the step response of continuous and identified discrete time model is presented in Fig. 3. As it can be seen the accuracy of the identification is very high, the responses are nearly the same.

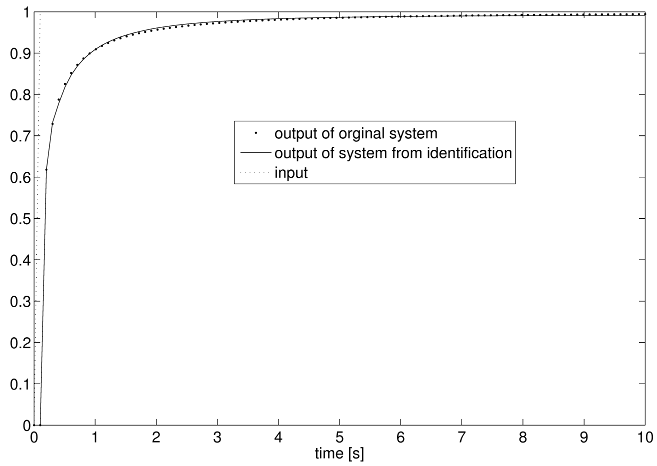

Fig. 3. Step response of continuous system for $\alpha = 0.5$ and indentificated model

6. Mathematical description of heating process

The heating process of a semi-infinite beam can be described by the following partial differential equation [18]

$$\frac{\partial^2}{\partial \lambda^2} T(t,\lambda) = a^2 \frac{\partial}{\partial t} T(t,\lambda), \quad (28)$$

with the following boundary conditions:

$$T(0,\lambda) = 0, \qquad T(t,0) = u(t),$$

where $T(t,\lambda)$ is a temperature of the beam at time instant t and coordinate λ, and a is a parameter which depends on beam parameters like heat conductibility and density.

Let us assume the following equation where the boundary conditions are the same as in (28):

$$\frac{\partial}{\partial \lambda} T(t,\lambda) = a \frac{\partial^{0.5}}{\partial t^{0.5}} T(t,\lambda). \quad (29)$$

By applying the derivative $\frac{\partial}{\partial \lambda}$ to both sides of the equation the following relation is obtained:

$$\frac{\partial^2}{\partial \lambda^2} T(t,\lambda) = a \frac{\partial^{0.5}}{\partial t^{0.5}} \frac{\partial}{\partial \lambda} T(t,\lambda). \quad (30)$$

Using again Eq. (29) we achieve

$$\frac{\partial^2}{\partial \lambda^2} T(t,\lambda) = a^2 \frac{\partial^{0.5}}{\partial t^{0.5}} \frac{\partial^{0.5}}{\partial t^{0.5}} T(t,\lambda). \quad (31)$$

This finally gives the traditional heat transfer partial differential Eq. (28).

Using the following notation

$$H(t,\lambda) = \frac{\partial}{\partial \lambda} T(t,\lambda), \quad (32)$$

where $H(t,\lambda)$ is the heat flux at time t and length coordinate λ, the following equation is obtained:

$$H(t,\lambda) = a \frac{\partial^{0.5}}{\partial t^{0.5}} T(t,\lambda). \quad (33)$$

Hence by first order differentiation with respect to λ of both side of previous equation and using Eq. (32) the following fractional order partial differential equation, describing heat flux transfer is achieved:

$$\frac{\partial}{\partial \lambda} H(t,\lambda) = a \frac{\partial^{0.5}}{\partial t^{0.5}} H(t,\lambda). \quad (34)$$

Applying the Laplace transformation with respect to t to this equation we obtain

$$\frac{\partial}{\partial \lambda} H(s,\lambda) = a s^{0.5} H(s,\lambda) -_0 \mathrm{D}_t^{-0.5} H(0,\lambda). \quad (35)$$

The solution of this equation (for $H(0,\lambda) = 0$) is given as follows

$$H(s,\lambda) = e^{a s^{0.5} \lambda} H(s,0), \quad (36)$$

from where the following relation describing the heat transfer with respect to the heat flux is obtained

$$T(s,\lambda) = \frac{1}{a s^{0.5}} e^{a s^{0.5} \lambda} H(s,0). \quad (37)$$

7. Practical example of ultracapacitor frequency domain modelling

Ultracapacitors are large capacity and power density electrical energy storage devices. This large capacity is due to a very complicated internal structure, which has effect in its dynamics. Many authors use the different RC models to perform modelling of ultracapacitor which are accurate only for a limited range of frequencies. A more effective approach is based on using the fractional order model which gives highly accurate results of modelling over a wide range of frequencies.

7.1. Ultracapacitor modelling. To model ultracapacitor the experimental setup (Fig. 4) containing the electronic circuit with ultracapacitor connected to the DS1104 Control Card was built.

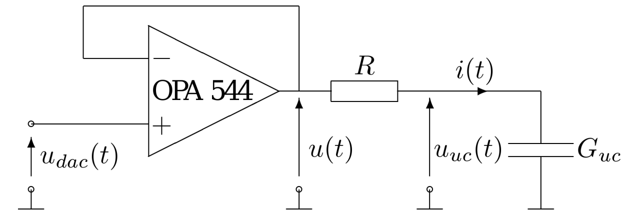

Fig. 4. Electronic circuit used in experiments with ultracapacitors

It is composed of the operational amplifier OPA544, resistor 180Ω and ultracapacitors produced by Panasonic® of nominal capacity 0.047F, 0.1F, 0.33F. OPA 544 is a high current operational amplifier and works in the voltage follower configuration. This circuit is used for both continuous and discrete modelling.

The identification was based on Bode diagrams matching. Bode diagram of the model was tuned to the diagram of the ultracapacitor achieved from measurements. As a result of this research, the authors obtained parameters of the model.

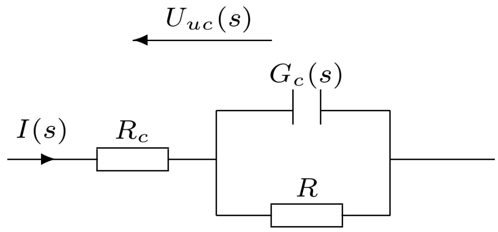

Fig. 5. Ultracapacitor equivalent model i.e. the G_{uc} element of Fig. 4

Spectral transfer function of the modelled system is defined as:

$$G_{uc}(j\omega) = \frac{U_{uc}(j\omega)}{I(j\omega)},$$

where $U_{uc}(j\omega)$ is a spectral transform of capacitor voltage and $I(j\omega)$ is a spectral transform of the capacitor current.

The Bode diagram was obtained from the following relations:

$$M(\omega) = 20\log\left(\frac{A_c(\omega)}{A_i(\omega)}\right), \qquad \varphi(\omega) = \varphi_i(\omega) - \varphi_u(\omega).$$

As a theoretical model of capacity of the ultracapcitor the following transfer function was used:

$$G_{uc}(s) = \frac{(Ts+1)^{\alpha}}{Cs} \tag{38}$$

and the whole transfer function of the ultracapacitor presented in Fig. 5 is:

$$G_{uc}(s) = \frac{U_{uc}(s)}{I(s)} = R_c + \frac{(Ts+1)^{\alpha}}{\frac{1}{R}(Ts+1)^{\alpha} + Cs}, \tag{39}$$

where R is self-discharge resistance and R_c is resistance of ultracapacitor.

The parameters achieved in the identification by diagrams matching are presented in Table 1.

Table 1

capacitor	R_c	T	C	R	α
0.047F	32Ω	5.1138	0.05	$0.11 M\Omega$	0.6
0.1F	38Ω	13.6628	0.1	$0.14 M\Omega$	0.6
0.33F	27Ω	52.7674	0.27	$0.47 M\Omega$	0.6

These parameters can be compared with physical parameters measured directly from ultracapacitors by means of step response of the circuit. The results of the comparison are shown in Table 2.

Table 2

capacitor	R_c	C	R
0.047F	32Ω	0.06	$0.06 M\Omega$
0.1F	42Ω	0.1	$0.15 M\Omega$
0.33F	28Ω	0.27	$0.24 M\Omega$

As may be seen, the parameters of the fractional order model are very close to the measured values.

The comparison of the measured data and theoretical Bode diagrams is presented in Figs. 6, 7 and 8.

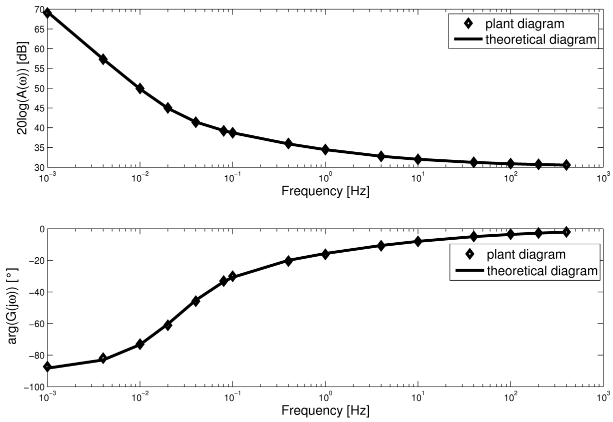

Fig. 6. Measured and theoretical Bode diagrams of ultracapacitor 0.047F

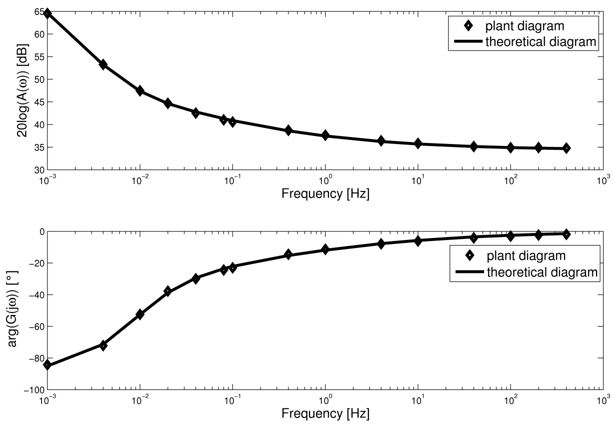

Fig. 7. Measured and theoretical Bode diagrams of ultracapacitor 0.1F

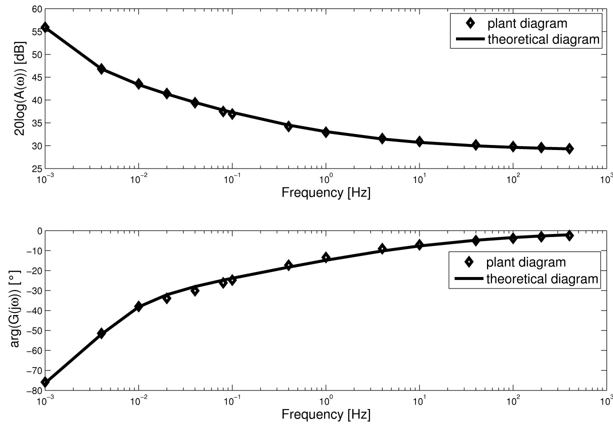

Fig. 8. Measured and theoretical Bode diagrams of ultracapacitor 0.33F

7.2. Meaning of T parameter. Knowledge about T parameter explains a very important phenomenon of an ultracapacitor. The meaning of this parameter is presented below.

The spectral transfer function of the system given by Equation (38) is given as follows:

$$G_c(j\omega) = \frac{(Tj\omega + 1)^\alpha}{Cj\omega}.$$

The magnitude of this spectral transfer function is

$$A_c(\omega) = \frac{((T\omega)^2 + 1)^{\alpha/2}}{C\omega}$$

this magnitude can be compared with magnitude of the traditional capacitor

$$\frac{((T\omega)^2 + 1)^{\alpha/2}}{C\omega} = \frac{1}{C'\omega} \tag{40}$$

which yields

$$C' = \frac{C}{((T\omega)^2 + 1)^{\alpha/2}}, \tag{41}$$

where C' is the capacity equivalent of the ultracapacitor for given frequency ω. This equivalent capacity illustrates what capacity the traditional capacitor should have in order to have the same magnitude for a desired value of frequency.

For any α we have

$$C' = \begin{cases} C & \text{for} \quad \omega \ll \dfrac{1}{T} \\ \dfrac{C}{(T\omega)^\alpha} & \text{for} \quad \omega \gg \dfrac{1}{T} \end{cases} \tag{42}$$

The frequency f_c for which the capacity equivalent decreases $2^{\alpha/2}$ times is given as follows

$$f_c = \frac{1}{2\pi T} \tag{43}$$

and values of them, for tested ultracapacitors, are summarized in Table 3.

Table 3

capacitor	T	$f_c[Hz]$
0.047F	5.1138	0.0311
0.1F	13.6628	0.0117
0.33F	52.7674	0.0030

Figures 9, 10 and 11 present values of the equivalent capacity as the function of frequency for the capacitors presented above. As it can be seen, the values of the capacity equivalent highly decrease when frequency of the sinusoidal signal increases. This can be a very important feature for engineers in the design process of systems that use ultracapacitors.

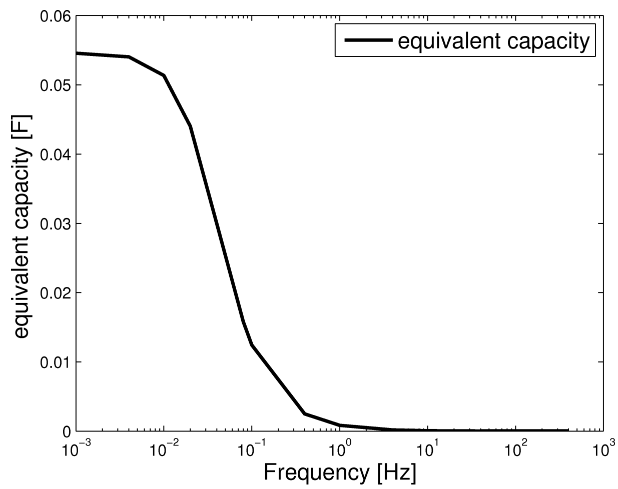

Fig. 9. Equivalent capacity C' of ultracapacitor 0.047F

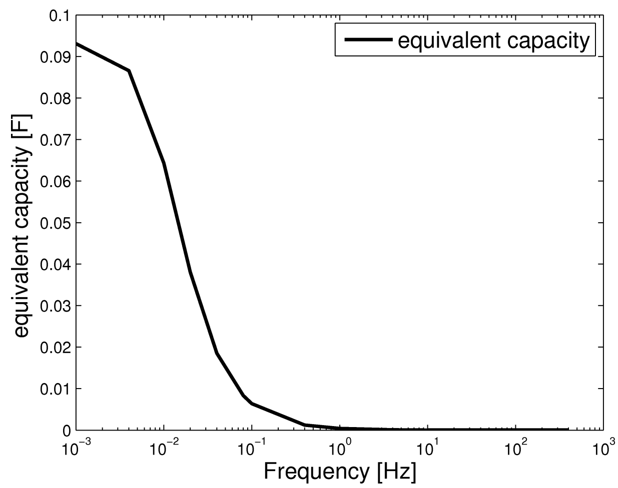

Fig. 10. Equivalent capacity C' of ultracapacitor 0.1F

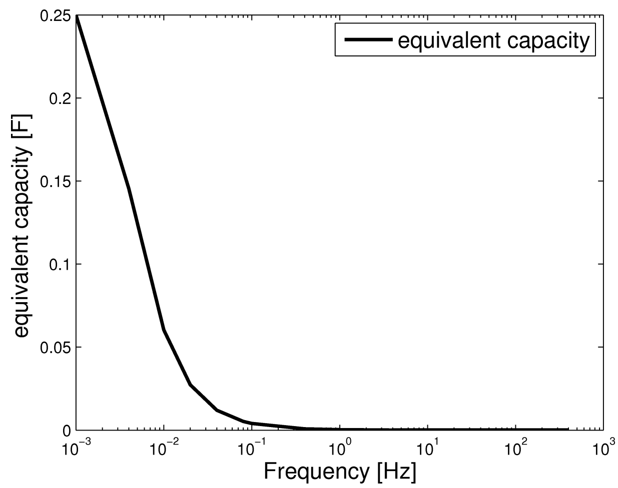

Fig. 11. Equivalent capacity C' of ultracapacitor 0.33F

7.3. Ultracapacitor discrete time identification. In parametric identification of this data the following discrete transfer function form is assumed as follows:

$$G(z) = \frac{b_2 z^2 \Delta^1(z) + b_1 z \Delta^{0.5}(z) + b_0}{z^2 \Delta^1(z) + a_1 z \Delta^{0.5}(z) + a_0} \tag{44}$$

which is rewritten in the form

$$\varphi_k = [\ -\Delta^{0.5} y_{k+1}, -y_k, \quad \Delta^1 u_k \Delta^{0.5} u_{k+1}, u_k\ ] \tag{45}$$

$$\theta^T = \begin{bmatrix} a_1 & a_0 & b_2 & b_1 & b_0 \end{bmatrix} \tag{46}$$

$$Y_k = \begin{bmatrix} \Delta^1 y_k \end{bmatrix} \tag{47}$$

By solving the equation (23) we have obtained the following discrete time model

$$A = \begin{bmatrix} 0 & 1 \\ -0.006333 & -0.037401 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.025055 & 0.004997 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.227795 \end{bmatrix}, \qquad \alpha = 0.5$$

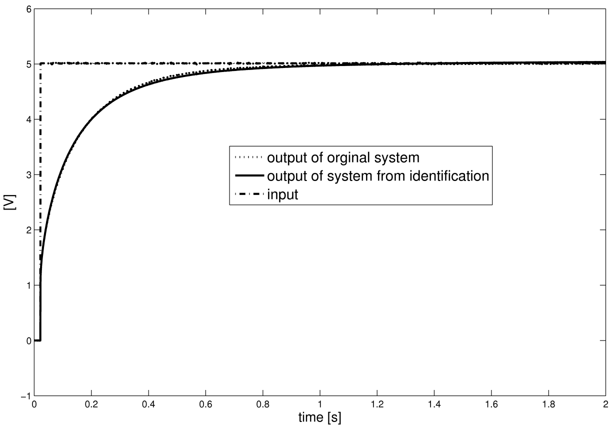

Fig. 12. Step response of the system with known and identified parameters

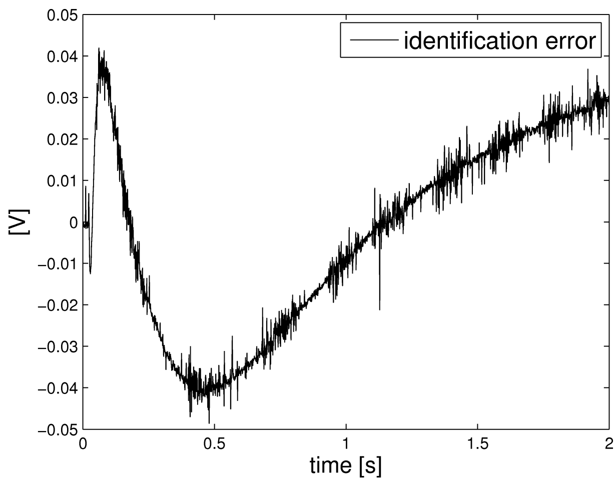

Fig. 13. Error between the step responses of the system with known and identified parameters

The comparison of the measured step response and identified the discrete time model step response is presented in Figs. 12 and 13. As it could be seen the accuracy of the identification is very high, the responses are nearly the same.

8. Experimental verification of heating process model

The heat distribution process modelling by fractional order PDEs and their respective counterparts in frequency domain has been verified by the experiments with real physical thermal system. The results obtained from the model proposed have been compared with those obtained from the experiment.

8.1. Experimental setup. The experimental setup contains:

1. dSPACE DS1103 PPC card with a PC
2. Electronic interface with OPA 549 power amplifier
3. thermoelectric (Peltier) module SCTB NORD TM-127-1.0-3.9-MS
4. 6 temperature sensors LM35DH

and its idea is presented in Fig. 14. The placement of the temperature sensors is depicted in Fig. 15.

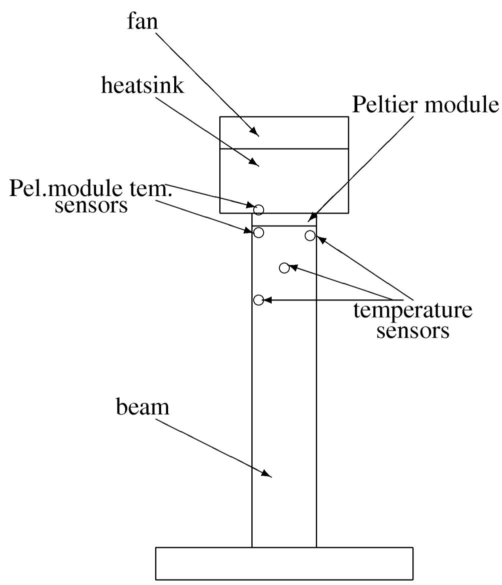

Fig. 14. The experimental setup

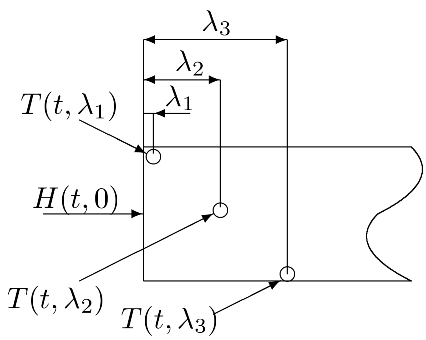

Fig. 15. Sensors placement on the beam

8.2. Modelling results. The transfer function based on Eq. (37) is given as follows (the additional parameters are used for modelling unknowns relations eg. current-heat flux):

$$G(\lambda, s) = \frac{T(s,\lambda)}{H(s,0)} = \frac{T_1}{s^{0.5}} e^{\lambda (T_2 s)^{0.5}}. \qquad (48)$$

For desired values of $\lambda = \{\lambda_1', \lambda_2', \lambda_3'\}$ corresponding to three sensors mounted on the beam, the set of transfer functions is obtained. Values of λ_i' are regularized for $\lambda_1' = 1$.

The transfer function (48) was derived with the assumption that the heat is not emitted to the outside of the beam. This does not happen in a real plant, so we have to adjust the transfer functions by replacing the fractional (0.5) order integrator by the fractional (0.5) order inertia unit. In such a case the transfer function has the following form:

$$G(\lambda, s) = \frac{T(s,\lambda)}{H(s,0)} = \frac{T_1}{(T_3 s)^{0.5} + 1} e^{\lambda (T_2 s)^{0.5}}. \qquad (49)$$

For $\lambda = \lambda_1$ the following parameters of the transfer function were achieved using Bode diagram matching:

$$T_1 = 1.758, \qquad T_2 = 12.875, \qquad T_3 = 88.799.$$

For other values of λ the same parameters were used. The λ values used are: $\lambda_1' = 1$, $\lambda_2' = 2.6$, $\lambda_3' = 4.55$. The results of modeling are presented in Fig. 16. It may be observed that the measured data for real thermal plant match quite accurately the values obtained from the model in the range of frequencies from $f = 10^{-3}$ Hz to $f = 0.25 \cdot 10^{-1}$ Hz. This is especially the case for $\lambda = \lambda_1$ (see top plots of magnitude and angle in Fig. 16). However, for other values of λ the accuracy of modelling is also very good (see middle and bottom magnitude and angle plots in Fig. 16).

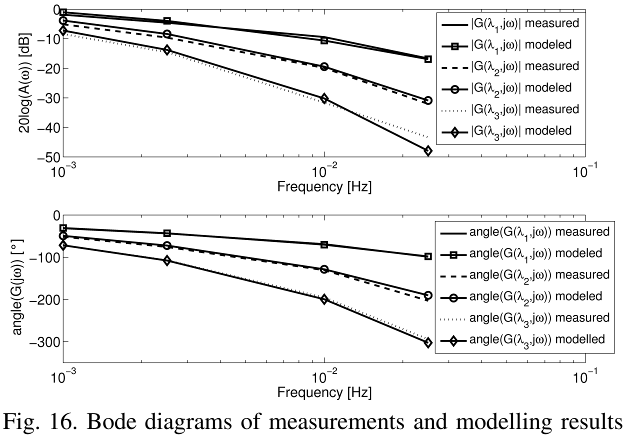

Fig. 16. Bode diagrams of measurements and modelling results

9. Conclusions

The paper presents some practical applications of fractional order calculus in the area of modelling physical systems. Firstly, a brief description of major theoretical developments in the area of modelling of physical systems with the dicrete-time fractional order models is given. After the short introduction to fractional order calculus the fractional order state-space system is recalled in this context and the identification problem solution is proposed. The major part of the paper describes the results of implementation of fractional order models to physical systems. The modelling of ultracapitors by the fractional order transfer function turned out to be much more accurate than the previously used methods of RC lines. Also the beam heating problem has been much more precisely described when the fractional order models had been used.

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