BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES
Vol. 58, No. 1, 2010

Robustness of adaptive discrete-time LQG control for first-order systems

A. KRÓLIKOWSKI* and D. HORLA

Institute of Control and Information Engineering, Poznań University of Technology, 3a Piotrowo St., 60-965 Poznań, Poland

Abstract. The discrete-time adaptive LQG control of first-order systems is considered from robustness point of view. Both stability and performance robustness are analyzed for different control system structures. A case of amplitude-constrained control is presented, and application of certainty equivalence for self-tuning implementation is also discussed.

Key words: first-order system, adaptive LQG control, robustness.

1. Introduction

Analysis and design of both adaptive and H_2 robust control for uncertain linear time invariant systems is an area of an extensive research, however the robustness issues in adaptive control context are not much researched especially in the LQG framework. Robustness aspects of adaptive control are discussed in [1] where an example of a first-order MRAS is analyzed in the presence of unmodelled dynamics. The averaging methods were shown to be useful for analysis of equilibria and local properties around equilibria. These methods can also be applied to analyze what happens when adaptive control systems are designed based on simplified models. Two algorithm of indirect self-tuning LQG control are also given for ARMAX system. The first one is based on the spectral factorization and the second one is based on Riccati equation.

In [2] robust direct and indirect adaptive control laws are presented for the pole placement design where the robustness is considered with respect to bounded disturbances or disturbances which account for the effect of unmodelled dynamics.

A robust constrained predictive control with a new adaptive parameter estimation algorithm is presented in [3].

In [4], the adaptive LQG control in discrete-time domain with loop transfer recovery is investigated. Correspondingly, the self-tuning LQG control in frequency domain is considered in [5] and [6] where the controller and estimator are found by polynomial equations. This approach is different from standard Kalman theory based on Riccati equations, however the polynomial implementation of the LQG controller is entirely equivalent, from a performance point of view, with a state-space Kalman filtering solution.

In this latter approach which is considered also below it is known that the application of certainty equivalence principle yields some identifiability problems even if the unknown system parameters belong to the finite set Θ. In this case, an adaptive LQG control, based on biasing the usual least-squares parameter estimation criterion with a term favoring parameters associated with lower optimal costs was introduced in [7].

To restore the optimality of an adaptive LQG control, the cost-biased least-squares parameter estimation method was also introduced in [8] for the case of compact set Θ. The case of infinite parameter set Θ is much more difficult and still remains unsolved.

The cost of uncertainty quantified in terms of closed-loop performance is investigated in [9] for LQG control of first-order system where the graphical results are obtained by evaluation of integrals from Parseval's theorem.

The problem addressed in this paper is the robustness aspects of adaptive LQG control for uncertain linear first-order systems. Stability and performance robustness are investigated for different configurations of the control system, i.e.: Kalman predictor, Kalman filter and output feedback controllers. The numerical results are presented, concerning how stability and performance depend on actual and estimated values of controlled system parameters in LQG control having first-order system dynamics. This explicit insight is provided by pictures which are believed to be original.

The case of an amplitude-constrained input is also considered for both Kalman predictor and Kalman filter-based controllers. It is worthy to note that there is no separation theorem for LQG control with hard control and/or state constraints [10], and obviously, the same holds for adaptive LQG control. The robustness properties of adaptive control systems are illustrated and compared via simulations for all considered controllers. It is shown that good robustness features can be preserved in the considered LQG control of first-order systems. Obviously, this is not true in general case [11, 12], because LQG designs based on Kalman filtering can exhibit arbitrarily poor stability margins. This means that even when the state feedback gains are appropriate for the actual system, the inaccurate state estimates fed back can cause reduced performance or stability properties. It is known that the loop transfer recovery technique can be used then to improve the robustness and performance.

*e-mail: Andrzej.Krolikowski@put.poznan.pl

As already mentioned, problems with convergence to optimality may occur in self-tuning control based on the certainty equivalence principle. There is a link between parametric robustness considerations presented in the paper and the implementation of self-tuning control. For example, the initial parameter estimates have to be chosen to be in the stability region while the limiting parameter estimates, if they lie in the robustness region, they determine the accuracy of steady-state performance.

2. Kalman predictor controller

Let the first-order system be described by

$$x_{t+1} + ax_t = bu_t + v_t, \tag{1}$$

$$y_t = x_t + w_t, \tag{2}$$

where v_t, w_t are white gaussian noises with zero mean and corresponding variances σ_v^2, σ_w^2. For simplicity it is assumed that $x_0 = 0$.

First, consider only the system (1), (2) without measurement noise and the controller of the form

$$u_t = f_{xx} x_t, \tag{3}$$

aiming at minimization of the stationary cost function

$$J_{xx} = \sigma_x^2 + q\sigma_u^2 = (1 + qf_{xx}^2)\sigma_x^2, \tag{4}$$

where the weight $q \geq 0$ and the variances σ_x^2, σ_u^2 are given by

$$\sigma_x^2 = \frac{\sigma_v^2}{1 - (bf_{xx} - a)^2}, \qquad \sigma_u^2 = f_{xx}^2 \sigma_x^2. \tag{5}$$

The form of σ_x^2 in (5) can be obtained by putting (3) into Eq. (1), next raising square both sides of this equation, and finally taking expectation.

Minimization of J_{xx} w.r.t. the feedback gain f_{xx} gives

$$f_{xx}^{\mathrm{opt}}(a,b) = \\ = \frac{q(a^2-1) - b^2 + \sqrt{q^2(1-a^2)^2 + 2qb^2(1+a^2) + b^4}}{2abq}. \tag{6}$$

Obviously, the same result can also be obtained by solving the Riccati equation

$$P_f^2 b^2 + P_f\left(q(1-a^2) - b^2\right) - q = 0 \tag{7}$$

and calculating

$$f_{xx}^{\mathrm{opt}}(a,b) = (b^2 P_f + q)^{-1} abP_f \tag{8}$$

for $P_f > 0$. The minimal cost is then

$$J_{xx}^{\min} = P_f \sigma_v^2. \tag{9}$$

Now, consider the system described by (1) and (2) with the Kalman predictor (KP)-based controller

$$u_t = f_{\widehat{x}x} \widehat{x}_t. \tag{10}$$

According to the notation used in the text, the first subscript ($\widehat{x}$) denotes the variable used for feedback, and the second subscript (x) denotes the variable used in the cost function. Note that $f_{\widehat{x}x} = f_{xx}$ due to the separation theorem. The KP yielding $\widehat{x}_t$ is

$$\widehat{x}_{t+1} = -a\widehat{x}_t + bu_t + k\widetilde{y}_t, \tag{11}$$

where $\widetilde{y}_t = y_t - \widehat{x}_t$ and $\widetilde{x}_t^{\mathrm{KP}} = x_t - \widehat{x}_t$ is the estimation error. From (1), (2) and (11) it can be found that the estimation error satisfies the following recursive equation

$$\widetilde{x}_{t+1}^{\mathrm{KP}} = -(a+k)\widetilde{x}_t^{\mathrm{KP}} + v_t - kw_t \tag{12}$$

and its variance is

$$\sigma_{\widetilde{x},\mathrm{KP}}^2 = \frac{\sigma_v^2 + k^2\sigma_w^2}{1 - (a+k)^2}. \tag{13}$$

Minimization of (13) yields the optimal predictor gain k^{opt}

$$k^{\mathrm{opt}}(a,b) = \\ = \frac{(1-a^2)\sigma_w^2 + \sigma_v^2 - \sqrt{((1-a^2)\sigma_w^2 + \sigma_v^2)^2 + 4a^2\sigma_v^2\sigma_w^2}}{2a\sigma_w^2}. \tag{14}$$

Obviously, the same result can be found by solving the Riccati equation

$$P_k^2 + P_k((1-a^2)\sigma_w^2 - \sigma_v^2) - \sigma_v^2\sigma_w^2 = 0 \tag{15}$$

and calculating

$$k^{\mathrm{opt}}(a,b) = -aP_k(P_k + \sigma_w^2)^{-1} \tag{16}$$

for $P_k > 0$; note that $P_k = \sigma_{\widetilde{x},\mathrm{KP}}^2$. The minimal cost is

$$J_{\widehat{x}x}^{\mathrm{KPmin}} = P_f\sigma_v^2 + P_k(f_{\widehat{x}x}^{\mathrm{opt}})^2(b^2P_f + q). \tag{17}$$

2.1. Input-constrained case. When $|u_t| \leq \alpha$ then the feedback controller is implemented by means of a saturation function

$$u_t = sat(f_{\widehat{x}x,\alpha}\widehat{x}_t; \alpha), \tag{18}$$

where the feedback gain $f_{\widehat{x}x,\alpha}$ can be derived using the algorithm proposed in [13] for system (1), (2) assuming that the non-gaussian probability density function for $\widehat{x}_t$ can be approximated by the gaussian function. Then, taking into account $\sigma_x^2 = \sigma_{\widehat{x}}^2 + \sigma_{\widetilde{x},\mathrm{KP}}^2$, the cost function (4) can be derived as [13, 14]

$$J_{\widehat{x}x,\alpha}^{\mathrm{KP}} = [1 + qf_{\widehat{x}x,\alpha}^2 g_1(\sigma)]\sigma_{\widehat{x}}^2 + \sigma_{\widetilde{x},\mathrm{KP}}^2, \tag{19}$$

where $g_1(\sigma)$ appearing in (19) as a result of approximation is defined by error functions $g_1(\sigma) = \mathrm{erf}(\alpha\sigma^{-1}2^{-\frac{1}{2}}) - \alpha\sigma^{-1}2^{\frac{1}{2}}\mathrm{ierfc}(\alpha\sigma^{-1}2^{-\frac{1}{2}})$ and $\sigma^2 = f_{\widehat{x}x,\alpha}^2\sigma_{\widehat{x}}^2$.

The variance $\sigma_{\widehat{x}}^2$ can be found as an iterative solution to the stationary closed-loop equation for a given constraint α

$$\sigma_{\widehat{x}}^2 = a^2\sigma_{\widehat{x}}^2 - 2abf_{\widehat{x}x,\alpha}g_2(\sigma)\sigma_{\widehat{x}}^2 + b^2f_{\widehat{x}x,\alpha}^2 g_1(\sigma)\sigma_{\widehat{x}}^2 + k^2\sigma_{\widetilde{y}}^2, \tag{20}$$

where the error function $g_2(\sigma) = \mathrm{erf}(\alpha\sigma^{-1}2^{-\frac{1}{2}})$ and $\sigma_{\widetilde{y}}^2 = \sigma_{\widetilde{x},\mathrm{KP}}^2 + \sigma_w^2$. The solution to the Eq. (20) is in turn used in the iterative algorithm which gives the feedback gain $f_{\widehat{x}x,\alpha}$.

3. Kalman filter controller

The Kalman filter (KF) yielding $\widehat{x}_{t/t}$ is

$$\widehat{x}_{t+1/t+1} = -a\widehat{x}_{t/t} + bu_t + \overline{k}(y_{t+1} + a\widehat{x}_{t/t} - bu_t) \tag{21}$$

or in terms of the KP output

$$\widehat{x}_{t+1/t+1} = \widehat{x}_{t+1} + \overline{k}\widetilde{y}_{t+1}. \tag{22}$$

From (1), (2) and (21) it can be found that the estimation error $\widetilde{x}_t^{\mathrm{KF}} = x_t - \widehat{x}_{t/t}$ satisfies the following recursive equation

$$\widetilde{x}_{t+1}^{\mathrm{KF}} = -a(1-\overline{k})\widetilde{x}_t^{\mathrm{KF}} + (1-\overline{k})v_t - \overline{k}w_{t+1} \tag{23}$$

and its variance is

$$\sigma^2_{\widetilde{x},\mathrm{KF}} = \frac{(1-\overline{k})^2\sigma_v^2 + \overline{k}^2\sigma_w^2}{1-a^2(1-\overline{k})^2}. \tag{24}$$

Minimization of (24) yields the optimal filter gain $\overline{k}^{\mathrm{opt}}$

$$\overline{k}^{\mathrm{opt}}(a,b) = = \frac{(a^2-1)\sigma_w^2 - \sigma_v^2 + \sqrt{((1-a^2)\sigma_w^2+\sigma_v^2)^2 + 4a^2\sigma_v^2\sigma_w^2}}{2a^2\sigma_w^2}. \tag{25}$$

It is easy to see that $k^{\mathrm{opt}} = -a\overline{k}^{\mathrm{opt}}$.

The KF-based controller is

$$u_t = f_{\widehat{x}x}\widehat{x}_{t/t} \tag{26}$$

and the minimal cost is now

$$J_{\widehat{x}x}^{\mathrm{KFmin}} = P_f(\sigma_v^2 + abf_{\widehat{x}x}^{\mathrm{opt}}P_{k/k}), \tag{27}$$

where

$$P_{k/k} = P_k - P_k^2(P_k + \sigma_w^2)^{-1}. \tag{28}$$

Note that here $P_{k/k} = \sigma^2_{\widetilde{x},\mathrm{KF}}$ where $\sigma^2_{\widetilde{x},\mathrm{KF}}$ is given by (24). Calculating $\Delta J^{\min} = J_{\widehat{x}x}^{\mathrm{KPmin}} - J_{\widehat{x}x}^{\mathrm{KFmin}}$ yields

$$\Delta J^{\min} = P_k^2(f_{\widehat{x}x}^{\mathrm{opt}})^2(b^2P_f + q)(P_k+\sigma_w^2)^{-1} > 0, \tag{29}$$

which means that the KF-based controller performs better than KP-based controller in whole range of the weight q.

3.1. Input-constrained case. Again, when $|u_t| \le \alpha$ then the feedback controller (26) is implemented analogously to (18) as

$$u_t = sat(f_{\widehat{x}x,\alpha}\widehat{x}_{t/t};\alpha). \tag{30}$$

Similarly to (20), the approximate equation for the variance $\sigma^2_{\widehat{x}/\widehat{x}}$ resulting from (21) and (30) has now the form

$$\begin{aligned}\sigma^2_{\widehat{x}/\widehat{x}} &= a^2\sigma^2_{\widehat{x}/\widehat{x}} - 2abf_{\widehat{x}x,\alpha}g_2(\sigma)\sigma^2_{\widehat{x}/\widehat{x}} + \\ &+ b^2f^2_{\widehat{x}x,\alpha}g_1(\sigma)\sigma^2_{\widehat{x}/\widehat{x}} + \overline{k}^2\sigma^2_{\widetilde{y}}\end{aligned} \tag{31}$$

and can be solved iteratively for a given constraint α. Analogously to (19), the cost function has now the form

$$J_{\widehat{x}x,\alpha}^{\mathrm{KF}} = [1 + qf^2_{\widehat{x}x,\alpha}g_1(\sigma)]\sigma^2_{\widehat{x}/\widehat{x}} + P_{k/k}, \tag{32}$$

where $P_{k/k}$ is given as in (28). Again as for (18), the iterative algorithm similar to that proposed in [10] can be used to derive the feedback gain $f_{\widehat{x}x,\alpha}$ which minimizes the cost function (32).

4. Output feedback controller

Consider the system (1), (2) and the output feedback control problem with the controller

$$u_t = f_{yx}y_t, \tag{33}$$

which has to minimize the stationary cost function

$$J_{yx} = \sigma_x^2 + q\sigma_u^2. \tag{34}$$

It can be shown [15] that the cost function (34) can be expressed as

$$J_{yx} = (1 + f_{yx}^2q)\sigma_x^2 + f_{yx}^2q\sigma_w^2, \tag{35}$$

where

$$\sigma_x^2 = \frac{b^2f_{yx}^2\sigma_w^2 + \sigma_v^2}{1-(bf_{yx}-a)^2}. \tag{36}$$

Minimization of J_{yx} w.r.t. f_{yx} gives

$$\begin{gathered}-f_{yx}^4ab^3q\sigma_w^2 + 4f_{yx}^3a^2b^2q\sigma_w^2 + \\ + f_{yx}^2(-4a^3bq\sigma_w^2 + abq\sigma_v^2 + ab^3\sigma_w^2 + 4abq\sigma_w^2) + \\ + f_{yx}(b^2\sigma_v^2 + q\sigma_v^2 - a^2q\sigma_v^2 + b^2\sigma_w^2 - a^2b^2\sigma_w^2 + \\ + q\sigma_w^2 - 2a^2q\sigma_w^2 + a^4q\sigma_w^2) - ab\sigma_v^2 = 0.\end{gathered} \tag{37}$$

The numerical solution of (37) will yield the optimal feedback gain f_{yx}^{opt}. Notice that if $\sigma_w^2 = 0$ then the solution is again f_{xx}^{opt} while putting $\sigma_v^2 = 0$ in (33), the trivial feedback gain $f_{yx} = 0$ is obtained.

Consider again the system (1), (2) under the controller

$$u_t = f_{yy}y_t. \tag{38}$$

The stationary cost function

$$J_{yy} = \sigma_y^2 + q\sigma_u^2 \tag{39}$$

is now considered where variances σ_y^2, σ_u^2 are given by

$$\sigma_y^2 = \frac{2abf_{yy}\sigma_w^2 + \overline{\sigma}^2}{1-(bf_{yy}-a)^2}, \qquad \sigma_u^2 = f_{yy}^2\sigma_y^2, \tag{40}$$

where $\overline{\sigma}^2 = \sigma_v^2 + \sigma_w^2(1-a^2)$. Solving

$$\min_{f_{yy}} J_{yy} \tag{41}$$

yields the following equation

$$\begin{gathered}f_{yy}^4ab^3q\sigma_w^2 + 4f_{yy}^3ab^2q\sigma_w^2 + \\ + f_{yy}^2(ab^3\sigma_w^2 + 3abq\sigma_w^2 - 3a^3bq\sigma_w^2 + abq\overline{\sigma}^2) + \\ + f_{yy}(q\overline{\sigma}^2 - a^2q\overline{\sigma}^2 + b^2\overline{\sigma}^2) + \\ + ab(1-a^2)\sigma_w^2 - ab\overline{\sigma}^2 = 0,\end{gathered} \tag{42}$$

whose numerical solution gives the optimal feedback gain f_{yy}^{opt}.

5. Performance robustness

Performance robustness is analyzed in terms of parameter estimates $\widehat{a} = \delta_a a$, $\widehat{b} = \delta_b b$. For the control law (3) the minimal value of the cost function (4), i.e. $J_{xx}^{\min}$, can be calculated using (5). The performance robustness can be formulated as a determination of an allowable region of parameter estimates $\widehat{a}, \widehat{b}$ assuring that the attainable value of the cost function $\widehat{J}_{xx}$ satisfies the inequality

$$\widehat{J}_{xx} \leq (1+\Delta) J_{xx}^{\min} \tag{43}$$

for a given $\Delta > 0$, where $\widehat{J}_{xx} = J_{xx}(\widehat{f}_{xx}^{\mathrm{opt}})$ and $\widehat{f}_{xx}^{\mathrm{opt}}$ is obtained from (6) using the estimates $\widehat{a}$, $\widehat{b}$.

To guarantee the positive value of $\widehat{J}_{xx}$ the uncertainty intervals for δ_a, δ_b should be taken as to the feedback gain $\widehat{f}_{xx}^{\mathrm{opt}}$ ensures the closed-loop stability (see the next Section).

For $a = -0.5$, $b = 0.5$, $q = 0.1$ and $\sigma_v^2 = \sigma_w^2 = 0.1$ we have from (4),(5),(6) $J_{xx}^{\min} = 0.1073$ and $f_{xx}^{\mathrm{opt}} = -0.7284$. The same value of $J_{xx}^{\min}$ can also be obtained from (9). The inequality (43) is illustrated in Fig. 1 for $\Delta = 0.05$ where the flat (and light) surface is set at the value equal to the right hand side of inequality (43). The intersection of this surface with other surface representing the cost function $\widehat{J}_{xx}$ yields the allowable region of parameter estimates, i.e. the region of performance robustness taken w.r.t. parametric uncertainties δ_a, δ_b. Obviously, the region of performance robustness is constrained. Notice that in all other simulations concerning the performance robustness the same idea of illustration holds and the singular case $\delta_a = \delta_b = 0$ is ignored.

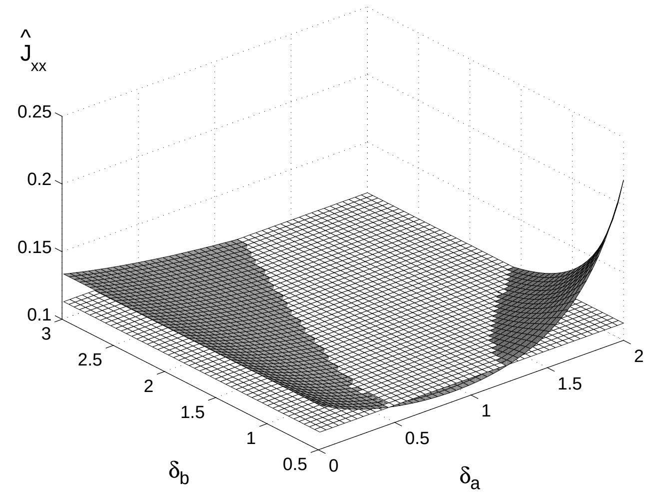

Fig. 1. Illustration of performance robustness w.r.t. J_{xx}

For the control law (10) the minimal value of the cost function (4) can be obtained using Eqs. (2), (10), (11) (or from (19) for $\alpha \to \infty$)

$$J_{\widehat{x}x}^{\mathrm{KPmin}} = \left(1 + q(f_{\widehat{x}x}^{\mathrm{opt}})^2\right)\sigma_{\widehat{x}}^2 + \sigma_{\widetilde{x},KP}^2, \tag{44}$$

where

$$\begin{aligned} \sigma_{\widehat{x}}^2 &= \frac{(k^{\mathrm{opt}})^2(\sigma_w^2 + \sigma_{\widetilde{x}}^2)}{1-(bf_{\widehat{x}x}^{\mathrm{opt}} - a)^2}, \\ \sigma_{\widetilde{x},\mathrm{KP}}^2 &= \frac{\sigma_v^2 + (k^{\mathrm{opt}})^2\sigma_w^2}{1-(a+k^{\mathrm{opt}})^2}. \end{aligned} \tag{45}$$

Analogously to (43), a condition for performance robustness can now be determined by inequality

$$\widehat{J}_{\widehat{x}x}^{\mathrm{KP}} \leq (1+\Delta) J_{\widehat{x}x}^{\mathrm{KPmin}} \tag{46}$$

for a given $\Delta > 0$, where $\widehat{J}_{\widehat{x}x}^{\mathrm{KP}} = J_{\widehat{x}x}^{\mathrm{KP}}(\widehat{f}_{\widehat{x}x}^{\mathrm{opt}}, \widehat{k}^{\mathrm{opt}})$, and the feedback and predictor gains $\widehat{f}_{\widehat{x}x}^{\mathrm{opt}}$, $\widehat{k}^{\mathrm{opt}}$ are calculated on the basis of (6) and (14) with use of estimates $\widehat{a}$, $\widehat{b}$.

To derive $\widehat{J}_{\widehat{x}x}^{\mathrm{KP}}$ the analysis of closed-loop system containing controller with gain $\widehat{f}_{\widehat{x}x}^{\mathrm{opt}}$ and KP (11) with gain $\widehat{k}^{\mathrm{opt}}$ has to be performed.

Performing some manipulations and using parameter estimates $\widehat{a}$, $\widehat{b}$ the following transfer function of the controller can be obtained

$$\widehat{G}_c(z) = \frac{u(z)}{y(z)} = \frac{\widehat{A}}{z+\widehat{B}}, \tag{47}$$

where $\widehat{A} = \widehat{k}^{\mathrm{opt}}(\widehat{a},\widehat{b})\widehat{f}_{xx}^{\mathrm{opt}}(\widehat{a},\widehat{b})$, $\widehat{B} = \widehat{k}^{\mathrm{opt}}(\widehat{a},\widehat{b}) + \widehat{a} - \widehat{b}\widehat{f}_{xx}^{\mathrm{opt}}(\widehat{a},\widehat{b})$ and $\widehat{a} = \delta_a a$, $\widehat{b} = \delta_b b$.

Combining the controller transfer function (47) with system transfer function we have the following expressions for $x(z)$ and $u(z)$ in the closed-loop system

$$x(z) = \frac{(z+\widehat{B})v(z) + b\widehat{A}w(z)}{z^2 + z(a+\widehat{B}) + a\widehat{B} - b\widehat{A}}, \tag{48}$$

$$u(z) = \frac{\widehat{A}v(z) + \widehat{A}(z+a)w(z)}{z^2 + z(a+\widehat{B}) + a\widehat{B} - b\widehat{A}}. \tag{49}$$

Using the above expressions the variances $\widehat{\sigma}_x^2$, $\widehat{\sigma}_u^2$ can be obtained which are necessary to calculate the following cost function

$$\widehat{J}_{\widehat{x}x}^{\mathrm{KP}} = \widehat{\sigma}_x^2 + q\widehat{\sigma}_u^2 \tag{50}$$

and consequently to verify the condition (46).

Inequality (46) is illustrated in Fig. 2 for $\Delta = 0.02$, $\sigma_v^2 = \sigma_w^2 = 1.0$ and other parameters as for Fig. 1. According to (46) we have then $J_{\widehat{x}x}^{\mathrm{KPmin}} = 1.2942$. The same value can be obtained from (52) for $\delta_a = \delta_b = 1.0$ or from (17). Moreover, we have $f_{\widehat{x}x}^{\mathrm{opt}} = f_{xx}^{\mathrm{opt}} = -0.7284$ and $k^{\mathrm{opt}} = 0.2655$.

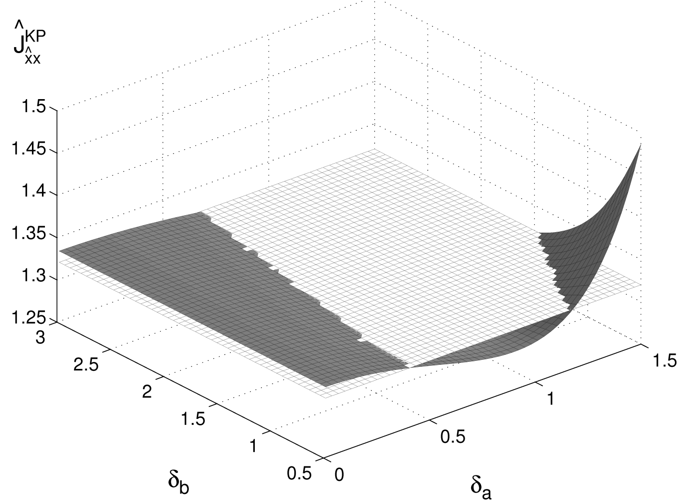

Fig. 2. Illustration of performance robustness w.r.t. $J^{\mathrm{KP}}_{\widehat{x}x}$

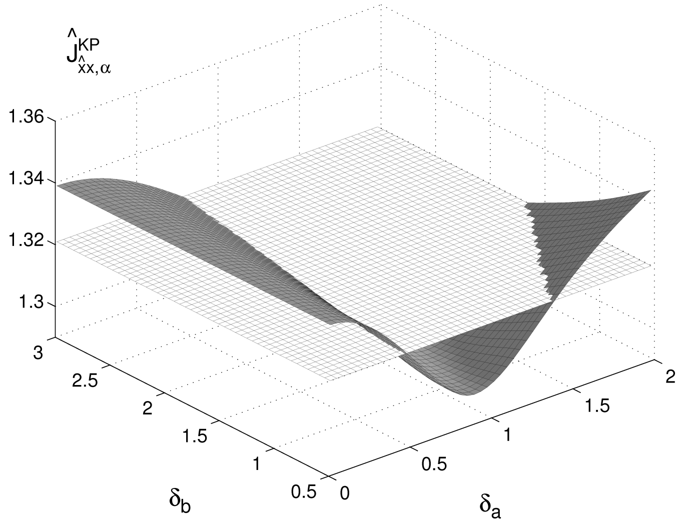

Fig. 4. Illustration of performance robustness w.r.t. $J^{\mathrm{KP}}_{\widehat{x}x,\alpha}$; $\alpha = 0.5$

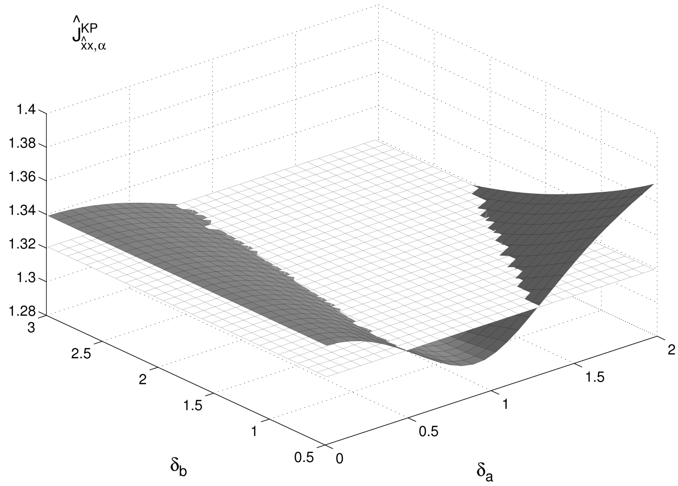

Fig. 5. Illustration of performance robustness w.r.t. $J^{\mathrm{KP}}_{\widehat{x}x,\alpha}$; $\alpha = 0.6$

Performance robustness under input constraint, i.e. for $J^{\mathrm{KP}}_{\widehat{x}x,\alpha}$ (19) has been investigated in simulation way where the feedback gain $\widehat{f}_{\widehat{x}x,\alpha}$ for estimates $\widehat{a}$, $\widehat{b}$ was derived iteratively using the algorithm given in [13]. To this end Eq. (20) was used and the cost function was calculated analogously to (50). Condition of robustness performance similar to (46) is illustrated in Figs. 3–5 for $\alpha = 0.4, 0.5, 0.6$, respectively where other parameters are as in Fig. 2. For $\delta_a = \delta_b = 1.0$ we have $J^{\mathrm{KPmin}}_{\widehat{x}x,\alpha} = 1.3005$, $f^{\mathrm{opt}}_{\widehat{x}x,\alpha} = -0.7333$ for $\alpha = 0.4$, $J^{\mathrm{KPmin}}_{\widehat{x}x,\alpha} = 1.2991$, $f^{\mathrm{opt}}_{\widehat{x}x,\alpha} = -0.7308$ for $\alpha = 0.5$, $J^{\mathrm{KPmin}}_{\widehat{x}x,\alpha} = 1.2985$, $f^{\mathrm{opt}}_{\widehat{x}x,\alpha} = -0.7295$ for $\alpha = 0.6$, and again $k^{\mathrm{opt}} = 0.2655$ in all cases. It should be noticed that $J^{\mathrm{KPmin}}_{\widehat{x}x,\alpha} > J^{\mathrm{KPmin}}_{\widehat{x}x}$ as a result of constraint. From Figs. 3–5 one can observe that the stronger the constraint (i.e. the smaller the value of α) the larger the area of performance robustness. For example, if $\delta_b = 0.5$ the corresponding robustness intervals for δ_a are: for $\alpha = 0.4$, $0.485 \le \delta_a \le 1.771$, for $\alpha = 0.5$, $0.485 \le \delta_a \le 1.369$, for $\alpha = 0.6$, $0.485 \le \delta_a \le 1.301$, for $\alpha = \infty$, $0.485 \le \delta_a \le 1.177$.

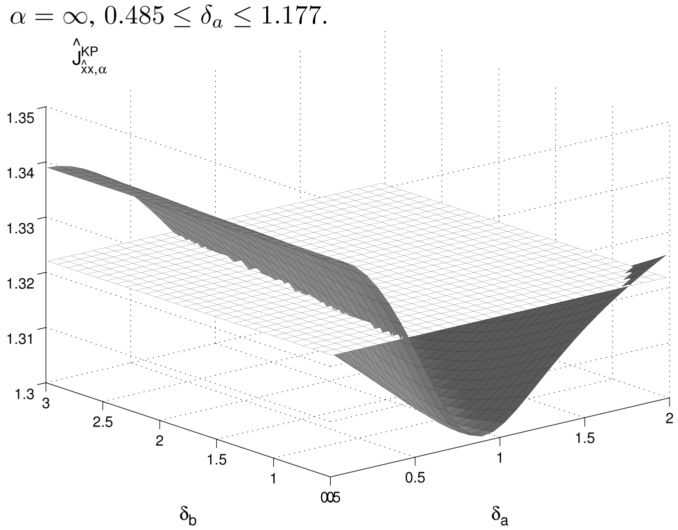

Fig. 3. Illustration of performance robustness w.r.t. $J^{\mathrm{KP}}_{\widehat{x}x,\alpha}$; $\alpha = 0.4$

In the case of the KF-based controller (26) the minimal value of the cost function, i.e. $J^{\mathrm{KFmin}}_{\widehat{x}x}(f^{\mathrm{opt}}_{\widehat{x}x}, \overline{k}^{\mathrm{opt}})$ can be calculated using

$$\sigma^2_{\widehat{x}/\widehat{x}} = \frac{(\overline{k}^{\mathrm{opt}})^2(\sigma_v^2 + \sigma_w^2 + \sigma^2_{\widetilde{x},\mathrm{KF}})}{1 - (bf^{\mathrm{opt}}_{\widehat{x}x} - a)^2}, \tag{51}$$

$$\sigma^2_{\widetilde{x},\mathrm{KF}} = \frac{(1 - \overline{k}^{\mathrm{opt}})^2\sigma_v^2 + (\overline{k}^{\mathrm{opt}})^2\sigma_w^2}{1 - a^2(1 - \overline{k}^{\mathrm{opt}})^2}. \tag{52}$$

To this end the formula (32) can be used putting $\alpha = \infty$. The expression (51) is obtained using Eqs. (21), (26), (1), (2), and noting that $\widetilde{x}^{\mathrm{KF}}_t = x_t - \widehat{x}_{t/t}$.

Analogously to (46), a condition for performance robustness can again be determined

$$\widehat{J}^{\mathrm{KF}}_{\widehat{x}x} \le (1 + \Delta) J^{\mathrm{KFmin}}_{\widehat{x}x}, \tag{53}$$

where $\widehat{J}^{\mathrm{KP}}_{\widehat{x}x} = J^{\mathrm{KP}}_{\widehat{x}x}(\widehat{f}^{\mathrm{opt}}_{\widehat{x}x}, \widehat{\overline{k}}^{\mathrm{opt}})$, and the feedback and predictor gains $\widehat{f}^{\mathrm{opt}}_{\widehat{x}x}$, $\widehat{\overline{k}}^{\mathrm{opt}}$ are calculated on the basis of (6) and (25) with use of estimates $\widehat{a}$, $\widehat{b}$.

The expressions for $x(z)$ and $u(z)$ are necessary to calculate $\widehat{J}^{\rm KF}_{\widehat{x}x}$. Correspondingly to (48) and (49) one can obtain

$$x(z) = \frac{(z+\widehat{B})v(z) + zb\widehat{A}w(z)}{z^2 + z(a+\widehat{B}-b\widehat{A}) + a\widehat{B}}, \tag{54}$$

$$u(z) = \frac{z\widehat{A}v(z) + z\widehat{A}(z+a)w(z)}{z^2 + z(a+\widehat{B}-b\widehat{A}) + a\widehat{B}}, \tag{55}$$

where $\widehat{A} = \widehat{\overline{k}}^{\rm opt}(\widehat{a},\widehat{b})\widehat{f}^{\rm opt}_{xx}(\widehat{a},\widehat{b})$, $\widehat{B} = (1 - \widehat{\overline{k}}^{\rm opt}(\widehat{a},\widehat{b}))(\widehat{a} - \widehat{b}\widehat{f}^{\rm opt}_{xx}(\widehat{a},\widehat{b}))$.

According to (44) and (51), (52) we have $J^{\rm KFmin}_{\widehat{x}x} = 1.1766$, i.e. the smaller value compared with the KP-based controller. Again, the same value can be obtained from (50) and (54), (55) for $\delta_a = \delta_b = 1$ or from (27). Moreover, we have $\overline{k}^{\rm opt} = 0.5311$.

The illustration of the performance robustness for KF-based controller is shown in Fig. 6 for the same parameters as in Fig. 2. One can observe a smaller region of performance robustness w.r.t. KP-based controller.

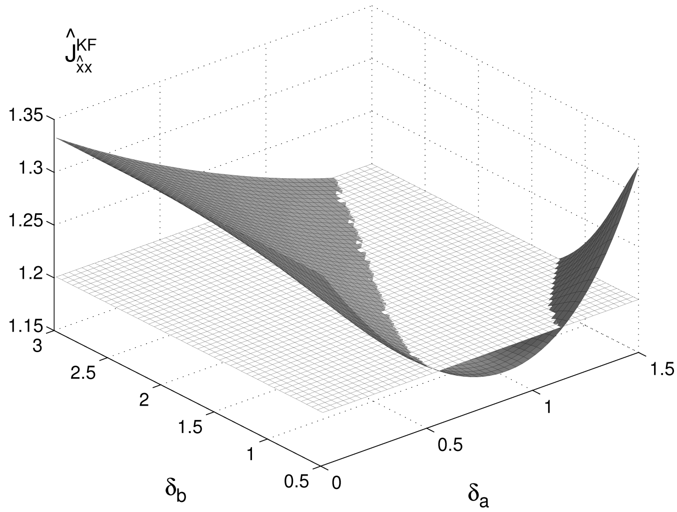

Fig. 6. Illustration of performance robustness w.r.t. $J^{\rm KF}_{\widehat{x}x}$

Performance robustness under input constraint, i.e. for $J^{\rm KF}_{\widehat{x}x,\alpha}$ (32) has been investigated in simulation way where the feedback gain $\widehat{f}_{\widehat{x}x,\alpha}$ for estimates $\widehat{a}$, $\widehat{b}$ was derived iteratively using again the algorithm given in [13]. To this end Eq. (31) was used and the cost function was calculated analogously to (50). Condition of robustness performance similar to (46) is illustrated in Figs. 7–9 for $\alpha = 0.4, 0.5, 0.6$, respectively where other parameters are as in Fig. 2. For $\delta_a = \delta_b = 1.0$ we have $J^{\rm KFmin}_{\widehat{x}x,\alpha} = 1.2223$, $f^{\rm opt}_{\widehat{x}x,\alpha} = -0.7464$ for $\alpha = 0.4$, $J^{\rm KFmin}_{\widehat{x}x,\alpha} = 1.2092$, $f^{\rm opt}_{\widehat{x}x,\alpha} = -0.7420$ for $\alpha = 0.5$, $J^{\rm KFmin}_{\widehat{x}x,\alpha} = 1.2000$, $f^{\rm opt}_{\widehat{x}x,\alpha} = -0.7383$ for $\alpha = 0.6$, and again $\overline{k}^{\rm opt} = 0.5311$ in all cases. It should be noticed that the control performance is better than the corresponding performance of control system with KP. Again, $J^{\rm KFmin}_{\widehat{x}x,\alpha} > J^{\rm KFmin}_{\widehat{x}x}$ as a result of constraint. Similarly to the KP case one can observe that if the constraint gets stronger then the area of performance robustness gets larger.

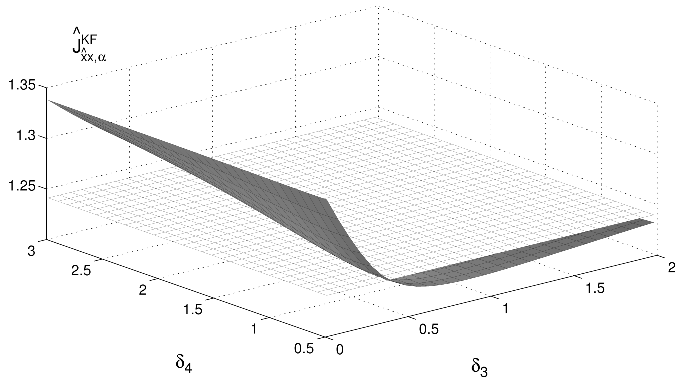

Fig. 7. Illustration of performance robustness w.r.t. $J^{\rm KF}_{\widehat{x}x,\alpha}$; $\alpha = 0.4$

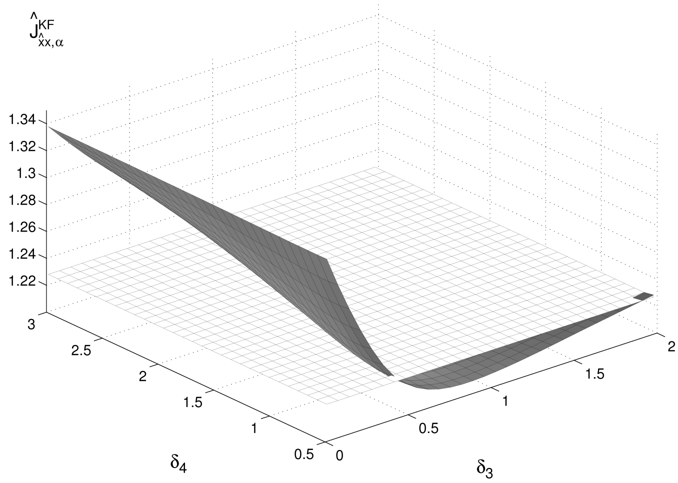

Fig. 8. Illustration of performance robustness w.r.t. $J^{\rm KF}_{\widehat{x}x,\alpha}$; $\alpha = 0.5$

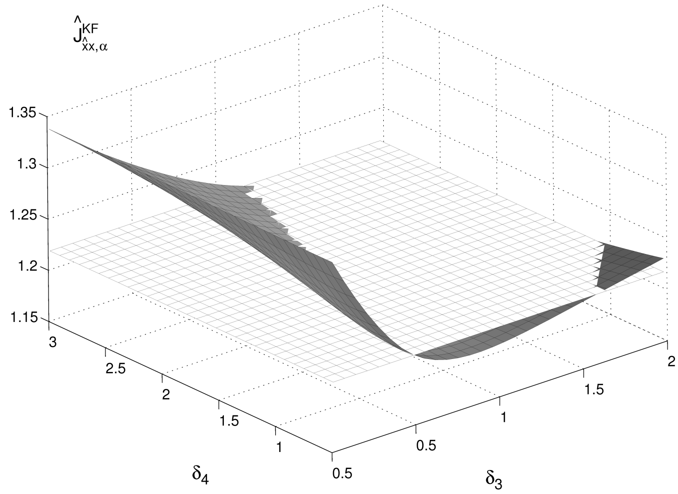

Fig. 9. Illustration of performance robustness w.r.t. $J^{\rm KF}_{\widehat{x}x,\alpha}$; $\alpha = 0.6$

Figures 10, 11 represent the performance robustness for J_{yx} and for J_{yy}, respectively, both under the same conditions as in Figs. 2, 6. For $\delta_a = \delta_b = 1.0$ and $\sigma_v^2 = \sigma_w^2 = 1.0$ we have $J^{\rm min}_{yx} = 1.1771$, $f^{\rm opt}_{yx} = -0.3930$ and $J^{\rm min}_{yy} = 2.1931$, $f^{\rm opt}_{yy} = -0.3993$. It can be observed that the controller (33) performs less robust w.r.t. the cost (35) than the controller (38) w.r.t. the cost (39), and the robustness region in Fig. 10 is smaller than the corresponding region in Fig. 6.

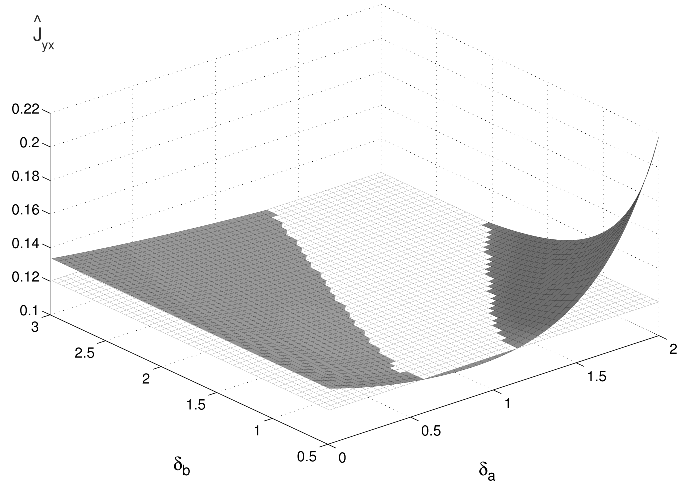

Fig. 10. Illustration of performance robustness w.r.t. J_{yx}

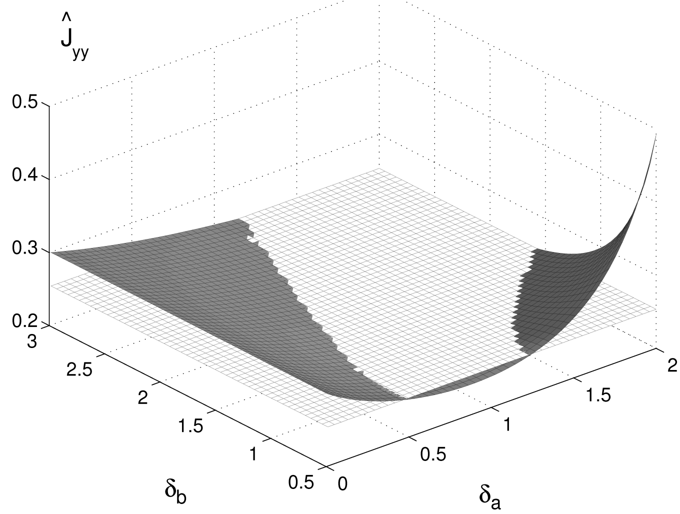

Fig. 11. Illustration of performance robustness w.r.t. J_{yy}

5.1. Adjustment of the weight q**.** Consider again the system (1) and the control law (3) for the minimization of the stationary cost function (4). The standard LQG approach gives the solution (8) where the explicit formula for P_f is

$$P_f = \frac{q(a^2-1)+b^2+\sqrt{q^2(1-a^2)^2+2qb^2(1+a^2)+b^4}}{2b^2}. \tag{56}$$

Obviously, (56) together with (8) give the same f_{xx}^{opt} as (6). To guarantee that the controller yields the minimal cost function the feedback gain for perturbed system, i.e. $f_{xx,\delta}$ has to fulfill the equation

$$f_{xx,\delta} = (\delta_b^2 b^2 P_{f,\delta} + q_\delta)^{-1}\delta_a\delta_b ab P_{f,\delta} = f_{xx}^{\mathrm{opt}}, \tag{57}$$

where $P_{f,\delta}$ is the solution of Riccati Eq. (7) for perturbed system, i.e. according to (56)

$$\begin{aligned} P_{f,\delta} = {} & \frac{q_\delta(\delta_a^2a^2-1)+\delta_b^2b^2}{2\delta_b^2b^2} + \\ & + \frac{\sqrt{q_\delta^2(1-\delta_a^2a^2)^2+2q_\delta\delta_b^2b^2(1+\delta_a^2a^2)+\delta_b^4b^4}}{2\delta_b^2b^2}. \end{aligned} \tag{58}$$

From (57) and (8) one can obtain

$$q_\delta = P_{f,\delta}\delta_b\left(\delta_a b^2 - \delta_b b^2 + \frac{\delta_a q}{P_f}\right), \tag{59}$$

which under $P_{f,\delta} > 0$ and $q_\delta > 0$ gives

$$\delta_b < \delta_a\left(1 + \frac{q}{P_f b^2}\right), \tag{60}$$

when $\delta_b > 0$, and

$$\delta_b > \delta_a\left(1 + \frac{q}{P_f b^2}\right), \tag{61}$$

when $\delta_b < 0$. It is easy to see that the uncertainties δ_a, δ_b must have the same sign. Moreover, their values should be such that $f_{xx,\delta}(\delta_a,\delta_b)$ ensures the closed-loop stability. By putting (58) into (59) a solution for q_δ can be found. A plot for q_δ obtained under condition (60) is shown in Fig. 12 for $a = -0.5$, $b = 0.5$, $q = 0.1$. Taking for example $\delta_b = \delta_a = 1.5$ one obtains $q_\delta = 0.0446$, $P_{f,\delta} = 0.2126$, and obviously $f_{xx,\delta} = -0.7284$ yielding the optimal cost.

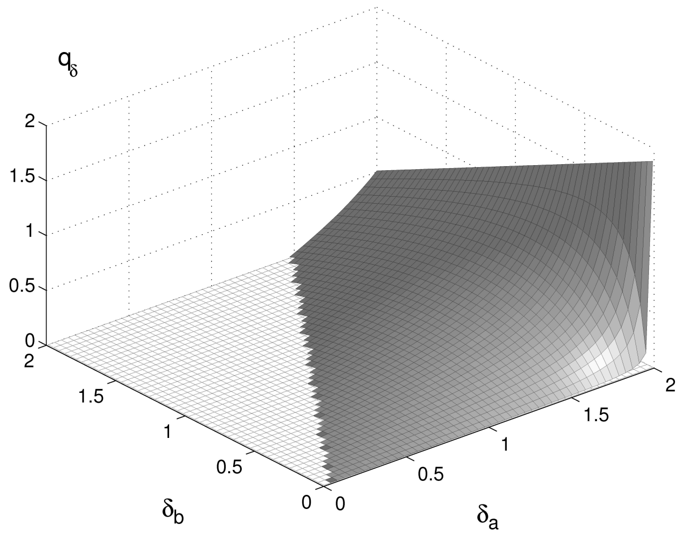

Fig. 12. Plot of the weight q_δ

In the light of above results one can see that the adjustment of the weight q could be used to compensate the effect of the parameter estimates bias represented by δ_a, δ_b on the suboptimal performance of the controller $u_t = f_{xx}(\widehat{a}_t,\widehat{b}_t)x_t$, thus to restore the optimality.

6. Stability robustness

From (48) one can find the characteristic equation of the closed-loop system with parametric uncertainties

$$z^2 + \alpha_1 z + \alpha_0 = 0, \tag{62}$$

where

$$\alpha_1 = a + \widehat{B},$$

$$\alpha_0 = a\widehat{B} - b\widehat{A}$$

and

$$\widehat{A} = \widehat{k}^{\mathrm{opt}}(\widehat{a},\widehat{b})\widehat{f}_{xx}^{\mathrm{opt}}(\widehat{a},\widehat{b}),$$

$$\widehat{B} = \widehat{a} + \widehat{k}^{\text{opt}}(\widehat{a}, \widehat{b}) - \widehat{b}\widehat{f}^{\text{opt}}_{xx}(\widehat{a}, \widehat{b}).$$

The closed-loop stability condition resulting from (62) for $q = 0.1$ and the true parameters $a = a^o = -0.5, b = b^o = 0.5$ is illustrated in Fig. 13, where the flat and light surface is set at the value 1 (stability region) while the dark surface corresponds to the largest absolute value of the root resulting from (62) for the given uncertainties δ_a, δ_b (instability region). It can be observed that for increasing uncertainty δ_b the admissible uncertainty δ_a is decreasing.

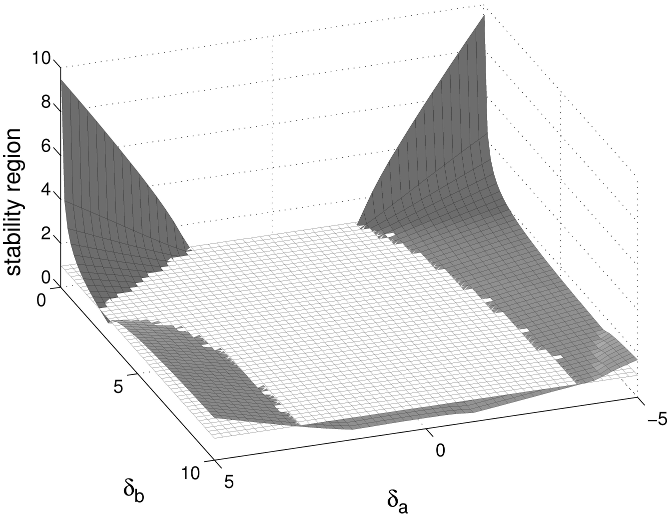

Fig. 13. Illustration of stability robustness; KP-based controller

In the case of KF-based controller, the corresponding characteristic equation of the closed-loop system follows from (54). Taking the notation of (62) we have $\alpha_1 = a + \widehat{B} - b\widehat{A}$, $\alpha_0 = a\widehat{B}$ with $\widehat{A}$, $\widehat{B}$ as in (54). The closed-loop stability condition resulting from (62) is illustrated in Fig. 14 for the same parameters as in Fig. 13. One can see that the stability region gets essentially smaller w.r.t. the KP-based controller. The corresponding stability region for controller (33) is shown in Fig. 15. It is seen that when output feedback is used an enlargement of robustness region is obtained w.r.t. the KF-based controller.

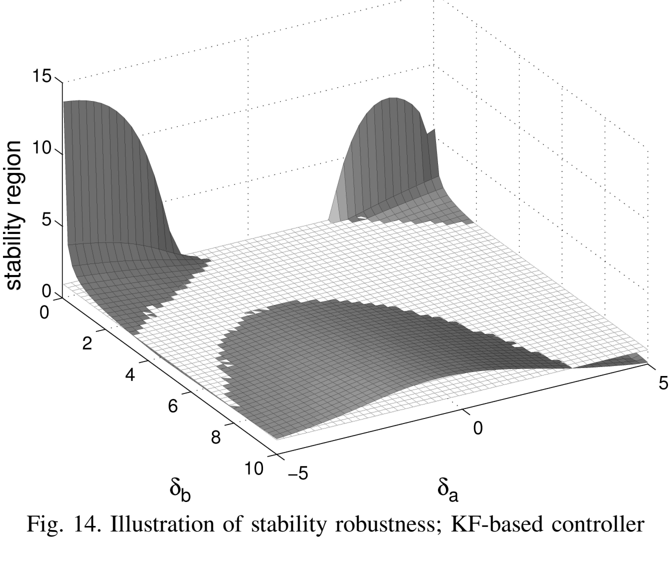

Fig. 14. Illustration of stability robustness; KF-based controller

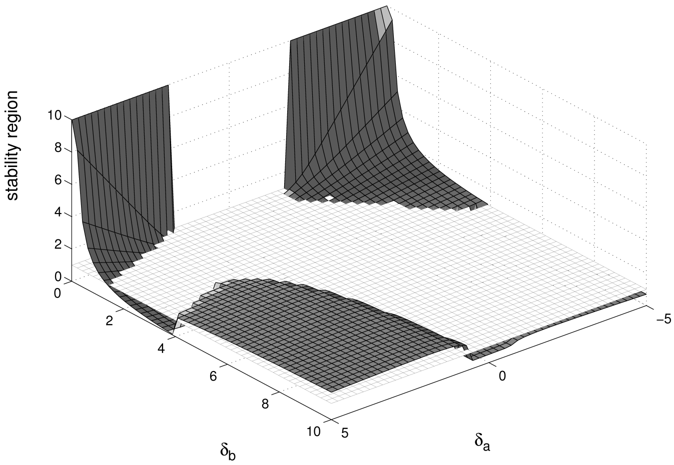

Fig. 15. Illustration of stability robustness; output feedback controller

7. Self-tuning implementation

In the self-tuning context the current control signal for the system (1) will be $u_t = f_{xx}(\widehat{a}_t, \widehat{b}_t)x_t$, where the estimates $\widehat{a}_t, \widehat{b}_t$ can be obtained from a recursive estimation scheme like, for example stochastic gradient algorithm.

It is interesting to notice [16] that the only way that the adaptive control law can asymptotically converge to the optimal control law $u_t = f^{\text{opt}}_{xx}(a, b)x_t$ is for the parameter estimates $(\widehat{a}_t, \widehat{b}_t)$ to converge precisely to the true parameters (a^o, b^o). However, as it was shown in [16] this does not happen in the case of the considered cost function (4), i.e. when $q \neq 0$. This observation is an intrinsic feature of adaptive LQG control. If $q = 0$ then $f_{xx}(\widehat{a}_t, \widehat{b}_t)$ converges to $f_{xx}(a^o, b^o)$ despite the estimates $\widehat{a}_t, \widehat{b}_t$ do not converge to the true values a^o, b^o but only to a multiple of the true values, i.e. $\lim_{t\to}(\widehat{a}_t, \widehat{b}_t) = (\delta a^o, \delta b^o)$ where δ is a random scalar. However, this is sufficient to ensure that $f_{xx}(\widehat{a}_t, \widehat{b}_t)$ converges to $f_{xx}(a^o, b^o)$.

Extra simulations showed that the larger the weight q in (4), the estimates converge farther away from the true system parameters.

Similar observations are true when the controllers (10) and (26) are combined with the extended KP (EKP) which provides also the parameter estimates $\widehat{a}_t, \widehat{b}_t$ (in the case of the controller (26) the EKP is used only for parameter estimates). The exemplary runs of the parameter estimates and feedback gain estimates are shown in Figs. 16, 17 for the controllers (10) and (26), respectively where $q = 0.1$ and $\sigma_v^2 = \sigma_w^2 = 1.0$. The initial parameter estimates are set at $\widehat{a}_0 = -0.1, \widehat{b} = 0.1$, i.e. they are in the stability regions shown in Figs. 13, 14. One can see that in both cases the parameter estimates do not converge to the true values of $a^o = -0.5$, $b^o = 0.5$ and the feedback gain estimates do not converge to the true value of $f^{\text{opt}}_{\widehat{x}\widehat{x}} = -0.7284$. The limiting values of parameter estimates $\widehat{a}, \widehat{b}$ are -0.5594, 0.5527 and -0.5721, 0.5878 for the KP-based and KF-based controllers, respectively. Correspondingly, one gets $\delta_a = 1.1118, \delta_b = 1.1054$ and $\delta_a = 1.1442, \delta_b = 1.1756$. From Figs. 2, 6 one can see that in both cases the estimates are in the robustness region, and the performance differs from optimal one not much than by 2%.

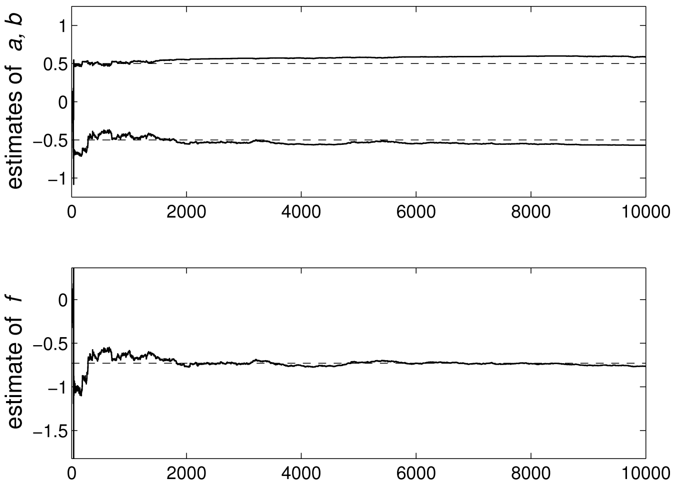

Fig. 16. Estimates in KP-based self-tuning control

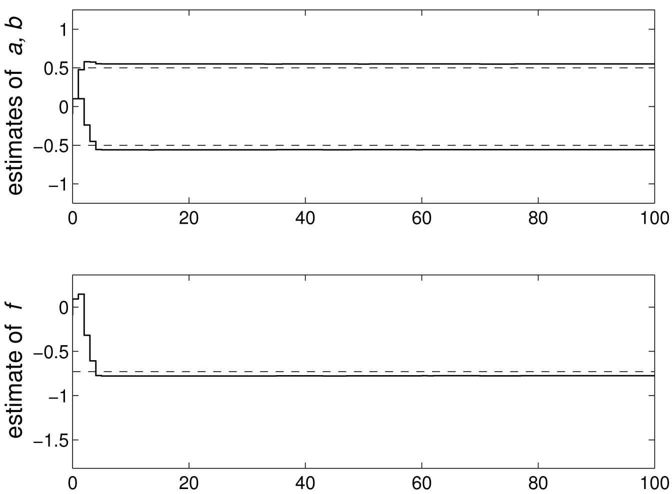

Fig. 17. Estimates in KF-based self-tuning control

8. Conclusions

Discrete-time adaptive LQG control of first-order systems is considered with emphasis on robustness analysis. Different feedback control configurations are taken into account. Stability and performance robustness of adaptive control for first-order system with parametric uncertainty in the LQG framework are analyzed and illustrated. Both unconstrained and constrained-input cases are investigated. The performance robustness conditions determined w.r.t. the optimal cost function are given and the question of adjustment of the control weight q used in order to restore the optimality is also presented. Problems with application of certainty equivalence in the self-tuning implementation for the proposed control algorithms are discussed.

REFERENCES

[1] K.J. Åström and B. Wittenmark, *Adaptive Control*, Addison Wesley, Massachusetts, 1989.

[2] I.D. Landau, R. Lozano, and M. M'Saad, *Adaptive Control*, Springer, Berlin, 1998.

[3] T.-H. Kim and T. Sugie, "A new adaptive parameter estimation algorithm for robust constrained predictive control", *Proc. 44th IEEE CDC and ECC Conf.* 2, 284–289 (2005).

[4] T.T. Tay and J.B. Moore, "Adaptive LQG controller with loop transfer recovery", *Int. J. Adapt. Contr. and Signal Proc.* 5, 135–149 (1991).

[5] T.J. Moir, "Frequency-domain approach to state-space LQG self-tuning control", *IEE Proc.* D 138 (4), 372–380 (1991).

[6] M.J. Grimble, "Controllers for LQG self-tuning applications with coloured measurement noise and dynamic costing", *IEE Proc. Pt.* D 133 (1), 19–29 (1986).

[7] P.R. Kumar, "Optimal adaptive control of linear-quadratic-gaussian systems", *SIAM J. Control and Optimization* 21 (2), 163–178 (1983).

[8] M.C. Campi and P.R. Kumar, "Optimal adaptive control of an LQG system", *Proc. 35th Conf. on Decision and Control* 1, 349–353 (1996).

[9] O.L.R. Jacobs, "Cost of uncertainty about controlled objects", *IEE Proc. Pt.* D 136 (4), 177–186 (1989).

[10] A.E.B. Lim , J.B. Moore and L. Faybusovich, "Separation theorem for linearly constrained LQG optimal control", *Systems Control Letters* 28, 227–235 (1996).

[11] J.M. Maciejowski, *Multivariable Feedback Design*, Addison-Wesley Publishing Company, Massachusetts, 1989.

[12] B.D.O. Anderson and J.B. Moore, *Optimal Control. Linear Quadratic Methods*, Prentice Hall, New Jersey, 1990.

[13] H.T. Toivonen, "Suboptimal control of discrete stochastic amplitude constrained systems", *Int. J. Control* 37 (3), 493–502 (1983).

[14] A. Królikowski, "Amplitude constrained adaptive LQG control of first-order systems", *Int. J. Adapt. Contr. and Signal Proc.* 9 (3), 285–299 (1995).

[15] P.M. Mäkilä and H.T. Toivonen, "Computational methods for parametric LQ problems – a survey", *IEEE Trans. Automat. Contr.* AC 32 (8), 658–671 (1987).

[16] W. Lin, P.R. Kumar, and T.I. Seidman, "Will the self-tuning approach work for general cost criteria?", *Systems Control Letters* 6, 77–85 (1985).