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Existence – uniqueness result for a certain equation of motion in fractional mechanics

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Abstract. The eigenfunction equation of fractional variational operator including left and right derivative of order α is solved using the fixed point theorem. Its exact and approximate solutions are studied in detail. The corresponding boundary conditions are derived by means of the composition rules of fractional operators and the theorem on a unique particular solution of the considered fractional differential equation is proved.

Key words: fractional derivative, fractional differential equation, Euler-Lagrange equation, fixed point theorem, approximate solution.

1. Introduction

In the paper we solve and study properties of the solutions of a certain fractional differential equations – one of Euler-Lagrange equations of fractional mechanics.

Fractional differential equations both ordinary and partial ones are applied in mathematical modeling of processes in physics, mechanics, control theory, biochemistry, bioengineering and economics [1–7]. Therefore the theory of fractional differential equations is an area intensively developed during last decades. The monographs [8–11] enclose a review of methods of solving which are an extension of procedures from differential equations theory. Recently, also equations including both – left and right fractional derivatives, are discussed. The existence-uniqueness results obtained in the present paper concern an eigenfunction equation of the composition of left and right derivative. Although rather basic, this equation is interesting from the point of view of further applications to general sequential linear equations of fractional order.

Let us point out that according to integration by parts formulas in fractional calculus [10, 12], we obtain equations mixing left and right operators whenever we apply the minimum action principle to fractional model. This approach was started in 1996 by Riewe [13, 14], developed by Agrawal and Klimek and investigated ever since [15–23]. The results by Tarasov [24, 25], Jumarie [26] and Klimek [27] show that the phenomenon of mixing derivatives can be prevented either by change of the differential geometry related to the space or by change of the underlying algebra of functions.

Apart from their possible applications, equations with left and right derivatives are an interesting and new field in fractional differential equations theory. Preliminary results can be found in papers [28–34]. Here we study an application of the fixed point theorem to the eigenfunction equation of fractional variational operator.

The paper is organized as follows. In Sec. 2 we recall definitions and some of the properties of fractional operators. This section is closed with a derivation of Euler-Lagrange equation for simple model of fractional mechanics. Main results of the paper are enclosed in Sec. 3, where we first obtain a general solution of eigenfunction equation of operator ${}^cD^{\alpha}_{b-}D^{\alpha}_{a+}$ using the fixed point theorem. Then in Subsec. 3.3 we study boundary conditions determining the unique particular solution for arbitrary fractional order $\alpha \in (n-1, n)$. Subsection 3.4 contains a study of analytic approximate solutions. We also discuss the dependence of the error of approximation on modulus of the eigenvalue and order α. The paper is closed with a discussion concerning application of the obtained eigenfunctions in the procedure of solving sequential linear fractional differential equations dependent on operator ${}^cD^{\alpha}_{b-}D^{\alpha}_{a+}$.

2. Fractional operators

We recall some basic definitions of fractional operators and their properties relevant to our further study [10].

Fractional Riemann-Liouville integrals of order α in finite interval $[a, b]$ are defined as follows for $\alpha > 0$:

$$I^{\alpha}_{a+}f(t) := \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(s)ds}{(t-s)^{1-\alpha}} \qquad t > a, \tag{1}$$

$$I^{\alpha}_{b-}f(t) := \frac{1}{\Gamma(\alpha)}\int_t^b \frac{f(s)ds}{(s-t)^{1-\alpha}} \qquad t < b, \tag{2}$$

where Γ is the Euler gamma function.

The fractional derivatives are defined using the notion of fractional integrals. We include the definition of left Riemann-Liouville and right Caputo derivatives.

Definition 2.1. (1) The left Riemann-Liouville derivative of order α with $\alpha \in (n-1, n)$ looks as follows $\Big($we have denoted the classical derivative as $D := \dfrac{d}{dt}\Big)$:

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$$D^{\alpha}_{a+} f(t) := D^n I^{n-\alpha}_{a+} f(t) \quad t > a. \tag{3}$$

(2) The right Caputo derivative of order $\alpha > 0$ is defined as follows:

$$\begin{aligned} &{}^{c}D^{\alpha}_{b-} f(t) := \\ &= D^{\alpha}_{b-} \left[f(t) - \sum_{k=0}^{n-1} \frac{D^k f(b)}{k!} \cdot (b-t)^k \right], \end{aligned} \tag{4}$$

where $n - 1 = [\alpha]$.

The fractional derivatives and integrals obey the composition rules. They are analogues of the fundamental theorem of classical integral calculus. We shall apply them to transform fractional differential equations into integral ones and to derive the corresponding boundary conditions.

Property 2.2. (1) Let $\alpha > 0$ and function $f \in L_p(a,b)$ with $p \in [1, \infty]$. Then the following equality

$$D^{\alpha}_{a+} I^{\alpha}_{a+} f(t) = f(t) \tag{5}$$

holds almost everywhere on $[a,b]$. When $f \in C[a,b]$, the equation is valid at any point $t \in [a,b]$.

For $f \in C_{\gamma}[a,b]$, it is fulfilled for $t \in (a,b]$.

(2) Let $\alpha > 0$ and $f \in L_{\infty}(a,b)$ or $f \in C[a,b]$. Then:

$${}^{c}D^{\alpha}_{b-} I^{\alpha}_{b-} f(t) = f(t). \tag{6}$$

In further considerations, we also apply the composition rule of type (6) for function $(t-a)^{\alpha-n}$ which belongs to the $C_{n-\alpha}[a,b]$ space. Let us check that it is fulfilled in interval $(a,b]$. Using definition (4) we obtain the formula:

$$\begin{aligned} &{}^{c}D^{\alpha}_{b-} I^{\alpha}_{b-} (t-a)^{\alpha-n} = D^{\alpha}_{b-} I^{\alpha}_{b-} (t-a)^{\alpha-n} - \\ &- \sum_{k=0}^{n-1} \frac{D^k I^{\alpha}_{b-} (t-a)^{\alpha-n} \mid_{t=b}}{k!} \cdot (b-t)^k. \end{aligned} \tag{7}$$

We calculate the elements of the sum. For $t \in (a,b]$ and $k = 0, \ldots, n-1$ we have

$$\begin{aligned} &D^k I^{\alpha}_{b-} (t-a)^{\alpha-n} = I^{\alpha-k}_{b-} (t-a)^{\alpha-n} = \\ &= \Gamma(\{\alpha\})(b-t)^{\alpha-k}(b-a)^{\alpha-n} \times \\ &\times {}_1\Psi_2 \left[\begin{matrix} (1,1) & & \\ (\{\alpha\}, -1) & (\alpha-k+1, 1) & \end{matrix} \mid -\frac{b-t}{b-a} \right]. \end{aligned}$$

The function on the right-hand side of the above formula is one of the Fox-Wright functions (compare the definition and properties in the monograph [10]). Clearly, at the end $t = b$, such functions vanish for $k = 0, \ldots, n-1$ i.e.

$$D^k I^{\alpha}_{b-} (t-a)^{\alpha-n} \mid_{t=b} = 0.$$

Thus, by virtue of Property 2.2 and the above derived equalities the following composition formula

$${}^{c}D^{\alpha}_{b-} I^{\alpha}_{b-} (t-a)^{\alpha-n} = (t-a)^{\alpha-n} \tag{8}$$

is valid in time interval $(a,b]$.

2.1. Euler-Lagrange equation for a simple fractional model. Let us recall a simple model in fractional mechanics, namely we shall derive the Euler-Lagrange equation for action S given as follows

$$S = \int_a^b \frac{1}{2} (D^{\alpha}_{a+} f \cdot D^{\alpha}_{a+} f - \lambda f \cdot f)\, dt. \tag{9}$$

According to results [13, 14, 17, 30], the corresponding Euler-Lagrange equation looks as follows

$$\frac{\partial L}{\partial f} + {}^{c}D^{\alpha}_{b-} \frac{\partial L}{\partial (D^{\alpha}_{a+} f)} = 0 \tag{10}$$

provided the boundary terms vanish:

$$\sum_{k=1}^{n} (-1)^k D^{k-1} D^{\alpha}_{a+} \frac{\partial L}{\partial (D^{\alpha}_{a+} f)} \cdot D^{n-k} I^{n-\alpha}_{a+} \eta(t) \mid^{b}_{t=a} = 0. \tag{11}$$

The above boundary conditions yield the following restrictions on variation η:

$$D^{n-k} I^{n-\alpha}_{a+} \eta(t) \mid_{t=a} = 0 \quad k = 1, \ldots, n, \tag{12}$$

$$D^{n-k} I^{n-\alpha}_{a+} \eta(t) \mid_{t=b} = 0 \quad k = 1, \ldots, n-1 \tag{13}$$

and for function f we get respectively:

$$D^{n-1} D^{\alpha}_{a+} \frac{\partial L}{\partial (D^{\alpha}_{a+} f)} \mid_{t=b} = 0 \tag{14}$$

$$D^{k-1} D^{\alpha}_{a+} \frac{\partial L}{\partial (D^{\alpha}_{a+} f)} \mid_{t=b} < \infty \quad k = 1, \ldots, n-1 \tag{15}$$

$$D^{k-1} D^{\alpha}_{a+} \frac{\partial L}{\partial (D^{\alpha}_{a+} f)} \mid_{t=a} < \infty \quad k = 1, \ldots, n. \tag{16}$$

In case $\alpha \longrightarrow 1^+$ we recover classical derivatives:

$$D^1_{a+} g(t) = Dg(t) \quad {}^{c}D^1_{b-} g(t) = -Dg(t) - Dg(b)$$

and Euler-Lagrange equation in a standard form:

$$\frac{\partial L}{\partial f} - D \frac{\partial L}{\partial (Df)} = 0$$

when boundary condition $D \frac{\partial L}{\partial (Df)} \mid_{t=b} = 0$ is fulfilled.

Equation (10) can be rewritten for our model in the form of

$$({}^{c}D^{\alpha}_{b-} D^{\alpha}_{a+} - \lambda) f(t) = 0 \qquad t \in [a,b]. \tag{17}$$

We also observe that the above equation for order $\alpha \longrightarrow 1^+$ becomes the harmonic oscillator equation

$$-(D^2 + \lambda) f(t) = 0,$$

provided boundary condition $D^2 f(b) = 0$ is obeyed, which means the final acceleration is equal to 0. Our aim is to solve the derived Euler-Lagrange equation for arbitrary real order $\alpha \in (n-1, n)$ and study the properties of its solutions. Such an equation is in fact an eigenfunction equation for operator ${}^{c}D^{\alpha}_{b-} D^{\alpha}_{a+}$ and eigenvalue λ. Thus, we shall denote its solution as $\mathcal{F}_{\lambda}$ and refer to them as eigenfunctions. We shall also call Eq. (17) an eigenfunction equation.

3. Eigenfunction equation of operator ${}^cD^\alpha_{b-}D^\alpha_{a+}$

We shall study an eigenfunction equation in finite time interval

$$({}^cD^\alpha_{b-}D^\alpha_{a+} - \lambda)\mathcal{F}_\lambda(t) = 0 \qquad t \in [a,b], \tag{18}$$

where $\lambda \in C$ is an arbitrary complex eigenvalue of operator ${}^cD^\alpha_{b-}D^\alpha_{a+}$ for real order $\alpha \in (n-1,n)$. Let us recall that the above equation is Euler-Lagrange equation for action (9) when the condition

$$D^{n-1}D^\alpha_{a+}\mathcal{F}_\lambda(t)\mid_{t=b}= 0 \tag{19}$$

is fulfilled. We transform the above fractional differential equation into its integral form. The first step is the application of the composition rules given in Property 2.2 and formula (8):

$${}^cD^\alpha_{b-}D^\alpha_{a+}(1 - \lambda I^\alpha_{a+}I^\alpha_{b-})\mathcal{F}_\lambda(t) = 0. \tag{20}$$

We omit the fractional differential operator using the notion of a stationary function for composition ${}^cD^\alpha_{b-}D^\alpha_{a+}$. The stationary function f^{st}_α fulfills the equation in the form of

$${}^cD^\alpha_{b-}D^\alpha_{a+}f^{st}_\alpha(t) = 0 \tag{21}$$

and is expressed as the following linear combination of power functions:

$$f^{st}_\alpha(t) = \sum_{k=-n}^{n-1} A_k(t-a)^{\alpha+k} \tag{22}$$

with A_k being arbitrary constant coefficients.

For order $\alpha \longrightarrow n^- \in N$, the stationary function becomes an arbitrary polynomial function of degree $2n-1$, which is the stationary function of derivative D^{2n}:

$$f^{st}_n(t) = \sum_{k=-n}^{n-1} A_k(t-a)^{n+k}. \tag{23}$$

When order $\alpha \longrightarrow (n-1)^+ \in N$, stationary function f^{st}_{n-1} is an arbitrary polynomial function of degree $2n-3$ - the stationary function of derivative D^{2n-2}:

$$f^{st}_{n-1}(t) = \sum_{k=-n+1}^{n-2} A_k(t-a)^{n-1+k}. \tag{24}$$

Let us note that when order α is a non-integer number, the above stationary function can be split into two parts: a continuous and a singular one

$$f^{st}_\alpha(t) = \bar{f}^{st}_\alpha(t) + A_{-n}(t-a)^{\alpha-n}, \tag{25}$$

where the continuous part is of the following form:

$$\bar{f}^{st}_\alpha(t) = \sum_{k=-n+1}^{n-1} A_k(t-a)^{\alpha+k}. \tag{26}$$

Using the notion of a stationary function we can now reformulate fractional differential Eq. (18) and write it as the following equivalent in the $C_{n-\alpha}$ space, integral fractional equation:

$$(1 - \lambda I^\alpha_{a+}I^\alpha_{b-})\mathcal{F}_\lambda(t) = f^{st}_\alpha(t) \qquad t \in [a,b]. \tag{27}$$

In what follows, we shall consider two cases separately: of continuous and of singular stationary functions. We shall derive the general solutions of Eq. (27) and therefore also of Eq. (18), transforming them into fixed point conditions on the $C[a,b]$ and $C_{n-\alpha}[a,b]$ spaces respectively. The application of the Banach theorem on a fixed point means the construction of contractive mapping. The proposed approach - the separate treatment of solving procedures, is connected to different conditions assuring the contraction properties of corresponding operators in spaces of functions continuous or singular and belonging to the $C_{n-\alpha}[a,b]$ space.

3.1. Continuous solutions of eigenfunction equation. In the present section, we shall discuss the application of the Banach theorem to solve Eq. (27) in the case when the stationary function is continuous in time interval $[a,b]$. Then this fractional integral equation becomes

$$(1 - \lambda I^\alpha_{a+}I^\alpha_{b-})\mathcal{F}_\lambda(t) = \bar{f}^{st}_\alpha(t). \tag{28}$$

In the space of continuous solutions, the above equation is equivalent to initial fractional differential Eq. (18) due to Property 2.2. Let us define the following integral operator acting on the space of functions continuous in time interval $[a,b]$ as mapping T:

$$Tg(t) := \lambda I^\alpha_{a+}I^\alpha_{b-}g(t) + \bar{f}^{st}_\alpha(t). \tag{29}$$

Equation (27) and therefore also initial Eq. (18), can be rewritten as a fixed point condition for mapping $T : C[a,b] \longrightarrow C[a,b]$

$$\mathcal{F}_\lambda(t) = T\mathcal{F}_\lambda(t). \tag{30}$$

Hence, each continuous solution of the considered equation is given as a fixed point of mapping T generated by continuous stationary function $\bar{f}^{st}_\alpha$. The set of solutions is described in the following proposition.

Proposition 3.1. Let $\alpha > 0$ and let $\bar{f}^{st}_\alpha$ be the arbitrary continuous stationary function of form (26) or (23,24) respectively. If inequality

$$\mid \lambda \mid \cdot \mid\mid I^\alpha_{a+}I^\alpha_{b-}1 \mid\mid < 1 \tag{31}$$

is fulfilled, then unique continuous solution $\mathcal{F}_\lambda$ of equation

$$({}^cD^\alpha_{b-}D^\alpha_{a+} - \lambda)\mathcal{F}_\lambda(t) = 0 \qquad t \in [a,b]$$

generated by stationary function $\bar{f}^{st}_\alpha$ exists and is given as the following series:

$$\mathcal{F}_\lambda(t) = \sum_{m=0}^{\infty} [\lambda I^\alpha_{a+}I^\alpha_{b-}]^m \bar{f}^{st}_\alpha(t) \qquad t \in [a,b]. \tag{32}$$

Proof. Let us recall the notion of supremum norm on the space of continuous functions $C[a,b]$:

$$|| f ||:= \max_{t\in[a,b]} | f(t) | . \tag{33}$$

Using this norm, we define the metric as follows

$$d(f,g) :=|| f - g || \tag{34}$$

for any pair of continuous functions $f, g \in C[a,b]$ and this function space with metric d, is a metric and complete space.

Let us note that mapping T given by (29) transforms the continuous function into a continuous function for arbitrary order $\alpha > 0$. We check when T is contractive in the space of continuous functions. Using the properties of integrals, we obtain the following inequalities for a pair of arbitrary functions $h, g \in C[a,b]$

$$|| T(h) - T(g) ||=|| \lambda I^{\alpha}_{a+} I^{\alpha}_{b-} (h-g) ||\leq \tag{35}$$

$$\leq| \lambda | \cdot || I^{\alpha}_{a+} I^{\alpha}_{b-} 1 || \cdot || h-g ||$$

and we observe that T is a contractive mapping when the following inequality holds

$$| \lambda | \cdot || I^{\alpha}_{a+} I^{\alpha}_{b-} 1 ||< 1. \tag{36}$$

Hence, from the Banach theorem on a fixed point, it follows that a unique solution in space $C[a,b]$, generated by stationary function $\bar{f}^{st}_{\alpha}$ exists. It fulfills equations (18, 27, 28) and is given as a limit of the iterations of mapping T:

$$\mathcal{F}_{\lambda}(t) = \lim_{m\longrightarrow\infty} T^m \psi(t), \tag{37}$$

where $t \in [a,b]$ and function ψ is an arbitrary function continuous in interval $[a,b]$. Formula (32) describing the solution as a series is a corollary of the above form of solution when $\psi \equiv 0$ and this ends the proof.

Let us now study condition (31). It clearly restricts the values of order α, λ and of the length of the time interval. If we assume $b-a=1$, then we arrive at the following form of condition (31):

$$| \lambda | \cdot\phi(\alpha) < 1,$$

where ϕ depends on order α and is given as the following series:

$$\phi(\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\alpha+1-k)\Gamma(\alpha+1+k)}.$$

We shall visualize the admissible values of $| \lambda |$ for the respective ranges of order α. The surfaces solving inequality (31) for $b-a=1$ are depicted in Figs. 1–3.

Analyzing the plots enclosed in these figures, we conclude that for a fixed length of time interval, the range of admissible eigenvalues increases when order α does.

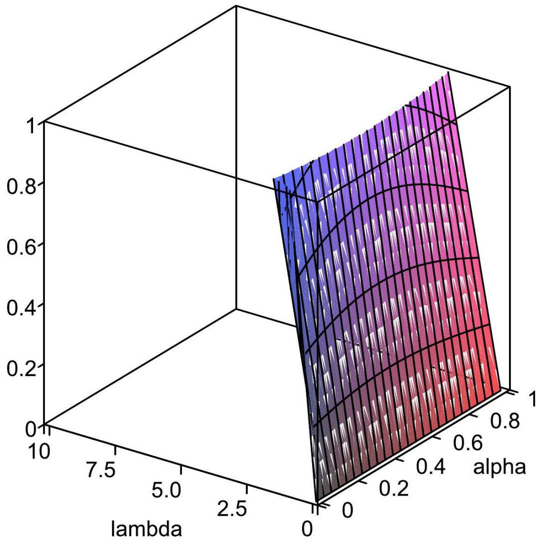

Fig. 1. Surface $| \lambda | \cdot\phi(\alpha) < 1$ for $\alpha \in= (0,1)$

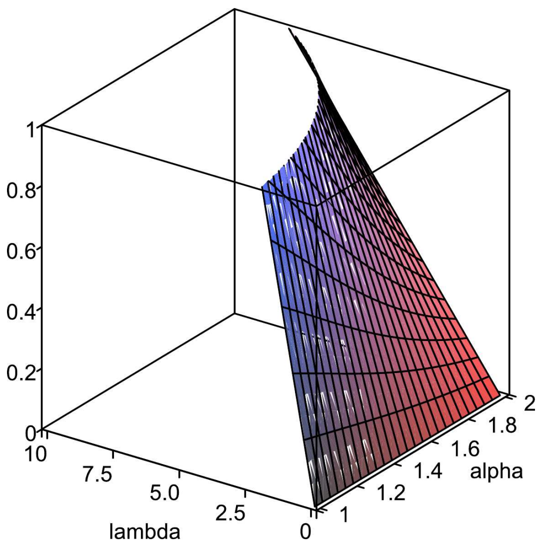

Fig. 2. Surface $| \lambda | \cdot\phi(\alpha) < 1$ for $\alpha \in= (1,2)$

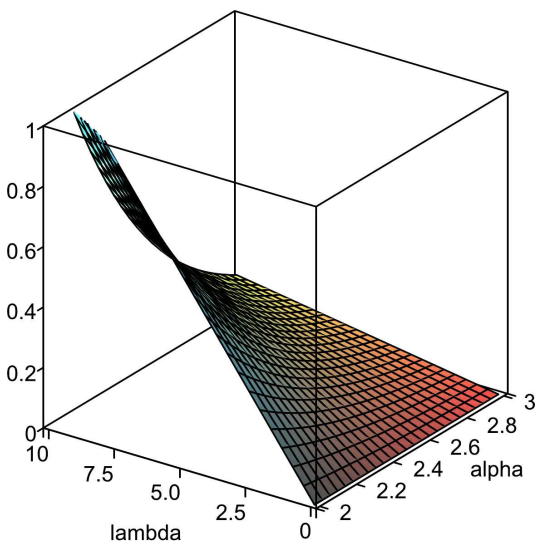

Fig. 3. Surface $| \lambda | \cdot\phi(\alpha) < 1$ for $\alpha \in= (2,3)$

3.2. Singular solutions of eigenfunction equation. In this section, we shall study solutions generated by a singular part of a stationary function. For given $\alpha \in (n-1, n)$, it belongs to the $C_{n-\alpha}[a, b]$ space which shall be described with its respective norm in the proof of the following lemma.

Lemma 3.2. Let $\alpha \in (n-1, n)$ and $\gamma \in (0, 1)$ then the fractional integration operator $I^{\alpha}_{a+} I^{\alpha}_{b-}$ is bounded in the $C_{\gamma}[a, b]$ space:

$$\| I^{\alpha}_{a+} I^{\alpha}_{b-} f \|_{C_{\gamma}} \leq K_1 \| f \|_{C_{\gamma}}, \tag{38}$$

where constant

$$K_1 = (b-a)^{2\alpha} \Gamma(1-\gamma) / \Gamma(\alpha+1) \Gamma(1-\gamma+\alpha).$$

Proof. Let us assume that function $f \in C_{\gamma}[a, b]$ and recall the corresponding norm:

$$\| f \|_{C_{\gamma}} := \max_{t \in [a,b]} | (t-a)^{\gamma} f(t) | . \tag{39}$$

For the right-sided integral $I^{\alpha}_{b-} f$, we obtain the inequality fulfilled for $t \in (a, b]$:

$$| I^{\alpha}_{b-} f(t) | \leq \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} | f(s) | ds \leq$$

$$\leq \frac{1}{\Gamma(\alpha)} \| f \|_{C_{\gamma}} \int_t^b (s-t)^{\alpha-1} (s-a)^{-\gamma} ds \leq$$

$$\leq \frac{(t-a)^{-\gamma}}{\Gamma(\alpha)} \| f \|_{C_{\gamma}} \int_t^b (s-t)^{\alpha-1} ds \leq$$

$$\leq \frac{(t-a)^{-\gamma}}{\Gamma(\alpha+1)} \| f \|_{C_{\gamma}} (b-a)^{\alpha}.$$

The above inequality implies that function $I^{\alpha}_{b-} f$ belongs to the $C_{\gamma}[a, b]$ space and its C_{γ} – norm obeys the formula:

$$\| I^{\alpha}_{b-} f \|_{C_{\gamma}} \leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \cdot \| f \|_{C_{\gamma}} .$$

Now we shall estimate the C_{γ} – norm for function $I^{\alpha}_{a+} I^{\alpha}_{b-} f$. Using the definition of this norm we calculate:

$$\| I^{\alpha}_{a+} I^{\alpha}_{b-} f \|_{C_{\gamma}} \leq$$

$$\leq \max_{t \in [a,b]} | (t-a)^{\gamma} I^{\alpha}_{a+} (t-a)^{-\gamma} \| I^{\alpha}_{b-} f \|_{C_{\gamma}} | \leq$$

$$\leq \| f \|_{C_{\gamma}} \cdot \max_{t \in [a,b]} | \frac{(t-a)^{\alpha} (b-a)^{\alpha} \Gamma(1-\gamma)}{\Gamma(\alpha+1) \Gamma(1-\gamma+\alpha)} | \leq$$

$$\leq \frac{(b-a)^{2\alpha} \Gamma(1-\gamma)}{\Gamma(\alpha+1) \Gamma(1-\gamma+\alpha)} \cdot \| f \|_{C_{\gamma}} .$$

From the above calculations, it follows that the considered double fractional integral operator is indeed bounded in the $C_{\gamma}[a, b]$ space:

$$\| I^{\alpha}_{a+} I^{\alpha}_{b-} f \|_{C_{\gamma}} \leq K_1 \| f \|_{C_{\gamma}},$$

where $K_1 = (b-a)^{2\alpha} \Gamma(1-\gamma) / \Gamma(\alpha+1) \Gamma(1-\gamma+\alpha)$ and this ends the proof.

Let us now denote as mapping T^s the following integral operator acting on space $C_{n-\alpha}[a, b]$:

$$T^s g(t) := \lambda I^{\alpha}_{a+} I^{\alpha}_{b-} g(t) + A_{-n} (t-a)^{\alpha-n}. \tag{40}$$

For $\gamma = n - \alpha$, constant K_1 from Lemma 3.2 is given as

$$K_1 = \frac{(b-a)^{2\alpha} \Gamma(\{\alpha\})}{\Gamma(\alpha+1) \Gamma(\alpha + \{\alpha\})}, \tag{41}$$

with $\{\alpha\}$ denoting the fractional part of order α.

We assume that function $g \in C_{n-\alpha}[a, b]$ and we check the $C_{n-\alpha}$ norm of its image $T^s g$ using Lemma 3.2. The following inequalities hold:

$$\begin{aligned} \| T^s g \|_{C_{n-\alpha}} &\leq | \lambda | \cdot \| I^{\alpha}_{a+} I^{\alpha}_{b-} g \|_{C_{n-\alpha}} + \\ &+ | A_{-n} | \cdot \| (t-a)^{\alpha-n} \|_{C_{n-\alpha}} \leq \\ &\leq | \lambda | \cdot K_1 \| g \|_{C_{n-\alpha}} + | A_{-n} | < \infty, \end{aligned} \tag{42}$$

where constant K_1 is determined by formula (41).

Mapping T^s is correctly defined as it maps functions from the $C_{n-\alpha}[a, b]$ space into image functions also belonging to the $C_{n-\alpha}[a, b]$ space.

For non-integer order α, Eq. (27) and therefore also initial Eq. (18) can be rewritten as a fixed point condition for mapping $T^s : C_{n-\alpha}[a, b] \longrightarrow C_{n-\alpha}[a, b]$

$$\mathcal{F}_{\lambda}(t) = T^s \mathcal{F}_{\lambda}(t). \tag{43}$$

We conclude that each singular solution of Eq. (18) from the $C_{n-\alpha}[a, b]$ space is given as a fixed point of mapping T^s. This subset of solutions is described in the following proposition.

Proposition 3.3. Let $\alpha \in (n-1, n)$ and let $f^{st}_{\alpha}(t) = A_{-n} (t-a)^{\alpha-n}$ be a singular stationary function of operator ${}^c D^{\alpha}_{b-} D^{\alpha}_{a+}$ with A_{-n} being an arbitrary constant. If inequality

$$| \lambda | < \frac{\Gamma(\alpha+1) \Gamma(\alpha + \{\alpha\})}{(b-a)^{2\alpha} \Gamma(\{\alpha\})} \tag{44}$$

is fulfilled, then unique singular solution $\mathcal{F}_{\lambda} \in C_{n-\alpha}[a, b]$ of Eq. (18), generated by the above stationary function f^{st}_{α} exists and is given by the following series:

$$\mathcal{F}_{\lambda}(t) = A_{-n} \sum_{m=0}^{\infty} (\lambda I^{\alpha}_{a+} I^{\alpha}_{b-})^m (t-a)^{\alpha-n}. \tag{45}$$

Proof. We have checked that mapping T^s acts as follows

$$T^s : C_{n-\alpha}[a, b] \longrightarrow C_{n-\alpha}[a, b].$$

Let us note that the $C_{n-\alpha}$-norm induces on space $C_{n-\alpha}[a, b]$ the following metric

$$d_s(g, h) := \| g - h \|_{C_{n-\alpha}}$$

for any pair of functions $g, h \in C_{n-\alpha}[a, b]$.

Space $C_{n-\alpha}[a, b]$ with metric d_s defined above is a metric and complete space, thus fulfills the assumptions of the Banach theorem on a fixed point. The next step is the investigation of properties of mapping T^s. We obtain the following inequality:

$$|| T^s g - T^s h ||_{C_{n-\alpha}} = || T^s (g-h) ||_{C_{n-\alpha}} =$$
$$= | \lambda | \cdot || I^{\alpha}_{a+} I^{\alpha}_{b-} (g-h) ||_{C_{n-\alpha}} \leq$$
$$\leq | \lambda | \cdot K_1 \, || g-h ||_{C_{n-\alpha}} .$$

The above inequality yields for metric d_s the relation:

$$d_s(T^s g, T^s h) \leq | \lambda | \cdot K_1 d_s(g,h).$$

We notice that mapping T^s is contractive on the $C_{n-\alpha}[a,b]$ space when the constants obey the condition:

$$| \lambda | \cdot K_1 < 1.$$

In this way, we obtain the inequality for eigenvalue λ, fractional order α and for the length of time interval $[a,b]$:

$$| \lambda | < \frac{\Gamma(\alpha+1)\Gamma(\alpha+\{\alpha\})}{(b-a)^{2\alpha}\Gamma(\{\alpha\})}.$$

When this assumption is fulfilled, mapping T^s is contractive on the $C_{n-\alpha}[a,b]$ space and using the Banach theorem, we conclude that a unique point obeying condition (43) exists. This fixed point is given as a limit of the iterations of mapping T^s:

$$\mathcal{F}_\lambda(t) = \lim_{m \longrightarrow \infty} (T^s)^m \psi(t)$$

for any function $\psi \in C_{n-\alpha}[a,b]$. Solution $\mathcal{F}_\lambda$ also is a unique solution of Eqs. (18), (27) in space $C_{n-\alpha}[a,b]$. Formula (45) describing the solution as a series is a limit of the iterations of mapping T^s when $\psi \equiv 0$ and this ends the proof.

3.3. Boundary conditions and particular solutions. The solutions described in the previous section include arbitrary constants. We shall now study the boundary conditions which determine these constants. Let us denote the components of solution $\mathcal{F}_\lambda$ generated by the $(t-a)^{\alpha+k}$ components of stationary functions as $\mathcal{F}^k_\lambda$. These particular solutions fulfill the following fractional integral equations for $k = -n, \ldots, n-1$:

$$\mathcal{F}^k_\lambda(t) = \lambda I^{\alpha}_{a+} I^{\alpha}_{b-} \mathcal{F}^k_\lambda + (t-a)^{\alpha+k}. \tag{46}$$

According to Propositions 3.1 and 3.3 they can be expressed as the sums of the following series:

$$\mathcal{F}^k_\lambda(t) = \sum_{m=0}^{\infty} (\lambda I^{\alpha}_{a+} I^{\alpha}_{b-})^m (t-a)^{\alpha+k} \tag{47}$$

for $k = -n, \ldots, n-1$. These series are convergent in interval $[a,b]$ when $k = -n+1, \ldots, n-1$ and in $(a,b]$ for $k = -n$. Their sums belong respectively to the $C[a,b]$ and $C_{n-\alpha}[a,b]$ spaces.

Let us now analyze the boundary values of eigenfunctions $\mathcal{F}^k_\lambda$. We calculate derivatives $D^{\alpha-l}_{a+}$ of component $\mathcal{F}^k_\lambda$. Using the composition rule from Property 2.2 [10], we obtain the following equality valid for $t \in [a,b]$:

$$\begin{aligned} &D^{\alpha-l}_{a+} \mathcal{F}^k_\lambda(t) = \\ &= \lambda I^{l}_{a+} I^{\alpha}_{b-} \mathcal{F}^k_\lambda(t) + \frac{\Gamma(\alpha+k+1)}{\Gamma(k+l+1)} (t-a)^{k+l}, \end{aligned} \tag{48}$$

where $l = 1, \ldots, n$. The above set of equalities yields the set of initial conditions for component solution $\mathcal{F}^k_\lambda$ fulfilled at $t = a$:

$$D^{\alpha-l}_{a+} \mathcal{F}^k_\lambda(t) \mid_{t=a} = \Gamma(\alpha+k+1)\delta_{-l,k} \tag{49}$$

for $k = -n, -n+1, \ldots, n-1$ and $l = 1, \ldots, n$.

Derivative D^{α}_{a+} of the component solution looks as follows:

$$D^{\alpha}_{a+} \mathcal{F}^k_\lambda(t) = \lambda I^{\alpha}_{b-} \mathcal{F}^k_\lambda(t) + \frac{\Gamma(\alpha+k+1)}{\Gamma(k+1)} (t-a)^k. \tag{50}$$

When we include derivatives D^l (of integer order $l = 0, \ldots, n-1$) we arrive at the formula:

$$\begin{aligned} &D^l D^{\alpha}_{a+} \mathcal{F}^k_\lambda(t) = \\ &= \lambda (-1)^l I^{\alpha-l}_{b-} \mathcal{F}^k_\lambda(t) + \frac{\Gamma(\alpha+k+1)}{\Gamma(k-l+1)} (t-a)^{k-l} \end{aligned} \tag{51}$$

valid in interval $[a,b]$.

This property of the component solution yields the following terminal conditions - fulfilled at end $t = b$:

$$D^l D^{\alpha}_{a+} \mathcal{F}^k_\lambda(t) \mid_{t=b} = \frac{\Gamma(\alpha+k+1)}{\Gamma(k-l+1)} (b-a)^{k-l}. \tag{52}$$

Now we shall derive particular solution $\mathcal{F}_\lambda$ of Eq. (18) obeying the set of boundary conditions. We conclude that the following set of conditions is consistent with our eigenfunction problem:

$$D^{\alpha+l}_{a+} \mathcal{F}_\lambda(t) \mid_{t=a} = A_l \quad l = -n, \ldots, -1, \tag{53}$$

$$D^l D^{\alpha}_{a+} \mathcal{F}_\lambda(t) \mid_{t=b} = A_l \quad l = 0, \ldots, n-1. \tag{54}$$

In order to construct a solution fulfilling the above boundary conditions, we shall define the set of coefficients A'_l by the recurrence relations:

$$A'_l = A_l \tag{55}$$

for $l = -n, \ldots, -1$,

$$A'_{n-j} = A_{n-j} - \sum_{i=1}^{j-1} \frac{(b-a)^i}{\Gamma(i+1)} A'_{n-j+i} \tag{56}$$

for $j = 2, \ldots, n-1$ and $A'_{n-1} = A_{n-1}$.

The following theorem represents the solution of Eq. (18) as the linear combination of basic solutions (46), (47). We construct them in such a way as to obtain particular solutions fulfilling the above set of boundary conditions.

Theorem 3.4. Let $\alpha \in (n-1, n)$. Then equation

$$({}^c D^{\alpha}_{b-} D^{\alpha}_{a+} - \lambda) \mathcal{F}_\lambda(t) = 0 \tag{57}$$

has in finite time interval $[a,b]$, a unique $C_{n-\alpha}[a,b]$ solution fulfilling the following boundary conditions:

$$D^{\alpha+l}_{a+} \mathcal{F}_\lambda(t) \mid_{t=a} = A_l \quad l = -n, \ldots, -1, \tag{58}$$

$$D^l D^{\alpha}_{a+} \mathcal{F}_\lambda(t) \mid_{t=b} = A_l \quad l = 0, \ldots, n-1, \tag{59}$$

provided inequality (31) is fulfilled. In the case when constant $A_{-n} \neq 0$, we replace condition (31) with the assumption

$$| \lambda | < \frac{\Gamma(\alpha+1)\Gamma(\alpha+\{\alpha\})}{(b-a)^{2\alpha}\Gamma(\{\alpha\})}. \tag{60}$$

The above solution is the following linear combination of basic solutions $\mathcal{F}_\lambda^k$ given in formulas (46), (47):

$$\mathcal{F}_\lambda(t) = \sum_{k=-n}^{n-1} \frac{A'_k}{\Gamma(\alpha+k+1)} \mathcal{F}_\lambda^k(t), \tag{61}$$

where coefficients A'_k are defined by recurrence relations (55)–(56).

Proof. From the results given in Propositions 3.1 and 3.3 it follows that the function given by formula (61) is a unique solution of fractional Eq. (18) in the $C_{n-\alpha}[a,b]$ space generated by the stationary function:

$$f_\alpha^{st}(t) = \sum_{k=-n}^{n-1} \frac{A'_k}{\Gamma(\alpha+k+1)}(t-a)^{\alpha+k}.$$

Let us check that this linear combination of the basic solutions obeys the boundary conditions given in formulas (58), (59). We begin with $l = -n, \ldots, -1$ and have the boundary conditions fulfilled at $t = a$:

$$D_{a+}^{\alpha+l} \mathcal{F}_\lambda(t) \mid_{t=a} =$$

$$= \sum_{k=-n}^{n-1} \frac{A'_k}{\Gamma(\alpha+k+1)} \cdot \Gamma(\alpha+k+1)\delta_{l,k} = A'_l = A_l.$$

Now we analyze the second subset of the boundary conditions:

$$D^l D_{a+}^{\alpha} \mathcal{F}_\lambda(t) \mid_{t=b} =$$

$$= \sum_{k=-n}^{n-1} \frac{A'_k}{\Gamma(\alpha+k+1)} \cdot \frac{\Gamma(\alpha+k+1)}{\Gamma(k-l+1)} \cdot (b-a)^{k-l} =$$

$$= \sum_{k=l}^{n-1} \frac{A'_k(b-a)^{k-l}}{\Gamma(k-l+1)} = A'_l + \sum_{k=l+1}^{n-1} \frac{A'_k(b-a)^{k-l}}{\Gamma(k-l+1)} =$$

$$= A_l - \sum_{i=1}^{n-l-1} \frac{A'_{l+i}(b-a)^i}{\Gamma(i+1)} + \sum_{k=l+1}^{n-1} \frac{A'_k(b-a)^{k-l}}{\Gamma(k-l+1)} = A_l.$$

We conclude that the above solution fulfills the assumed boundary conditions and that ends the proof.

Remark. Let us recall that when we consider eigenfunction Eq. (57) as an equation of motion for action (9) we have an additional restriction:

$$D^{n-1} D_{a+}^{\alpha} \mathcal{F}_\lambda(t) \mid_{t=b} = A_{n-1} = A'_{n-1} = 0.$$

3.4. Approximate continuous solutions. We shall now study the approximate continuous solutions of problem (18). The solutions described in Proposition 3.1 and Theorem 3.4 are constructed using the Banach theorem on a fixed point which gives us a good control of the error of approximation. The corresponding estimates of error are enclosed in the following proposition.

Proposition 3.5. Let $\alpha > 0$ and let $\bar{f}_\alpha^{st}$ be an arbitrary continuous stationary function of operator ${}^c D_{b-}^{\alpha} D_{a+}^{\alpha}$ of form (26) or (23), (24) respectively. If inequality (31) is fulfilled, then approximate solution f_{ap} of Eq. (18) given by the finite sum:

$$f_{ap}(t) = \sum_{l=0}^{m} [\lambda I_{a+}^{\alpha} I_{b-}^{\alpha}]^l \bar{f}_\alpha^{st}(t) \tag{62}$$

yields the following error of approximation for continuous solution $\mathcal{F}_\lambda$:

$$\| f_{ap} - \mathcal{F}_\lambda \| \leq \frac{L^m}{1-L} \cdot \| \bar{f}_\alpha^{st} \|, \tag{63}$$

where constant $L := \mid \lambda \mid \cdot \| I_{a+}^{\alpha} I_{b-}^{\alpha} 1 \|$.

The proof is the straightforward corollary of the Banach theorem on a fixed point.

Inequality (63) yields the condition for the error to be smaller than ϵ, namely from the inequality

$$\| f_{ap} - \mathcal{F}_\lambda \| \leq \frac{L^m}{1-L} \cdot \| \bar{f}_\alpha^{st} \| \leq \epsilon,$$

it follows

$$m \geq \frac{\log(\epsilon) + \log(1-L) - \log(\| \bar{f}_\alpha^{st} \|)}{\log(L)}.$$

The above formula is simplest for the normalized stationary function $\| \bar{f}_\alpha^{st} \| = 1$. In this case, we obtain the condition for the number of iterations:

$$m \geq \frac{\log(\epsilon) + \log(1-L)}{\log(L)}.$$

Let us denote the function on the right-hand side of the above inequality as follows:

$$\Delta(\epsilon, L) := \frac{\log(\epsilon) + \log(1-L)}{\log(L)}.$$

We notice that the above function yields the lower bound for the number of iterations m and it is a decreasing function of ϵ. Thus, we shall fix the value of $\epsilon = 0.01$, restrict the length of time interval $b - a = 1$ and analyze the influence of order α and $\mid \lambda \mid$. In the first example illustrated by Fig. 4, we draw the graphs of function $\Delta(0.01, L)$ for chosen values of $\mid \lambda \mid$ when $\alpha \in (0, 1)$.

We observe that function $\Delta(0.01, L)$ has a maximum in interval $(0, 1)$ for every analyzed value of modulus $\mid \lambda \mid$. In addition, if $\mid \lambda \mid$ increases, then so does the lower bound for m.

Now we check the number of iterations necessary to have an error of approximation smaller than $\epsilon = 0.01$ for fixed range $\alpha \in (1, 2)$. The graphs in Fig. 5 illustrate the dependence of the lower bound of m on parameters $\mid \lambda \mid$ and α. Contrary to the case $\alpha \in (0, 1)$, in this example the lower bound of m is a decreasing function of order α for each chosen modulus of the eigenvalue. Similarly to the previous example we observe that it is an increasing function of $\mid \lambda \mid$.

Analyzing the results illustrated in Fig. 6 for range of order $\alpha \in (2, 3)$, we notice that the behaviour of the lower bound of m is analogous to that given in Fig. 5 for $\alpha \in (1, 2)$. For each given λ, we also observe that $\Delta(0.01, L)$ is a decreasing function in a set $(1, 2) \cup (2, 3)$.

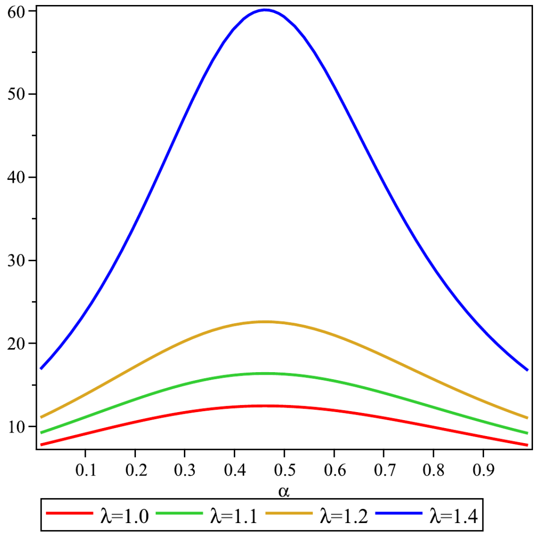

Fig. 4. Function $\Delta(0.01, L)$ for $\alpha \in (0, 1)$

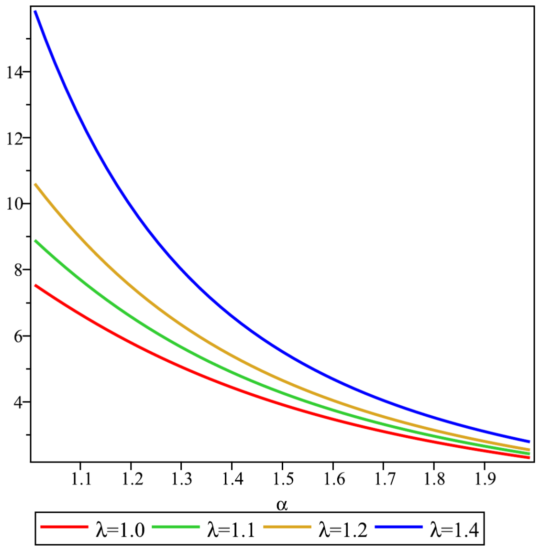

Fig. 5. Function $\Delta(0.01, L)$ for $\alpha \in (1, 2)$

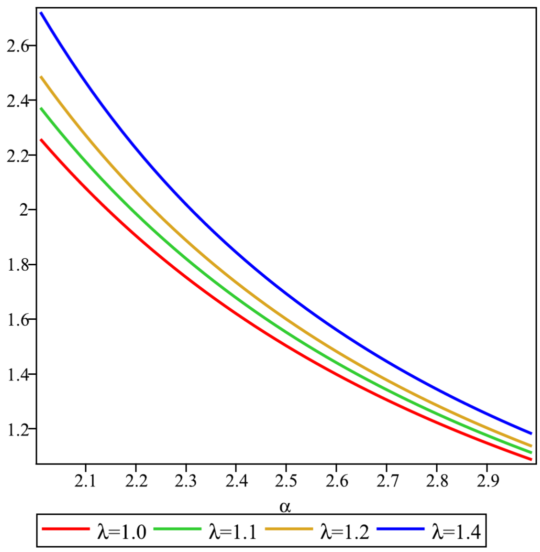

Fig. 6. Function $\Delta(0.01, L)$ for $\alpha \in (2, 3)$

4. Concluding remarks

In the paper we studied an eigenfunction equation of a variational operator and derived its solution using the Banach fixed point theorem. The study included analysis of properties of exact and approximate solutions. The surfaces of admissible eigenvalues were illustrated by examples for a given range of fractional order α. For approximate solutions, we investigated the behaviour of the lower bound of the number of iterations restricting the error of approximation.

In our opinion, this simple fractional differential equation is of interest as its solutions can be used in the construction of solutions of general linear sequential equations containing operator ${}^cD^{\alpha}_{a+}D^{\alpha}_{b-}$. Clearly, when we consider in finite time interval $[a, b]$ an equation in the form of:

$$L({}^cD^{\alpha}_{a+}D^{\alpha}_{b-})f(t) = 0,$$

where L is a polynomial function

$$L(\lambda) := \lambda^N + \sum_{k=0}^{N-1} A_k \lambda^k$$

we can factorize the above equation as follows:

$$L({}^cD^{\alpha}_{a+}D^{\alpha}_{b-})f(t) = \prod_{k=1}^{N}({}^cD^{\alpha}_{a+}D^{\alpha}_{b-} - \lambda_k)f(t) = 0.$$

In the case when $\{\lambda_k \in C \quad k = 1, \ldots, N\}$ is a set of simple roots of the characteristic equation $L(\lambda) = 0$ we immediately construct the general solution as the sum

$$f(t) = \sum_{k=1}^{N} \mathcal{F}_{\lambda_k}$$

dependent on $N \times n$ constants with $\mathcal{F}_{\lambda_k}$ given in Propositions 3.1 and 3.3, provided condition

$$\max_{k=1,\ldots,N} \mid \lambda_k \mid \cdot \mid\mid I^{\alpha}_{a+}I^{\alpha}_{b-}1 \mid\mid < 1$$

is fulfilled. The proof and further applications of the obtained results will be enclosed in a subsequent paper.

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